Fermi pseudopotential in arbitrary dimensions

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(Received 13 August 1990)

The theory of zero-range potentials is investigated in an arbitrary number of dimensions. Except for the trivial one-dimensional case the zero-range potentials are described by nonlocal operators called Fermi pseudopotentials. It is shown that in odd dimensions the Fermi pseudopotentials involve a very simple regularization operator. In even dimensions with the help of dimensional regularization, explicit formulas for the Fermi pseudopotentials are derived. The Green's functions, the propagators, and the exact solutions of the Lippmann-Schwinger equations are derived in explicit forms. In odd dimensions $d=1$ and 3 the Fermi pseudopotentials can be applied to describe multiphoton processes of atoms and molecules with very short-range interactions. In even dimension $d=2$ the Fermi pseudopotential can be applied to describe tunneling from laser-driven quantum wells. Physical applications involving higher d are also possible.

I. INTRODUCTION

The simplest zero-range potential is given by a δ function. Such a potential has been first introduced and used by Fermi in 1936 in the investigation of the scattering of neutrons with bound hydrogen atoms.[1] In later applications of this zero-range potential in nuclear physics[2–4] it has been recognized that in three dimensions a tempered operator of the form $(\partial/\partial r)r$ is required in order to make the Schrödinger equation well behaved. Because of this property a zero-range potential of the δ type tempered by a proper operator has been called the Fermi pseudopotential. In one dimension the δ potential has been extensively used as an exactly soluble model of a many-body theory.[5] The one- and the three-dimensional Fermi pseudopotentials have been apllied to quark tunneling,[6] H photodetachment and multiphoton ionization,[7–11] harmonic generation by strong laser light,[12] periodic lattices,[13] and in the theory of transport phenomena.[14] The five-dimensional Fermi pseudopotential has been used in the problem of narrow resonances in infinite, linear, uniform arrays (Yagi-Uda antenna array).[15] A theory of a light particle moving in the field of two heavy projectiles has also been treated by zero-range potentials.[16]

Despite these very extensive applications in different branches of physics, no systematic discussion of the Fermi pseudopotential in arbitrary number of dimensions can be found in the literature. With the exception of Ref. 15, in which the form of the pseudopotential has been investigated in five dimensions and possible tempering operators conjectured for dimensions 2, 4, and 6, no explicit derivation of these potentials and their Green's functions appear to have been published so far.

It is the purpose of this paper to fill this gap and to provide a more or less complete description of Fermi pseudopotentials, and their tempering operators in any number of dimensions.

Just formally, the δ potential can be written in the following form:

$$V(\mathbf{r})=-a_d\delta^{(d)}(\mathbf{r}) , \tag{1.1}$$

where d is the number of dimensions and a_d is the strength of this potential that can vary in form with d. As has been recognized in the early investigations of this zero-range interaction the formal form of the potential (1.1) can lead to singularities which have to be tempered by a proper regularization operator $\hat{R}_d$. In such a case a well-behaved Hamiltonian can be obtained if instead of the formula (1.1), a regularized potential, the Fermi pseudopotential denoted by V_F is used. The Fermi pseudopotential is accordingly defined by the following relation:

$$V_F(\mathbf{r})=-a_d\delta^{(d)}(\mathbf{r})\hat{R}_d . \tag{1.2}$$

In this paper a systematic discussion of this pseudopotential and its regularization operators $\hat{R}_d$ is going to be performed. It will be shown that if the time integration and the limit of $\mathbf{r}\to\mathbf{0}$ are performed in a certain order no regularization is required for an odd number of dimensions. In this case exact and explicit expressions for the energy Green's functions are derived. As a results exact wave functions (including scattering amplitudes) in any number of odd dimensions can be obtained. Explicit formulas are derived for the time-dependent propagators in one and three dimensions. In order to exchange the ordering of the time integration and the limit of $\mathbf{r}=\mathbf{0}$, regularization operators are required. The form and the structure of these regularization operators are derived.

In even number of dimensions the singularities associated with the zero-range potential are much more severe. With the help of dimensional regularization it is shown that the Fermi pseudopotential can be used in odd dimensions with highly nontrivial regularization operators $\hat{R}_d$. The explicit formula for the Green's function in two dimensions derived in this paper opens a new class of possible applications. Dynamical properties of quantum wells on a surface are just an example of such possible applications.[17]

In general zero-range potentials are widely used in model calculations involving very short-range interactions of atoms and molecules.[18]

This paper is organized in the following way. In Sec. II we derive the exact expression for the energy Green's function using Laplace transform. From this expression valid in any number of dimensions an exact solution of the Lippmann-Schwinger equations is obtained. In Sec. III the scattering and the bound-state problem in odd dimensions are discussed. Explicit formulas for $d=1$, 3, and 5 are derived. In Sec. IV explicit expressions for the energy Green's function are obtained in odd dimensions. An exact derivation of the temporal propagator in $d=1$ and 3 is presented. In Sec. V the problem of the Fermi pseudopotentials in odd dimensions is investigated. Proper regularization operators are derived and discussed. In Sec. VI the Green's function in even dimensions is derived. With the help of dimensional regularization explicit expressions for the regularized propagator are obtained. In Sec. VII a detailed discussion of the two-dimensional Fermi potential is performed. Scattering amplitudes, the bound state, and regularization operators are investigated. Finally Sec. VIII contains some conclusions. The paper ends with two Appendices devoted to technical derivations of the regularization operators.

II. THE ENERGY GREEN'S FUNCTION

Instead of working with the Schrödinger equation, we chose to study the integral equation for the full Green's function (propagator) with the zero-range potential. In d dimensions this equation has the following form:

$$K(\mathbf{r},t;\mathbf{r}_0,0)=K_0(\mathbf{r},t;\mathbf{r}_0,0)+\frac{i}{\hbar}\int_0^t ds\int d^dr' K_0(\mathbf{r},t;\mathbf{r}',s)V(\mathbf{r}')\times K(\mathbf{r}',s;\mathbf{r}_0,0) \qquad (2.1)$$

where we have set for convenience the initial time at $t=0$. The free propagator in d dimension has the following well-known form:

$$K_0(\mathbf{r},t;\mathbf{r}_0,0)=\left[\frac{m}{2\hbar\pi it}\right]^{d/2}\exp\left[\frac{im(\mathbf{r}-\mathbf{r}_0)^2}{2\hbar t}\right], \qquad (2.2)$$

where $\mathbf{r}$ and $\mathbf{r}_0$ are d-dimensional vectors. In the following we shall solve this integral equation applying the Laplace transform to an initial value problem. We define the Laplace-transformed Green's function by the following relation:

$$G(\mathbf{r};\mathbf{r}_0)=\int_0^\infty dt\, e^{-zt}K(\mathbf{r},t;\mathbf{r}_0,0) \quad \text{for } z\in\mathbb{C}^1 . \qquad (2.3)$$

For notational convenience we have dropped the complex parameter z from the left-hand side of this relation. Using the Laplace transform and the explicit form of the δ-function potential we map the integral equation into an algebraic equation which can be solved exactly. As a result we obtain that

$$G(\mathbf{r};\mathbf{r}_0)=G_0(\mathbf{r};\mathbf{r}_0)+G_1(\mathbf{r};\mathbf{r}_0)\ , \qquad (2.4)$$

where G_0 is the Laplace transform of the free propagator K_0 and G_1 is defined by the following algebraic expression:

$$G_1(\mathbf{r};\mathbf{r}_0)=\frac{ia_d}{\hbar}\,\frac{G_0(\mathbf{r};\mathbf{0})G_0(\mathbf{0};\mathbf{r}_0)}{1-(ia_d/\hbar)G_0(\mathbf{0};\mathbf{0})} . \qquad (2.5)$$

At this point we make a very important remark that in order to calculate $G_0(\mathbf{0};\mathbf{0})$ we have first set $\mathbf{r}=\mathbf{r}_0=\mathbf{0}$ in Eq. (2.2) and then performed the Laplace transform with respect to time. If this order is preserved no regularization operator $\hat{R}_d$ is required at this stage of the calculations. Equations (2.4) and (2.5) give the exact expression for the energy propagator in the presence of zero-range potential. We see that this exact solution is a sum of the free propagator G_0 with a term G_1 which is given by a simple algebraic expression (2.5). We stress again at this point that this expression should be calculated preserving the proper order of the integration and the limiting procedure. If this order is preserved, all the relevant functions can be evaluated using standard integral tables.[19] The Laplace transform of the free propagator is

$$\begin{aligned}G_0(\mathbf{r},\mathbf{r}_0)&=2e^{(i\pi/4)(-d/2-1)}\pi^{-d/2}\alpha^{d/4+1/2}\\&\times z^{d/4-1/2}R^{-d/2+1}\\&\times K_{-d/2+1}(2R\exp(-i\pi/4)\sqrt{\alpha z}\,)\\&\qquad\text{with } z\in\mathbb{C}^1\ ,\end{aligned} \qquad (2.6)$$

where $R=|\mathbf{r}-\mathbf{r}_0|$, $\alpha=m/2\hbar$, and $K_{-d/2+1}$ is the modified Bessel function.[19]

In the same way we obtain (according to our ordering of operations)

$$\begin{aligned}G_0(\mathbf{0};\mathbf{0})&=\int_0^\infty dt\, e^{-zt}\left[\frac{\alpha}{\pi it}\right]^{d/2}\\&=\left[\frac{\alpha}{\pi i}\right]^{d/2}z^{d/2-1}\Gamma\left[-\frac{d}{2}+1\right].\end{aligned} \qquad (2.7)$$

Note that due to the given ordering of time integration and the limiting procedure $G_0(\mathbf{0};\mathbf{0})$ cannot be obtained from $G_0(\mathbf{r},\mathbf{r}_0)$ as a simple limit of $\mathbf{r}=\mathbf{r}_0=\mathbf{0}$. In fact, this limit, if performed in (2.6), turns out to be singular. On the other hand, $G_0(\mathbf{0};\mathbf{0})$ from (2.7) is given by a Γ function which is an analytic function with simple poles for $-d/2+2=0,-1,-2,\ldots$, i.e., has a pole singularity only in even dimensions. In odd dimensions this expression is regular everywhere and Eqs. (2.4)–(2.7) describe an explicit solution for the Green's function in any odd number of dimensions. Note that if the proper order of integration and limiting procedure is followed, the zero-range potential given by (1.1) can be inserted into the integral Eq. (2.1). In such a case the Fermi pseudopotential (2.2) is not necessary. But this is true only in an odd number of dimensions.

In the case of an even number of dimensions, Eq. (2.7) is singular and the formal form of the potential (1.1) cannot be used anymore. We shall study the properties and the form of the Fermi pseudopotential in an odd number of dimensions in Sec. VI.

The exact and the explicit solution for the Green's function G, given by Eqs. (2.4) and Eq. (2.5), allows an exact solution of the scattering problem in the presence of this zero-range potential. The outgoing scattered wave function can be easily calculated from the Lippmann-Schwinger equation using the Green's function from Eq. (2.4). As a result we obtain that the scattered wave is

$$\psi_{\mathbf{k}}^{(+)}(\mathbf{r})=e^{i\mathbf{k}\cdot\mathbf{r}}+\frac{ia_d}{\hbar}\frac{G_0(\mathbf{r};0)}{1-(ia_d/\hbar)G_0(0;0)} , \tag{2.8}$$

where $\mathbf{k}$ is the wave vector of the incoming plane wave and the complex parameter z has to evaluate at the point $z=e^{i\pi/2}k^2/4\alpha$. The incoming scattered wave $\psi_{\mathbf{k}}^{(-)}(\mathbf{r})$ can be obtained by a complex conjugation of Eq. (2.8). The asymptotic scattered wave in d dimensions have the following well-known form:

$$\frac{e^{ikr}}{r^{d/2-1/2}} . \tag{2.9}$$

Calculating the asymptotic behavior of the scattering solution for $r\to\infty$, we obtain from Eq. (2.8), the scattering amplitude $f_{\mathbf{k}}^{(d)}$ in an arbitrary dimension d:

$$\lim_{r\to\infty}\psi_{\mathbf{k}}^{(+)}(\mathbf{r})=e^{-i\mathbf{k}\cdot\mathbf{r}}+f_{\mathbf{k}}^{(d)}\frac{e^{ikr}}{r^{d/2-1/2}} . \tag{2.10}$$

The actual form of the scattering amplitude follows from the asymptotic properties of the modified Bessel functions in the formula (2.6).

From the analytic properties of the scattering amplitude in the complex-k plane, it is possible to establish possible bound states of the particle coupled by a zero-range potential. We shall investigate this problem in the following section.

III. SCATTERING AND BOUND STATES IN ODD DIMENSIONS

In this section we shall investigate the scattering amplitude and possible bound states in odd dimensions. In order to do this we shall set $d=2n+1$ with $n=0,1,\ldots$ in all the relevant expressions derived in the preceding section. From the asymptotic properties of the modified Bessel function for $r\to\infty$ we calculate the scattering amplitude (2.10).[19] Simple algebraic manipulation leads to

$$f_{\mathbf{k}}^{(d)}=\frac{-i^{-n}k^{n-1}(2n-1)!!2^{-n}}{k^{2n-1}+i\dfrac{(2n-1)!!\pi^n\hbar}{a_{2n+1}\alpha}} , \tag{3.1}$$

where we have used the relation $\Gamma(\frac{1}{2}-n)=[(-1)^n 2^n\pi^{1/2}]/(2n-1)!!$.

Let us for the completeness of our arguments write down the scattering amplitude and the scattering outgoing wave functions for $d=1$, 3, and 5.

A. One-dimensional scattering

In this case from Eqs. (3.1) and (2.8) for $n=0$, which is equivalent to $d=1$, we obtain the following expressions:

$$\psi_k^{(+)}(x)=e^{ikx}+f_k^{(1)}e^{i|x|k} , \tag{3.2a}$$

$$f_k^{(1)}=\frac{2a\alpha i}{\hbar}\frac{1}{k-2a\alpha i/\hbar} , \tag{3.2b}$$

where for shorthand notation we have denoted $a=a_1$. From Eq. (3.2) we see that the scattering amplitude has a single pole in the physical k half plane, if $a\geq 0$. This pole of the scattering amplitude corresponds to the energy of the single bound states, with the residue proportional to the wave function of the bound state. The energy of this single bound state is

$$E_0=-\frac{m}{2\hbar^2}a^2 . \tag{3.3}$$

A single bound state with the same energy can be obtained from a one-dimensional square well with length L and energy depth V_0 in the limit of $L\to 0$ and $V_0\to\infty$ with fixed $a=V_0L$. The coupling constant a has in this case the interpretation of the "area" of the potential. We generalize this relation to any number of dimensions. For arbitrary d the "area" of the potential a_d can always be written in the following form:

$$a_d=V_0L^d , \tag{3.4}$$

where V_0 is a coupling constant of dimension energy and where L is a characteristic length. We shall use this important relation in the section devoted to the Fermi pseudopotential in even dimensions.

B. Three-dimensional scattering

In this case from Eq. (3.1) and Eq. (2.8) for $n=1$, which is equivalent to $d=3$, we obtain the following expressions:

$$\psi_{\mathbf{k}}^{(+)}(\mathbf{r})=e^{i\mathbf{k}\cdot\mathbf{r}}+f_k^{(3)}\frac{e^{ikr}}{r} , \tag{3.5a}$$

$$f_k^{(3)}=i\left[k+\frac{i\hbar\pi}{a_3\alpha}\right]^{-1} . \tag{3.5b}$$

Note that $\psi_{\mathbf{k}}^{(+)}(\mathbf{r})$ given by Eq. (3.5a) is the exact (not the asymptotic) outgoing solution of the Lippmann-Schwinger equation in three dimensions. The scattering amplitude has a single pole in the physical k half plane, if $a_3\leq 0$. Note that in the three dimensions in order to sustain a single bound state, the coupling constant of the δ interaction has to change the sign comparing to the one-dimensional case. The energy of this single bound state is

$$E_0=-\frac{2\hbar^6\pi^2}{m^3a_3^2} . \tag{3.6}$$

If we impose an additional condition that this energy should be equal to the bound-state energy in one dimension we obtain the following relation between the coupling constants a and a_3:

$$a_3=-\frac{2\pi\hbar^4}{m^2a} . \tag{3.7}$$

If we assume following Eq. (3.4) that $a=V_0L$ and $a_3=-V_0L^3$ (with $V_0>0$) we obtain from Eq. (3.7) that

$$L=\frac{\hbar}{(mV_0)^{1/2}}(2\pi)^{1/4} . \tag{3.8}$$

C. Five-dimensional scattering

In this case from Eq. (3.1) and Eq. (2.8) for $n=2$, which is equivalent to $d=5$, we obtain the following expressions:

$$\psi_{\mathbf{k}}^{(+)}(\mathbf{r})=e^{i\mathbf{k}\cdot\mathbf{r}}+2^{-1/2}a_5\pi^{-5/2}(-ik)^{3/2}\alpha r^{-3/2}\times K_{3/2}(-irk)\left[1-\frac{ia_5\alpha k^3}{6\hbar\pi^2}\right]^{-1} , \tag{3.9a}$$

$$f_k^{(5)}=-\frac{ia_5k\alpha}{2\hbar\pi^2}\left[1-\frac{ia_5\alpha}{6\hbar\pi^2}k^3\right]^{-1} . \tag{3.9b}$$

where $\mathbf{k}$ is a five-dimensional vector with length k. Note that $\psi_{\mathbf{k}}^{(+)}(\mathbf{r})$ given by Eq. (3.9a) is the exact (not asymptotic) outgoing solution of the Lippmann-Schwinger equation in five dimensions. The scattering amplitude has a pole in the physical k half plane, if $a_5\geq 0$. The energy of this bound state is

$$E_0=-\left[\frac{2^5\hbar^{10}\pi^5}{m^5a_5^2\Gamma^2(-\frac{3}{2})}\right]^{1/3} . \tag{3.10}$$

If we impose an additional condition that this energy should be equal to the bound-state energy in one dimension we obtain the following relation between the coupling constants a and a_5:

$$a_5=\frac{16\hbar^8\pi^{5/2}}{m^4a^3\Gamma(-\frac{3}{2})} . \tag{3.11}$$

If we assume following Eq. (3.4) that $a=V_0L$ and $a_5=V_0L^5$ (with $V_0\geq 0$) we obtain from Eq. (3.11) that

$$L=\frac{\hbar}{(mV_0)^{1/2}}\left[\frac{16\pi^{5/2}}{\Gamma(-\frac{3}{2})}\right]^{1/8} , \tag{3.12}$$

where we have kept the Γ function in order to indicate the dimensionality dependence.

In conclusion of this section, let us investigate the classical limit of the scattering cross section in any odd dimension: $d=2n+1$. From Eq. (3.1) we obtain that

$$\lim_{\hbar\to 0}\left[\frac{d\sigma}{d\Omega}\right]\propto\hbar^{2n}p^{-2n}[(2n-1)!!2^{-n}]^2 , \tag{3.13}$$

where $p=\hbar k$ is the momentum of the incoming scattered particle. From this relation we conclude that in $d=1$ and in the classical limit the zero-range potential acts like a mirror with an order $\hbar$ term getting through the mirror.[21] For $d>1$ the cross section vanishes in the classical limit.

IV. GREEN'S FUNCTION IN ODD DIMENSIONS

In odd dimensions ($d=2n+1$) the zero-range potential Green's function (2.3) is given as a sum of potential-free propagator G_0 and G_1 defined by the following formula:

$$G_1(\mathbf{r};\mathbf{r}_0)=4\frac{ia_{2n+1}}{\hbar}\left[\frac{e^{-i\pi/4}}{\pi}\right]^{2n+1}\alpha^{n+3/2}z^{n-1/2}(rr_0)^{-n+1/2}\frac{K_{-n+1/2}(2re^{-i\pi/4}\sqrt{\alpha z})K_{-n+1/2}(2r_0e^{-i\pi/4}\sqrt{\alpha z})}{1-\frac{ia_{2n+1}}{\hbar}z^{n-1/2}\Gamma(-n+\frac{1}{2})(\alpha/\pi i)^{n+1/2}} , \tag{4.1}$$

where r and r_0 are lengths of n-dimensional vectors $\mathbf{r}$ and $\mathbf{r}_0$. This formula gives the exact expression for the energy Green's function with the zero-range interaction. The time-dependent Green's function (propagator) for this interaction can be written in the following form:

$$K(\mathbf{r},t;\mathbf{r}_0,0)=K_0(\mathbf{r},t;\mathbf{r}_0,0)+K_1(\mathbf{r},t;\mathbf{r}_0,0) , \tag{4.2}$$

where K_0 is the free propagator given by Eq. (2.2) and K_1 is the inverse Laplace transform of the expression (4.1).

In the following we shall present an explicit derivation of K_1 in one and three dimensions.

A. One-dimensional propagator

Taking the inverse transform of Eq. (2.3) and using Eq. (4.1) with $n=0$ we obtain

$$K_1(xt;x_00)=\frac{\alpha a}{\hbar}\oint dz\frac{1}{2\pi i}\frac{e^{zt}}{\sqrt{z}}\frac{\exp[-2(|x|+|x_0|)\sqrt{z\alpha}\exp(-i\pi/4)]}{\sqrt{z}-(i\alpha/\hbar)(\alpha/i)^{1/2}} \tag{4.3}$$

where the z integration is along the imaginary axes with all the singularities on the left if $t\geq 0$. Changing the integration variable $z=-v^2$ and deforming accordingly the integration contour, it is possible to rewrite the integration (4.3) as an integration over a real axis v:

$$K_1=\frac{2a\alpha}{\hbar}\int_{-\infty}^{\infty}dv\frac{1}{2\pi}\frac{\exp[-v^2t-2(|x|+|x_0|)v(\alpha/i)^{1/2}]}{iv-(i\alpha/\hbar)(\alpha/i)^{1/2}} . \tag{4.4}$$

The denominator in this integral can be removed with the help of the following integral trick: $1/b=\int_0^{\infty}d\lambda\, e^{-\lambda b}$. As a result of this step the v integration involves only Gaussian functions and can be performed exactly. As a result we obtain

$$K_1=\frac{a\alpha}{\hbar}\int_0^\infty d\lambda\frac{1}{\sqrt{\pi t}}\exp\left[\frac{i\lambda a}{\hbar}\left[\frac{\alpha}{i}\right]^{1/2}-\frac{[\lambda+2(\alpha/i)^{1/2}(|x|+|x_0|)]^2}{4t}\right] . \tag{4.5}$$

This integral can be expressed in terms of the complex error function $\mathrm{erfc}(u)=(2/\pi)\int_u^\infty d\lambda\, e^{-\lambda^2}$. As a result we obtain the following final and exact expression for the one-dimensional propagator in the presence of zero-range potential:

$$K(x,t;x_0,0)=\left[\frac{\alpha}{\pi i t}\right]^{1/2}\exp\left[\frac{i\alpha(x-x_0)^2}{t}\right]+\frac{a\alpha}{\hbar}\exp(i\zeta^2\alpha/t+u^2)\mathrm{erfc}(u) , \tag{4.6}$$

where $\zeta=|x|+|x_0|$ and $u=(\alpha/i)^{1/2}(\zeta/\sqrt{t}-iat/\hbar)$.

This formula has been derived in the literature by an explicit summation over the complete set of states[20] or in the context of the Feymnan path-integral formalism.[21–23]

A different representation for the δ-potential propagation can be obtained if one recognizes that Eq. (4.5) with a change of the integration variable $\lambda=2(\alpha/i)^{1/2}\zeta'$ leads to a weighted distribution of potential-free propagators.

$$K(x,t;x_0,0)=K_0(x,t;x_0,0)+\frac{2a\alpha}{\hbar}\int_0^{\infty\sqrt{-i}}d\zeta'\exp\left[\frac{a\alpha\zeta'}{\hbar}\right]K_0(z+\zeta',t;0,0) , \tag{4.7}$$

where K_0 is given by (2.2) and $\zeta=|x|+|x_0|$.

B. Three-dimensional propagator

Performing the inverse transform of Eq. (2.3) and using Eq. (4.1) for $n=1$), which is equivalent to $d=3$, we obtain the following expressions:

$$K_1(\mathbf{r},t;\mathbf{r}_0,0)=e^{-i\pi/4}\frac{\alpha^{1/2}}{2\pi r r_0}\oint dz\frac{1}{2\pi i}\frac{\exp[-zt-2(r+r_0)\sqrt{z\alpha}e^{-i\pi/4}]}{\sqrt{z}+\exp(i\pi/4)\pi\hbar/(2a_3\alpha^{3/2})} . \tag{4.8}$$

This integral can be written in the following form:

$$K_1(\mathbf{r},t;\mathbf{r}_0,0)=-\frac{1}{4\pi r r_0}\frac{\partial}{\partial r}\oint dz\frac{1}{2\pi i}\frac{\exp[-zt-2(r+r_0)\sqrt{z\alpha}e^{-i\pi/4}]}{\sqrt{z}\,[\sqrt{z}+\exp(i\pi/4)\pi\hbar/(2a_3\alpha^{3/2})]} . \tag{4.9}$$

We recognize in this integral the one-dimensional integral (4.3) provided that $|x|$ and $|x_0|$ are identified with r and r_0 and the three-dimensional coupling constant a_3 satisfies the condition (3.7). As a result of this identification we obtain the following exact expression for the three-dimensional propagator in the presence of zero-range potential:

$$K(\mathbf{r},t;\mathbf{r}_0,0)=\left[\frac{\alpha}{\pi i t}\right]^{3/2}\exp[i\alpha(\mathbf{r}-\mathbf{r}_0)^2/t]-\frac{1}{4\pi r r_0}\frac{\partial}{\partial r}[\exp(i\zeta^2\alpha/t+u^2)\mathrm{erfc}(u)] , \tag{4.10}$$

where now $\zeta=r+r_0$ and $u=(\alpha/i)^{1/2}(\zeta/\sqrt{t}-iat/\hbar)$.

The three-dimensional propagator can be written also in the integral representation form (4.7)

$$K(\mathbf{r},t;\mathbf{r}_0,0)=K_0(\mathbf{r},t;\mathbf{r}_0,0)-\frac{1}{2\pi r r_0}\frac{\partial}{\partial r}\int_0^{\infty\sqrt{-i}}dz'\exp(2a\alpha\zeta'/\hbar)\times K_0(\zeta+\zeta',t;0,0) . \tag{4.11}$$

V. REGULARIZATION IN ODD DIMENSIONS

Our derivations of the scattering wave functions and propagators have been so far trouble-free under the condition of taking the limit of $\mathbf{r}$ and $\mathbf{r}_0$ to zero first and integrating over the time next, in order to calculate (2.7). With this order of steps the potential given by Eq. (1.1) could be used without any difficulties provided that the number of dimensions has been odd.

If nevertheless we insist that the interpretation and limiting procedure should be interchangable we need to modify this potential in the form given by Eq. (1.2) in order to calculate $G_0(0;0)$ from the expression (2.6) in such a way that no singularities in r and r_0 do occur. Only the use of this Fermi pseudopotential allows for a free interchange of integration with the limiting procedure. The singularities at the origin result from the definition of the modified Bessel function which has the following form:[19]

$$K_{n+1/2}(y)=\sqrt{\pi/2y}\,e^{-y}\sum_{l=0}^{n}\frac{(n+l)!}{l!(n-l)!(2y)^l} . \tag{5.1}$$

In order to remove the singularities we introduce the regularization operator $\hat{R}_d$ defined by the following condition:

$$G_0(0;0)=\lim_{r\to\infty}\hat{R}_d(r)G_0(\mathbf{r};0)\ , \tag{5.2}$$

where the left-hand side of this equation is given by the formula (2.7) and $\hat{R}_d$ acts on $G_0(\mathbf{r};0)$ given by Eq. (2.6).

We note that in one dimension from Eq. (2.5) we have

$$G_0(x;0)=\left[\frac{\alpha}{i}\right]^{1/2} z^{-1/2}\exp(-2|x|\sqrt{z\alpha}e^{-i\pi/4}) \tag{5.3a}$$

and the formula (2.6) gives

$$G_0(0;0)=\left[\frac{\alpha}{i}\right]^{1/2} z^{-1/2}\ . \tag{5.3b}$$

We see that in this case $G_0(0;0)=\lim_{x\to 0}G_0(x;0)$ without any singularities and accordingly the regularization operator in $d=1$ is not required. This means that in one dimension one can in a safe way interchange the time integration with the limiting procedure of taking $x\to 0$. For $d>1$ this cannot be done and as a result in dimensions higher than one a nontrivial regularization operator is required in order to assure the relation (5.2). The explicit form of the regularization operator can be derived using the formula (5.1) for the modified Bessel function. The derivation is presented in Appendix A and here we just quote the final result. In odd dimensions, the regularization operator has the following form:

$$\hat{R}_{2n+1}=\gamma_{2n+1}\frac{\partial^{2n-1}}{\partial r^{2n-1}}r^{2n-1} \quad \text{for } n>1 \text{ and } d=2n+1\ , \tag{5.4}$$

where the coefficient γ_{2n+1} is given by the following formula:

$$\gamma_{2n+1}=\frac{\pi^{-1/2}\Gamma(\frac{1}{2}-n)}{\sum_{l=0}^{n-1}(-1)^{l+n}2^{-l+n}\frac{(n-1+l)!(2n-1)!}{l!(n-1-k)!(n+l)!}}\ . \tag{5.5}$$

As an example, let us write explicit expressions for three Fermi pseudopotentials (1.2). We write

$$V_F(x)=-a\delta(x) \quad \text{for } d=1\ , \tag{5.6a}$$

$$V_F(\mathbf{r})=\frac{2\pi\hbar^4}{m^2 a}\delta^{(3)}(\mathbf{r})\frac{\partial}{\partial r}r \quad \text{for } d=3\ , \tag{5.6b}$$

$$V_F(\mathbf{r})=-\frac{\hbar^8 8\pi^{5/2}}{3m^4a^3\Gamma(-\frac{3}{2})}\delta^{(5)}(\mathbf{r})\frac{\partial^3}{\partial r^3}r^3 \quad \text{for } d=5\ , \tag{5.6c}$$

with coupling constants leading to the bound state with equal energy.

VI. THE GREEN'S FUNCTION IN EVEN DIMENSIONS

So far our discussion has been limited only to odd dimensions. The expression (2.7) has been regular in odd dimensions and had only single poles in even dimensions. Such a situation is well known in quantum field theory which leads to loop-integral divergences in a form of simple poles in $d-4$.[24] With the help of dimensional regularization it is possible to extract from the loop integrals finite parts and remove the divergent parts by the renormalization counter terms.[25] We shall apply this procedure of dimensional regularization to the problem of zero-range potential in an even number of dimensions. In order to perform this procedure we note first that the Green's function $G_0(0;0)$ from Eq. (2.7) is always multiplied by the coupling constant a_d, which, according to the formula (3.4), contains a term L^d. Because of this we shall investigate a regular quantity $L^dG_0(0;0)$ for $d=2(n-\epsilon)$ and perform at the end of the limit $\epsilon\to 0$ in order to establish the regular part of the Green's function in $d=2n$. This is the essence of the dimensional regularization. Following this procedure we obtain

$$L^{2n}G_0(0;0)=L^{2n}z^{n-1}\left[\frac{\alpha}{\pi i}\right]^n\left[\frac{\pi i}{\alpha z L^2}\right]^{\epsilon}\times\Gamma(-n+1+\epsilon)\ . \tag{6.1}$$

We expand around $d=2n$ using the formula[25]

$$\Gamma(-n+1+\epsilon)=\frac{(-1)^{n-1}}{(n-1)!}\left[\frac{1}{\epsilon}+\psi(n)+O(\epsilon)\right]\ , \tag{6.2a}$$

where

$$\psi(n)=1+\frac{1}{2}+\cdots+\frac{1}{n-1}-\gamma\ ,$$

γ being the Euler constant [$\psi(1)=-\gamma=-0.5772\ldots$]. In addition to this expansion we use

$$\left[\frac{\pi i}{\alpha z L^2}\right]^{\epsilon}=\exp\left[\epsilon\ln\left[\frac{\pi}{\alpha z L^2}\right]\right]=1+\epsilon\ln\frac{\pi}{\alpha z L^2}+O(\epsilon^2)\ . \tag{6.2b}$$

Combining (6.2a) and (6.2b) in the formula (6.1) we obtain the following expression for the regular part of the Green's function in the limit of $\epsilon\to 0$:

$$[a_dG_0(0;0)]_{\text{reg}}=a_{2n}z^{n-1}\left[\frac{\alpha}{\pi i}\right]^n\frac{(-1)^{n-1}}{(n-1)!}\times\left[\psi(n)+\ln\frac{\pi i}{\alpha z L^2}\right]\ . \tag{6.3}$$

After the dimensional regularization, the regular part of $G_0(0;0)$ entering the formula (2.5) is well behaved in even dimensions. The formula (2.5) with the expression (6.3) gives an exact and an explicit solution of the zero-range potential in even numbers of dimension. We note, as in the case of odd dimensions, that the explicit form of the propagator (6.3) cannot be obtained from the formula (2.6) by a simple limit of $\mathbf{r}$ and $\mathbf{r}_0\to\mathbf{0}$. In order to reproduce the formula (6.3) a proper regularization operator $\hat{R}_d$ has to be derived. We shall discuss the form of this regularization operator and the expression for the Fermi pseudopotential in even dimensions in the following section.

VII. ZERO-RANGE POTENTIAL IN TWO DIMENSIONS

In view of possible physical applications involving a zero-range interaction in two dimensions (quantum wells, superlattices, two-dimensional Krönig-Penney models, etc.) we shall discuss in this section the properties of the zero-range interaction in two dimensions in more detail. For $d=2$ from Eq. (2.5) we obtain

$$G_0(\mathbf{r};0)=2\left[\frac{\alpha}{\pi i}\right]\frac{1}{\sqrt{r}}K_0(2re^{-i\pi/4}\sqrt{\alpha z}\,)\ , \tag{7.1}$$

and

$$[G_0(0;0)]_{reg}=\left[\frac{\alpha}{\pi i}\right]\left[\psi(1)+\ln\left[\frac{\pi i}{\alpha z L^2}\right]\right]. \tag{7.2}$$

From these relations we obtain according to the formula (2.7) the following exact expression for the outgoing solution of the Lippmann-Schwinger equation:

$$\psi_{\mathbf{k}}^{(+)}(\mathbf{r})=e^{i\mathbf{k}\cdot\mathbf{r}}+\frac{\dfrac{2\alpha a_2}{\pi\hbar\sqrt{r}}K_0(-irk)}{1-\dfrac{a_s\alpha}{\pi\hbar}\left[\psi(1)+\ln\left[\dfrac{4\pi}{k^2L}e^{i\pi}\right]\right]}\ . \tag{7.3}$$

From the asymptotic behavior of the modified Bessel function K_0 we obtain from (7.3) the following expression from the scattering in two dimensions:

$$\lim_{r\to\infty}\psi_{\mathbf{k}}^{(+)}(\mathbf{r})=e^{i\mathbf{k}\cdot\mathbf{r}}+f_{\mathbf{k}}^{(2)}\frac{e^{ikr}}{\sqrt{r}}\ , \tag{7.4}$$

where the scattering amplitude is

$$f_{\mathbf{k}}^{(2)}=\frac{\alpha e^{i\pi/4}\pi^{-1/2}k^{-1/2}2^{1/2}}{1-\dfrac{\alpha a_2}{\pi\hbar}\left[\psi(1)+\ln\dfrac{4\pi}{k^2L}+i\pi\right]}\ . \tag{7.5}$$

The pole of the scattering amplitude in the physical k half plane corresponds to the bound state. From Eq. (7.5) it follows that such a pole existence for any real value of a_2. The energy of this bound state is

$$E_0=-\frac{\hbar^2 2\pi}{mL^2}\exp\left[-\frac{2\pi\hbar^2}{ma_2}+\psi(1)\right]. \tag{7.6}$$

At this point we return to the problem of the Fermi pseudopotential (1.2) in two dimensions. According to the discussion from Sec. V, we shall introduce the regularization operator $\hat{R}_2$ which provides the following relation:

$$[G_0(0;0)]_{reg}=\lim_{r\to\infty}\hat{R}_2(r)G_0(\mathbf{r};0)\ , \tag{7.7}$$

where the left-hand side of this equation is the formula (7.2) obtained after dimensional regularization. The Green's function $G_0(\mathbf{r};0)$ is given by the expression (7.1). The explicit formula for the regularization operator can be derived using the following formula for the modified Bessel function:[19]

$$K_0(y)=-\ln\left[\frac{y}{2}\right]I_0(y)+\sum_{l=0}^{\infty}\frac{y^{2l}}{2^{2l}(l!)^2}\psi(l+1)\ . \tag{7.8}$$

The characteristic property of even dimensions is reflected by a logarithmic divergence in the formula (7.8). This logarithmic divergence has to be removed by the regularization operator in such a way that the resulting expression is finite and equal to Eq. (7.2). The derivation is presented in Appendix B and here we just quote the final result. In two dimensions, the regularization operator has the following form:

$$\hat{R}_2=-r(\ln\beta_2 r)^2\frac{\partial}{\partial r}(\ln\beta_2 r)^{-1}\ , \tag{7.9a}$$

where

$$\beta_2=\frac{1}{L}\sqrt{\pi}\exp[-\psi(1)/2]\ . \tag{7.9b}$$

As a result of this procedure we obtain the following expression for the Fermi pseudopotential in two dimensions:

$$V_F(\mathbf{r})=-a_2\delta^{(2)}(\mathbf{r})\times\left[1-\ln\left[\sqrt{\pi}\frac{r}{L}\exp\{-[\psi(1)/2]\}\right]r\frac{\partial}{\partial r}\right]. \tag{7.10}$$

The generalization of these results to arbitrary even dimensions is tedious but straightforward. We note that for even dimensions larger than two the modified Bessel function in (2.6) contains also polynomial singularities. From the properties of these functions it follows that the Green's function $G_0(\mathbf{r};0)$ on the top of the logarithmic singularity contains also polynomial singularities. In order to obtain the regular expression (6.3) the regularization operator $\hat{R}_{2n}$ must contain terms removing both the logarithmic and the polynomial divergences. We have seen that polynomial divergences can be handled with the help of the operator (5.4) while the logarithmic divergence can be removed with the help of the operator (7.9). It is clear that for $d>2$, the regularization operator $\hat{R}_{2n}$ will contain terms (5.4) and terms (7.9) properly scaled with coefficients γ and β in order to reproduce the results (6.3). As an example, in four dimensions the logarithmic part of the divergence can be removed using the operator (7.9a) but with $\beta_4=(\sqrt{\pi}/L)\exp[-\psi(1)+\frac{1}{2}\psi(2)]$. We do not reproduce here the explicit expression, for arbitrary even d, because it is too complicated and probably not very useful after the dimensional regularization of $G_0(0;0)$ given by Eq. (6.3) is performed.

VIII. CONCLUSIONS

The purpose of this paper has been a systematic discussion of the zero-range potential in any number of dimensions. We have shown that except for the one-dimensional case the zero-range potential leads to singularities which have to be tempered by a proper regularization operator. The zero-range potential with this regularization operator called the Fermi pseudopotential has been derived and discussed in any number of dimensions. We have shown that in odd dimensions the regularization can be avoided by maintaining a certain order of integration and evaluation at the origin. In even dimensions the situation is much more complicated, and only after dimensional regularization can the Fermi pseudopotential be defined. We have derived exact expressions for the Green's function, propagator, and the scattering amplitude in any number of dimensions. In view of growing applications in multiphoton detachment of atoms, the difference between the $d=1$ and the $d=3$ Fermi pseudopotential becomes important.

In even dimensions after dimensional regularization the Green's function can be obtained. We have derived the explicit expression for the Fermi pseudopotential in $d=2$. This potential could be used to study the dynamical effects in quantum wells.

The theory that we have investigated in this paper is not limited to the Schrödinger equation. The Maxwell equation in the paraxial approximation have the formal structure of the Schrödinger equation with the interaction potential replaced by the dielectric index of the medium. Zero-range interactions could be used in such a case to study propagation effects in different configurations and different geometries.

It is relatively easy to generalize our solutions if, for example, the particle interacting with zero-range potential is in addition coupled to an external classical electromagnetic field. In this case the free propagator K_0 in Eq. (2.1) has to be replaced by the propagator involving the external field. If the field has no spatial dependence, this propagator becomes the well-known Volkov propagator.[7] The zero-range interaction cannot be solved exactly in this case, and Eq. (2.1) can be reduced to a simple integral equation for one time-dependent function.[26] The dimensionality of the problem has important consequences as far as the structure of this equation is concerned.[10–12] Because of the regularization operators for $d>1$ the gauge problems and the minimal-coupling terms involved in zero-range interactions are more complicated.[11,26,27]

ACKNOWLEDGMENTS

The author would like to thank Professor M. O. Scully for his invitation and for the hospitality extended to him at the University of New Mexico, where this work has been done. I would like to thank Professor W. Becker and Professor C. Chandler for valuable discussions about the zero-range interactions. This work was partially supported by the Office of Naval Research.

APPENDIX A

Using the definition (5.1) we can write the Green's function (2.5) in the following form:

$$G_0(\mathbf{r};\mathbf{0})=\exp(-2r\sqrt{\alpha z/i}\,)\sum_{l=0}^{n-1}\frac{b_l z^{-l/2}}{r^{n+l}}, \tag{A1}$$

where the explicit form of the b_l coefficients follows from Eqs. (5.1) and (2.6), On the other hand, we know that $G_0(\mathbf{0};\mathbf{0})$ is given by Eq. (2.7). In order to take the limit of $r\to 0$ in the expression (A1), we have to remove all the polynomial singulaities in (A1). The highest singularity of this type is given by the last term in the sum (A1) and is r^{-2n+1}. Multiplying by this term and differentiating $2n-1$ times we obtain from (A1) an expression which has no polynomial singularities. Let us see how this works on the l term in the sum (A1)

$$\lim_{r\to 0}\gamma_{2n+1}\frac{\partial^{2n-1}}{\partial r^{2n-1}}[b_l r^{(n-1-l)}z^{-l/2}\exp(-2r\sqrt{\alpha z/i}\,)]$$
$$=\gamma_{2n+1}z^{n/2}b_l(-2\sqrt{\alpha/i}\,)^{n+l}\,. \tag{A2}$$

As a result of this operation we obtain

$$\lim_{r\to 0}\gamma_{2n+1}\frac{\partial^{2n-1}}{\partial r^{2n-1}}r^{2n-1}G_0(\mathbf{r};0)$$
$$=\gamma_{2n+1}\sum_{l=0}^{n-1}z^{n/2}b_l(-2\sqrt{\alpha/i}\,)^{n+1}\,. \tag{A3}$$

Comparing (A3) with the expression (2.6) evaluated for $d=2n+1$ we obtain the formula (5.5) for γ_{2n+1}. Note that this is a purely numerical factor entirely independent from the dynamical properties of the zero-range potential.

APPENDIX B

Using the definition (7.8), we can write the Green's function in the two dimensions in the following form:

$$G_0(\mathbf{r};\mathbf{0})=\frac{2\alpha}{\pi i}\left[\{-\ln r-\tfrac{1}{2}\ln[\alpha z\exp(-i\pi/2)]\}\right.$$
$$\left.\times\sum_{l=0}^{\infty}\phi_l(r)+\sum_{l=0}^{\infty}\phi_l(r)\psi(l+1)\right], \tag{B1}$$

where

$$\phi_l(r)=\frac{[r^2\alpha z\exp(-i\pi/2)]^l}{(l!)^2}\,. \tag{B2}$$

From the definitions (B1) and (B2) it is clear that the only singularity in the limit of $r\to 0$ comes from the $\ln r$ term. We introduce the regularization operator in the following form:

$$\hat{R}_d G_0(\mathbf{r};\mathbf{0})=\lim_{r\to 0}\left[G_0(\mathbf{r};0)-r\ln r\frac{dG_0(\mathbf{r};\mathbf{0})}{dr}\right]$$
$$+\frac{2\alpha}{\pi i}\ln\beta 2\,, \tag{B3}$$

where the parameter β_2 will be determined later. If we denote the limit of the first two terms in Eq. (B3) by F, using (B1) and (B2) we obtain

$$F=\frac{2\alpha}{\pi i}\lim_{r\to 0}\Bigg[-\tfrac{1}{2}\ln[\alpha z\exp(-i\pi/2)]+\psi(1)+\{r\ln^2 r+\tfrac{1}{2}\ln[\alpha z\exp(-i\pi/2)]r\ln r\}\times\sum_{l=0}^{\infty}\frac{d\phi_l(r)}{dr}+r\ln r\sum_{l=0}^{\infty}\frac{d\phi_l(r)}{dr}\psi(l+1)\Bigg]. \tag{B4}$$

From the definition of $\phi_l(r)$ [Eq. (B2)] and from the fact that $\lim_{r\to 0} r^l \ln r=0$ we obtain that the limit in (B4) is finite and is equal to

$$F=\frac{2\alpha}{\pi i}\{-\tfrac{1}{2}\ln[\alpha z\exp(-i\pi/2)]+\psi(1)\}\ . \tag{B5}$$

After regularization the expression should be equal to the formula (7.2) obtained by the dimensional regularization method.

This equality allows us to fix the parameter β_2 in (B2). In this case we derive that β_2 is given by Eq. (7.9b) and that the regularization procedure from Eq. (B2) can be written in the form presented by Eq. (7.9a).

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