Asymptotic probability distribution for a supercritical bifurcation swept periodically in time

Manuel O. Caceres,* A. Becker, and L. Kramer
Theoretische Physik, Physikalisches Institut der Universität Bayreuth, D-8580 Bayreuth, Federal Republic of Germany
(Received 18 October 1990)

By using a path-integral approach we have studied the asymptotic probability distribution of a periodically swept supercritical bifurcation. The steepest-descent approximation has been used with the corresponding time-dependent Onsager-Machlup Lagrangian of the Fokker-Planck equation. We prove by using the Lyapunov function the uniqueness of the asymptotic time-periodic probability distribution for periodically forced Markov processes; then the mixing property for these types of stochastic processes is proved. An iterative matrix procedure is introduced to calculate the long-time behavior of the probability distribution. Monte Carlo simulations were performed in order to show the agreement between the path-integral approach and the numerical solution of the corresponding periodically forced stochastic differential equation. A discussion on the problem of calculating the weak-noise Graham-Tel invariant measure is presented.

I. INTRODUCTION

A. General framework

The study of time-dependent modulation is of interest in the analysis of pattern-formation phenomena.[1,2] In fact, in electrohydrodynamic convection in liquid crystals the control parameter (voltage) is usually time periodic.[3] Moreover the influence of the stochastic forces is of great importance in the evolution of time-dependent structures.[4,5]

Recently, in connection with the analysis of the so-called stochastic resonance, the periodically forced stochastic processes have been studied from the experimental[6] and theoretical points of view.[7–10]

Different approaches have been used in order to get an approximate description for this type of nonstationary stochastic processes. But due to the nonlinearity and time dependence of the corresponding Fokker-Planck operator, the description of this kind of stochastic process presents a formidable mathematical problem.

Two types of approaches have been used in order to get the asymptotic probability distribution.

(i) Graham and Tel[11] introduced the idea of working with an equivalent two-dimensional stochastic differential equation (SDE) by introducing in addition to the stochastic variable x the phase $y=\Omega t$ as an independent variable (here Ω is the frequency of the external periodic force). Then the invariant measure (the stationary probability distribution in the variables x and y) in the limit of weak noise, can be found by a mapping to a Hamiltonian system. In principle the weak-noise invariant measure can be analyzed by studying the associated nonequilibrium potential $\Phi(x,y)$, which is defined variationally by minimization of an action functional $S([x],[y])$. Under the condition of C^2 continuity the nonequilibrium potential satisfies an equation of the Hamilton-Jacobi type at zero energy. Under this continuity assumption there is a correspondence between the weak-noise limit and an associated mechanical system. However, they found that it is not possible to get a smooth nonequilibrium potential $\Phi(x,y)$ unless the Hamiltonian is integrable at zero energy.[12] Similar results were found by Jauslin[13] who proved, using Melnikov's criterion, the nondifferentiability of the weak-noise nonequilibrium potential. In the Appendix of the present paper we show in a *simple physical way* that any small perturbation expansion (in both limits: the adiabatic $\Omega\rightarrow 0$, and the high-frequency $\Omega\rightarrow\infty$) for the Hamilton-Jacobi equation gives a non-well-defined invariant measure $\exp[-\Phi(x,y)/d]$, where d is the noise.

Using the two-dimensional Fokker-Planck-equation, Jung and Hänggi[9] showed connections between a Floquet expansion for the solution of the time-dependent one-dimensional Fokker-Planck equation and the eigenvalues of the associated two-dimensional operator. Then under the conditions of initially homogeneous phase distribution they were able to get the asymptotic probability distribution for the periodically modulated asymmetrical bistable potential. In order to get this invariant measure they used the continuous-matrix-fraction method[14] to work our numerically the corresponding eigenvalues.

(ii) The second method (to get the asymptotic probability distribution) is based on the use of a time-dependent perturbation theory[7,8] with the strength of the external periodic force as small parameter. The perturbation is carried out using a Floquet-like expansion with the set of eigenfunctions of a suitable time-independent one-dimensional Fokker-Planck operator. Unfortunately this set of eigenfunctions is unknown due to the high nonlinearity of the equivalent potential in the time-independent "unperturbed Schrödinger" operator[14] and hard numerical work is required.

For the system in which we are interested the small parameter is the noise[1,15(a)] and the bifurcation is supercritical.[15(b)] Our final aim is to study the correlation function for these types of stochastic processes as a function of the strength of the external periodic force. So we need to know the one-time probability distribution and the propagator as a function of this parameter, even when it is not small. Eventually we will consider the asymptotic behavior of the correlation function.

The normal form of the dynamical equation associated with the time-periodically driven bifurcation is

$$\dot{x}\,' = -[a + b\cos(\Omega t')]x' - g x'^3 + \sqrt{d}\,\xi(t') , \qquad (1.1)$$

with $b > 0$ and $g > 0$. The primes are used to distinguish from the scaled quantities introduced subsequently. In (1.1) d is the noise amplitude and $\xi(t')$ is a Gaussian stochastic process characterized by

$$\langle \xi(t') \rangle = 0 , \quad \langle \xi(t'_1)\xi(t'_2) \rangle = \delta(t'_1 - t'_2) . \qquad (1.2)$$

Our aim is to analyze ensemble objects characterized by the SDE (1.1) at very long time. In order to do this we need to calculate the asymptotic one-time probability distribution $P_{as}(x',t')$ and the conditional probability distribution (CPD) $P(x'_2,t'_2|x'_1,t'_1)$. We are going to do this by using the path-integral approach (with a time-dependent Lagrangian) in the steepest-descent approximation (SDA). So in the present paper we choose a substantially different approach in comparison with the works commented on before, because neither (i) nor (ii) can be implemented in a simple and successful way to the SDE (1.1). As a matter of fact, the first one (i) is only devoted to the asymptotic probability distribution (the invariant measure in the variables x,y) and the second one (ii) is a perturbation in the strength of the periodic force (which is not a small parameter for the system we wish to solve).

Our path-integral scheme gives in a straightforward way both objects: the propagator (the CPD) and, as we will prove, the asymptotic one-time probability distribution. We will show that the SDA is good enough to describe the essential physical behavior occurring in periodically modulated nonlinear instabilities. We have checked the SDA with a Monte Carlo simulation and found excellent agreement between both asymptotic distributions. The analysis of the cummulants of the SDE (1.1) is planned to be published elsewhere.[16]

B. Scaling

Owing to the structure of (1.1) we see that if $b > |a|$ the effect of the periodic modulation is to drive the system from monostable ($x = 0$) to bistable with instantaneous local (degenerate) minima of the corresponding potential at positions

$$x'_{\pm}(t') = \pm\sqrt{-[a + b\cos(\Omega t')]/g} \qquad (1.3)$$

when $\cos(\Omega t') < -a/b$.

We can scale the time in units of Ω as

$$t = \Omega t' \qquad (1.4)$$

and the "space" in units of the maximum amplitude of the attractor [i.e., $x'_+(\pi/\Omega)$] as

$$x = \sqrt{g/(b-a)}\,x' . \qquad (1.5)$$

Using (1.4) and (1.5) in Eq. (1.1) we get the dimensionless SDE:

$$\dot{x} = -[A + B\cos(t)]x - (B - A)x^3 + \sqrt{\epsilon}\,\xi(t) , \qquad (1.6)$$

where $A \equiv a/\Omega$, $B \equiv b/\Omega$, $\epsilon \equiv dg/\Omega(b-a)$. We will be interested only in the case $B > |A|$.

The outline of the paper is as follows. In Sec. II we use the Lyapunov function of the Fokker-Planck equation to prove the uniqueness and existence of the asymptotic one-time probability distribution, showing in this way the mixing property for this type of periodically modulated Markov processes. This adds to the discussion of the mixing property in recent works.[9,7] In Sec. III we introduce the path integral formulation and its SDA for time-dependent Lagrangians. In Sec. IV A we give a straightforward method to get the asymptotic one-time probability distribution $P_{as}(x,t)$ in an iterative manner. In Sec. IV B we plot $P_{as}(x,t)$ for different values of the parameters and noise and give a short discussion of our path-integral results. In Sec. IV C we introduce some basic elements for the Monte Carlo simulation and compare the histogram of the asymptotic one-time probability distribution with the path-integral approach. The Appendix is devoted to showing that, for the present model (1.1), any attempt of a perturbation expansion for the solution of the associated Hamilton-Jacobi equation leads to a non-well-defined weak-noise Graham-Tel invariant measure.

II. UNIQUENESS OF THE TIME ASYMPTOTIC DISTRIBUTION

We want to calculate the CPD, $P(x,t|x_0,t_0)$, and the asymptotic one-time probability distribution $P_{as}(x,t)$ of the stochastic variable x characterized by the SDE (1.6). The evolution of this nonstationary Markov process is described by the Fokker-Planck equation:[14]

$$\partial_t W(x,t) = [-\partial_x K(x,t) + (\epsilon/2)\partial_x^2]W(x,t) = \vartheta_{FP}(x,t,\partial_x)W(x,t) , \qquad (2.1)$$

where

$$K(x,t) = -[A + B\cos(t)]x - (B - A)x^3 . \qquad (2.2)$$

There is no ambiguity in the Fokker-Planck operator ϑ_{FP}. It is unique because the SDE (1.6) is additive.[14]

The functional $\mathcal{H}(t)$ defined as

$$\mathcal{H}(t) = \int W_1 \ln(W_1/W_2)dx \qquad (2.3)$$

is a Lyapunov function[14,17] of the Fokker-Planck equation. Here W_1 and W_2 are two possible solutions of (2.1) satisfying *natural boundary conditions* [$W_i(x,t) \to 0$; $\lim x \to \mp\infty$] and *normalization to 1* [$\int W_i(x,t)dx = 1$].

Then using that $\mathcal{H}(t) \geq 0$ and $\partial_t \mathcal{H}(t) \leq 0$, it is possible to see that for the long-time limit, the condition[14,17]

$$\lim_{t\to\infty} [W_1(x,t) - W_2(x,t)] = 0 \qquad (2.4)$$

holds.

Clearly Eq. (2.1) is invariant under the discrete time-translation transformation (in dimensional units $T = 2\pi$):

$$\mathcal{T}: t \to t+T \ . \tag{2.5}$$

Let $W_1(x,t)=P(x,t|x_1,t_1)$ be a solution of the above Fokker-Planck equation. Then the distribution $W_1(x,t+T)$ is also a solution of (2.1):

$$[\partial_t - \vartheta_{\rm FP}(x,\partial_x,t)]W_1(x,t+T)=0 \ . \tag{2.6}$$

In general $W_1(x,t)\neq W_1(x,t+T)$, but in the long-time limit both solutions approach each other, i.e., using (2.4) we get

$$\lim_{t\to\infty}[W_1(x,t)-W_1(x,t+T)]=0 \ . \tag{2.7}$$

This means that $W_1(x,t)$ becomes periodic in the long-time limit. The same result can be applied to another arbitrary solution $W_2(x,t)=P(x,t|x_2t_2)$. From (2.4) and (2.7) we conclude that in the asymptotic limit ($t\to\infty$) any solution of the Fokker-Planck equation (2.1) becomes periodic in time with period T and converges to an unique asymptotic distribution which is independent of the *initial conditions.* In what follows we are going to call this distribution the asymptotic time-periodic distribution (ATPD) $P_{\rm as}(x,t)$.

Owing to this conclusion we can assure that any such nonstationary Markov process which is invariant under the transformation $\mathcal{T}$ and satisfies *natural boundary conditions and normalization 1* is mixing.

This is so because the 2-time joint probability distribution $P(x_2,t_2;x_1,t_1)=P(x_2,t_2|x_1t_1)P(x_1,t_1)$ goes in the long-time limit (i.e., $t_2-t_1\to\infty$ and $t_1\to\infty$) to

$$P(x_2,t_2;x_1,t_1)\to P_{\rm as}(x_2,t_2)P_{\rm as}(x_1,t_1) \ . \tag{2.8}$$

Then in this limit the correlation function $\chi(t_1,t_2)$ behaves as

$$\chi(t_1,t_2)\equiv[\langle x(t_1)x(t_2)\rangle-\langle x(t_1)\rangle\langle x(t_2)\rangle]\to 0 \ , \tag{2.9}$$

which is in accordance with the conjectures of Ref. 7.

It is worth mentioning that if we transform the SDE (1.6) into a two-dimensional set of equations (i.e., using the phase transformation $y=\Omega t$) the corresponding two-dimensional functional

$$\mathcal{H}(t)=\int\int W_1\ln(W_1/W_2)dx\,dy \tag{2.10}$$

cannot be used anymore as a Lyapunov function because the diffusion matrix in the corresponding two-dimensional Fokker-Planck equation is not positive definite. Of course this is connected with the existence of a purely imaginary branch of eigenvalues in the corresponding two-dimensional Fokker-Planck operator $\vartheta_{\rm FP}(x,y,\partial_x,\partial_y)$. Only if we impose phase homogeneous initial conditions [i.e., $\int W(x,y,t=0)dx=1/2\pi$] the stationary probability distribution of the two-dimensional Fokker-Planck system will be proportional to the $P_{\rm as}(x,t)$.[9]

We conclude that the ergodic properties of the SDE (1.6) are proved by Eqs. (2.8) ad (2.9).

III. THE PATH-INTEGRAL SOLUTION

In order to obtain a path-integral solution of (2.1) we need to specify an α discretization. Then the associated Lagrangian, $L(x,\dot{x},t)$, is not unique. However, this is not a real problem because if we use the α discretization in a consistent way the path-integral propagator will be unique.[18] A different task is the approximation of the path-integral solution by its SDA. Then we need to calculate the most probable path, that is, the solution of the equation of motion:[18,19]

$$\left.\frac{\delta S[x]}{\delta x(t)}\right|_{x=x_c}=\left.\left[\frac{\partial L}{\partial x}-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right]\right|_{x=x_c}=0 \ . \tag{3.1}$$

It has been proved[20] for time-independent Markov processes that the most probable path is characterized by the Onsager-Machlup Lagrangian. The same conclusion can be achieved for nonstationary Markov processes. Therefore from now on we are going to use the time-dependent Onsager-Machlup Lagrangian,

$$L_{\rm OM}(x,\dot{x},t)=\tfrac{1}{2}[\dot{x}-K(x,t)]^2+(\epsilon/2)\partial_x K(x,t) \ , \tag{3.2}$$

which corresponds to the $\alpha=\frac{1}{2}$ discretization.

The propagator or CPD of (2.1) can be written[18,19] as

$$P(x_f,t_f|x_0,t_0)=\int_{x_0}^{x_f}\mathcal{D}[x]\exp\left[\frac{-S[x]}{\epsilon}\right] \ . \tag{3.3}$$

Here the path-integral sums the contributions of all trajectories $x(t)$ satisfying the boundary condition

$$x(t_0)=x_0 \ , \quad x(t_f)=x_f \tag{3.4}$$

and the action functional is given by

$$S[x]=\int_{t_0}^{t_f}dt\,L_{\rm OM}(x,\dot{x},t) \ . \tag{3.5}$$

Now we introduce the time-dependent potential $U(x,t)$:

$$\begin{aligned}K(x,t)&=-\partial_x U(x,t) \ ,\\ U(x,t)&=+\tfrac{1}{2}[A+B\cos(t)]x^2+\tfrac{1}{4}(B-A)x^4 \ .\end{aligned} \tag{3.6}$$

Integrating by parts the cross term $\dot{x}K(x,t)$ one gets the action functional

$$\begin{aligned}S[x]=U(x_f,t_f)-U(x_0,t_0)&\\ +\int_{t_0}^{t_f}\tfrac{1}{2}dt[\dot{x}^2+K(x,t)^2+\epsilon\partial_x K(x,t)&\\ -2\partial_t U(x,t)] \ .&\end{aligned} \tag{3.7}$$

In the limit of small ϵ noise the main contribution to the path integral, (3.3), comes from trajectories close to the most probable one, $x_c(t)$, which minimizes the action $S[x]$.

Using (3.7) we get the following equation of motion:

$$\ddot{x}_c(t)=\left.\frac{\partial}{\partial x}V(x,t)\right|_{x=x_c} \ , \tag{3.8}$$

$$V(x,t)=\tfrac{1}{2}K(x,t)^2+\tfrac{1}{2}\epsilon\partial_x K(x,t)-\partial_t U(x,t) \ , \tag{3.9}$$

where the solution must satisfy the boundary condition (3.4). Then $x_c(t)$ can be considered as the path of a particle of mass 1 in the *classical* potential $-V(x,t)$.

Making an expansion of $S[x]$ around $x_c(t)$ the linear contribution vanishes since $x_c(t)$ is the solution of (3.8); the second order in $\delta x(t)$ gives a Gaussian path integral which can be evaluated. The result for the propagator in the SDA is[18,21]

$$P(x_f,t_f|x_0,t_0)=\frac{1}{\sqrt{2\pi\epsilon B(t_0)}}\exp\left[\frac{-S[x_c]}{\epsilon}\right] . \tag{3.10}$$

Here the function $B(\tau)$ satisfies the equation

$$\ddot{B}(\tau)-f(\tau)B(\tau)=0 \tag{3.11a}$$

with boundary conditions

$$B(t_f)=0 , \quad \dot{B}(t_f)=-1 , \tag{3.11b}$$

and where

$$f(\tau)\equiv\partial_x^2 V(x,\tau)|_{x=x_c(\tau)} . \tag{3.12}$$

The *classical* potential $-V(x,t)$ is given by Eqs. (3.9), (3.6), and (2.2). Due to the nonlinearity of this potential the most probable path x_c must be found numerically. Then Eq. (3.11a) can be solved in a similar way. Alternatively we can express the quantity $B(t_0)$, needed in Eq. (3.10), in terms of the van Vleck determinant. Considering $x_c(\tau)$ as a function of $\dot{x}_c(t_f)$, for a fixed final position $x_c(t_f)$, we can write the solution of (3.11a) satisfying the boundary condition (3.11b) as

$$B(\tau)=-\frac{dx_c(\tau)}{d\dot{x}_c(t_f)} . \tag{3.13}$$

The quantity $B(t_0)$ is then

$$B(t_0)=-\frac{dx_c(t_0)}{d\dot{x}_c(t_f)}=-\left[\frac{d\dot{x}_c(t_f)}{dx_c(t_0)}\right]^{-1} . \tag{3.14}$$

Using the identities

$$\frac{\partial S}{\partial x_c(t_f)}=-\dot{x}_c(t_f)-K(x_f,t_f)$$

$$\frac{\partial S}{\partial x_c(t_0)}=\dot{x}_c(t_0)+K(x_0,t_0) ,$$

we get for $B(t_0)$ the expression

$$\begin{aligned}B(t_0)&=-\left[\frac{d\dot{x}_c(t_f)}{dx_c(t_0)}\right]^{-1}\\&=\left[-\frac{\partial^2 S([x_c])}{\partial x_c(t_0)\partial x_c(t_f)}\right]^{-1}=\frac{dx_c(t_f)}{d\dot{x}_c(t_0)} .\end{aligned} \tag{3.15}$$

Therefore $B(t_0)$ can be obtained by calculating the derivative of the final position with respect to the initial velocity of solutions of (3.8) with fixed initial position. This reduces the numerical work.

Experimentally we get information of ensemble objects like correlation functions, cummulants, etc. in the asymptotic limit. Then we are interested in the statistical properties of the nonstationary process, Eq. (1.6), in its asymptotic regime. We will show how the ATPD, $P_{as}(x,t)$, can be obtained from the SDA given in (3.10). In principle this is not a simple task because we need to calculate the most probable path x_c for different parameters and boundary conditions: x_f,t_f,x_i,t_i. Nevertheless we are going to show that using the fact that the solution (3.3) is invariant under the transformation $\mathcal{T}$ [see (2.5)], we only need to calculate the propagator from $t_0=0$ to times $\tau\le T$ (we remind that $T=2\pi$ is the scaled period of the external driving force). We compare this ATPD obtained from the path-integral formalism with a Monte Carlo simulation and we find good agreement.

IV. THE ASYMPTOTIC TIME-PERIODIC DISTRIBUTION

A. The matrix scheme

Our next step is to introduce a matrix scheme to get $P_{as}(x,t)$ from the path-integral propagator. We will need the most probable path only with boundary conditions inside the time interval $[0,T]$ and we will iterate the Chapman-Kolmogorov equation:

$$P(x_i,t_i|x_0,t_0)=\int P(x_i,t_i|x_j,t_j)P(x_j,t_j|x_0,t_0)dx_j . \tag{4.1}$$

In particular if $t_i=\tau+nT$ and $n\gg1$ Eq. (4.1) gives the ATPD $P_{as}(x,t)$. We start the iteration with the Chapman-Kolmogorov equation from time T to $2T$.

$$\begin{aligned}P(x_2,2T|0,0)&=\int P(x_2,2T|x_1,1T)P(x_1,1T|0,0)dx_1\\&=\int P(x_2,1T|x_1,0)P(x_1,1T|0,0)dx_1 .\end{aligned} \tag{4.2}$$

The last equation is true because the propagator is invariant under $\mathcal{T}$ [i.e., $P(x,t-T|x_0,t_0-T)=P(x,t|x_0,t_0)$; this can also be seen by setting the transformation $\tau=t+T$ into the path-integral solution (3.3)]. The nth iteration gives for $P(x_n,nT|0,0)$ the expression

$$P(x_n,nT|0,0)=\int\cdots\int dx_1dx_2\cdots dx_{n-1}\times\prod_{i=1}^{n}P(x_i,T|x_{i-1},0) \tag{4.3}$$

with $x_0\equiv0$.

Then we can obtain the asymptotic regime by increasing the number of iterations. The last step, in order to get the ATPD, is to use Eq. (4.1) with $t_j=nT$ and $t_i=\tau+nT$, where $0\le\tau\le T$:

$$P_{as}(x,\tau)=\lim_{n\to\infty}\int\cdots\int dx_1dx_2\cdots dx_nP(x,\tau|x_n,0)\prod_{i=1}^{n}P(x_i,T|x_{i-1},0) \tag{4.4}$$

This equation gives the ATPD as a product of n one-period propagators multiplied by the propagator from 0 to τ ($\tau\le T$) which corresponds to a simple multiplication of matrices. We have checked the convergence of the above method by using the identity

$$P_{as}(x,0)=P_{as}(x,T)=\int P(x,T|x_j,0)P_{as}(x_j,0)dx_j\ , \tag{4.5}$$

which is valid for the stochastic process (1.6). The numerical convergence shows that depending on the parameters A, B, and ϵ the number of iteration n is different. But for the set of parameters we have used, with n between 5 and 20 we reach the asymptotic regime.

B. Discussion

Pattern formation is associated here with the *breaking* of a single-peak probability distribution centered around zero. The occurrence of a *bimodality* in the probability distribution expresses the fact that the *order parameter* (i.e., the amplitude x) in a supercritical bifurcation has a value different from zero.

For a sudden jump of the control parameter from a monostable to a bistable potential, the study of the decay from the unstable state to the stable one gives quantitative information on the appearance of the pattern.[22] For particular models of smooth sweep of the control parameter the analysis of the relaxation from the unstable state can be extended and it is possible to define a generalized Suzuki's onset time.[5] In contrast, for periodically modulated stochastic processes, a relaxation picture does not work anymore because there is not a proper scaling onset time. The competition between the scales $1/A$, $1/B$, and $1/\epsilon$ plays an essential role in periodically modulated stochastic processes. To understand pattern formation in this type of processes we need to search for the breaking of the *monomodality* in the asymptotic probability distribution, i.e., the occurrence of additional saddle points and maxima in the x-t plane. This will represent the appearance of macroscopic structure, which will be periodically repeated. The interesting problem of the coherence of the patterns at subsequent periods are planned to be analyzed elsewhere.[16] This question was studied experimentally by Meyer, Ahlers, Cannell,[1(b)] and Swift and Hohenberg[23] studied it by numerical simulation.

Let us remind the reader that if $B>|A|$ the potential $U(x,t)$ will be bistable during the interval of time satisfying $\cos(t)<-A/B$. Depending on the parameters A, B, and ϵ three qualitatively different behaviors can be inferred from the analysis of our results. The first one is when $P_{as}(x,t)$ has a *monomodality* which is repeated periodically in time. The second one is when $P_{as}(x,t)$ always shows a *bimodality* during the whole period of time. The third one is when $P_{as}(x,t)$ changes periodically from a monomodality to a bimodality distribution, i.e., a periodically pulsating pattern formation.

In Figs. 1–3 we show the altitude charts of the ATPD for different values of A, B, and ϵ noise by using the SDA in the path-integral approach. The time is scaled in units of Ω and the "space" x in such a way that the position of the attractor at time $t=\pi$ is located at ±1 [see Eq. (1.5)]. We have plotted $\ln[P_{as}(x,t)+1]$ during one period of time for values of x inside the nontrivial domain. Due to the scaling of x we expect that the maxima of the distribution, if the ATPD has a bimodality, will be around the values ∓1.

In Fig. 1 we have fixed the parameter A ($=1$) and noise ϵ ($=0.1$). Then if we increase the parameter B ($>A$) the monomodality is destroyed. For $B=1.1$ and 1.5 [Figs. 1(a) and 1(b)], respectively) the topological structure of $P_{as}(x,t)$ is similar, the ATPD preserves its maximum around zero and the distribution is widened around the time π. From Fig. 1(c) ($B=2$) we see that for times around $3\pi/2$ a periodic pattern starts to emerge, i.e., there are two saddle points at times near π. This is the result of a cooperative effect between the nonlinear and the external periodic forces. The pattern will appear after the maximum deformation of the time-dependent potential $U(x,t=\pi)$. The ATPD shows two maxima, in x, at amplitude smaller than 1. If $B=4$ a different topological structure appears (there are four saddle points during one period of time). Figure 1(d) shows a typical periodicity pulsating pattern-formation process, the maxima ($x\cong\mp1$) are after π because the particles cannot follow instantaneously the time-dependent potential $U(x,t)$ (see the comment on the adiabatic approximation given in the Appendix) and nearly disappear after $t=3\pi/2$.

In Fig. 2 we show the same graphs for $A=-1$. The noise ϵ and the set of values of B ($=1.1,1.5,2,4$) are the same as in Fig. 1. The topology of $P_{as}(x,t)$ is different from that of Fig. 1. For times around 2π (or 0) the ATPD shows a bimodality if $B\cong|A|$. During the time interval $[\pi/2,3\pi/2]$ the probability to be in the origin is small for all the values of B. If $B\gg|A|$ [Fig. 2(d)] the periodic external force is dominant and the behavior of $P_{as}(x,t)$ is a typical periodically pulsating pattern process, similar to the case shown in Fig. 1(d).

At $A=0$ there is a crossover in the behavior of $P_{as}(x,t)$ because the integral over one period of the linear coefficient $A(t)=-[A+B\cos(t)]$ of the SDE (1.6) changes sign. If $A<0$ the time-dependent potential $U(x,t)$ will be more bistable than monostable. Then for sufficiently small ϵ noise $P_{as}(x,t)$ exhibits a time-dependent bimodality during the whole period. If $A>0$ the conclusion is similar, but in the reverse sense, i.e., $P_{as}(x,t)$ shows a time-dependent monomodality for suffiently small ϵ noise. For example, if we put $\epsilon\cong0.001$ and use the same parameters A and B as in Fig. 1(c) [or 2(c)] we will get a monomodality (or bimodality) for the ATPD during the full period. In Fig. 3 we show this destruction of the periodically pulsating pattern formation due to the reduction of the ϵ-noise amplitude.

It is interesting that due to the cooperative effect between the nonlinear force and the exterial time-periodic modulation, there exists a region in parameter space where $P_{as}(x,t)$ shows a bimodality even at times when the time-dependent potential $U(x,t)$ is monostable [see Figs. 1(c) and 1(d)]. Moreover for $A<0$ and $B\cong|A|$ one has situations where the bimodality remains for all times. Physically this reflect the fact that the *particles* need time to relax.

If the frequency is small enough ($\Omega\rightarrow 0$) we expect a certain adiabatic behavior. We show in the Appendix that the first contribution to the nonequilibrium potential $\Phi(x,y)$ at order $(\Omega/a)^0$ gives just the adiabatic distribution (A10). On the other hand, in the high-frequency limit ($\Omega\rightarrow\infty$), a time-independent asymptotic distribution should appear. In the Appendix we have also shown that the first contribution to $\Phi(x,t)$, now in order (a/Ω),[1] gives a time-independent distribution (A15). Nevertheless, a perturbative expansion solution, of the weak-noise Graham-Tel invariant measure, in the small parameter Ω/a (or a/Ω) cannot be obtained. Therefore, by using the weak-noise Hamilton-Jacobi approach, the asymptotic probability distribution cannot be improved beyond its trivial approximation.

On the other hand, the SDA for the path-integral solution of the time-periodic Fokker-Planck operator, gives very good agreement with the Monte Carlo simulation

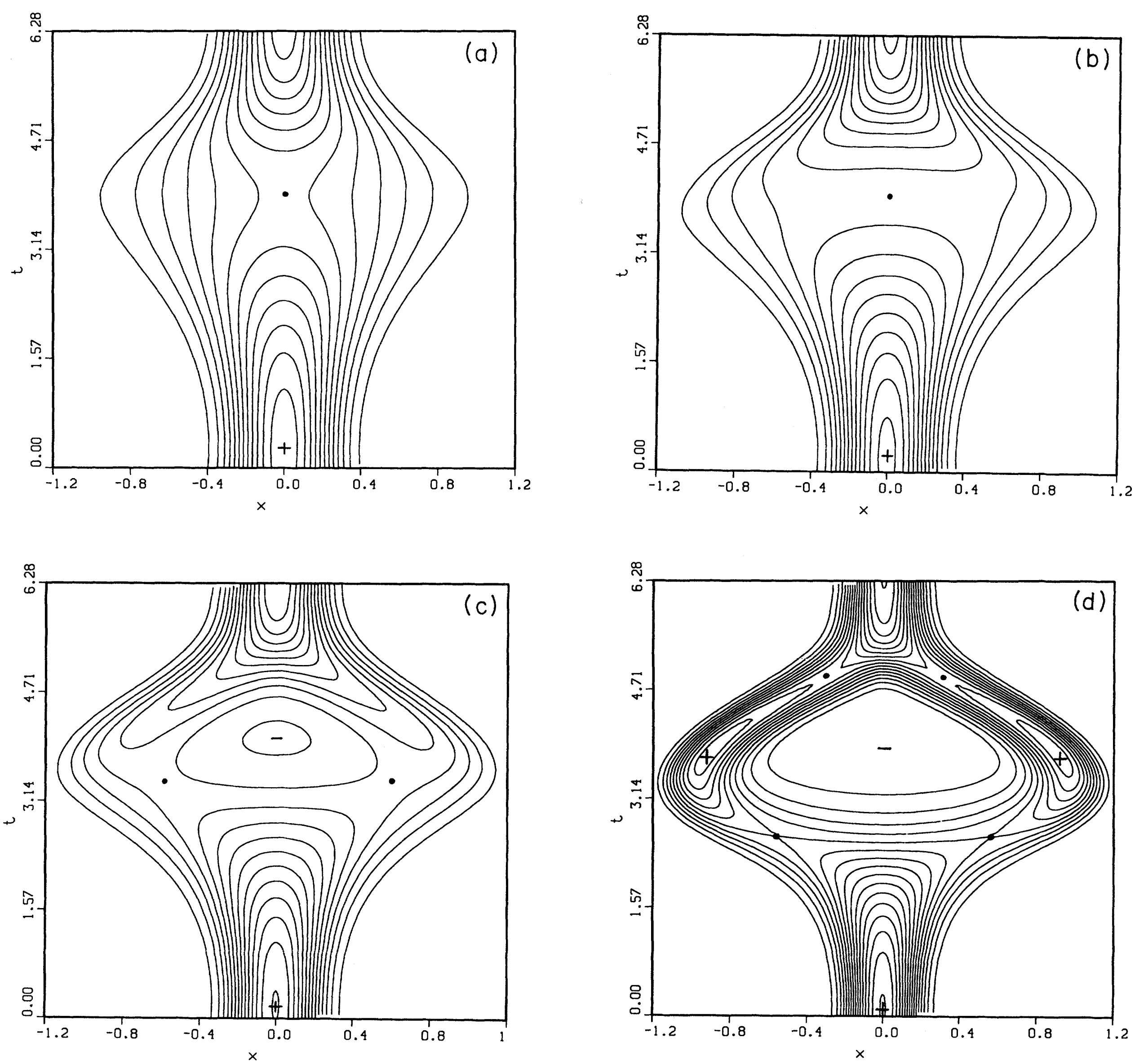

FIG. 1. Altitude charts of $\ln[P_{as}(x,t)+1]$ for noise intensity $\epsilon=0.1$ and $A=1$. Different values of B $(=1.1,1.5,2,4)$ are plotted in (a), (b), (c), and (d), respectively. The solid dots indicate saddle points and the + (−) signs denote the regions of high (low) probability. The contour lines are equidistant with $\Delta=0.1$.

for both limits (low and high frequency). This fact lends additional support to the use of the path-integral formalism in the one-dimensional configuration phase space.

C. Monte Carlo simulation

In order to check the accuracy of our SDA (3.10), for the path-integral propagator we have performed a Monte Carlo simulation of the SDE, Eq. (1.6), to get a histogram of $P_{as}(x,t)$.

For the simulation of our nonlinear SDE we used a Heun algorithm.[24] This algorithm discretizes a SDE:

$$\dot{x}=f(x,t)+\sqrt{\epsilon}\,\xi(t) \tag{4.6}$$

(note that in our case, Eq. (1.6), $f(x,t) = -[A+B\cos(t)]x-(B-A)x^3$) in the form

$$x(t_{i+1})=x(t_i)+\tfrac{1}{2}[f(x(t_i),t_i)+f(\hat{x}(t_{i+1}),t_{i+1})]h+\sqrt{\epsilon h}\,w_i \tag{4.7}$$

with the predictor step

$$\hat{x}(t_{i+1})=x(t_i)+f(x(t_i),t_i)h+\sqrt{\epsilon h}\,w_i\ . \tag{4.8}$$

Here h is the time step $t_{i+1}-t_i$ and the w_i's are independent Gaussian distributed random variables with zero

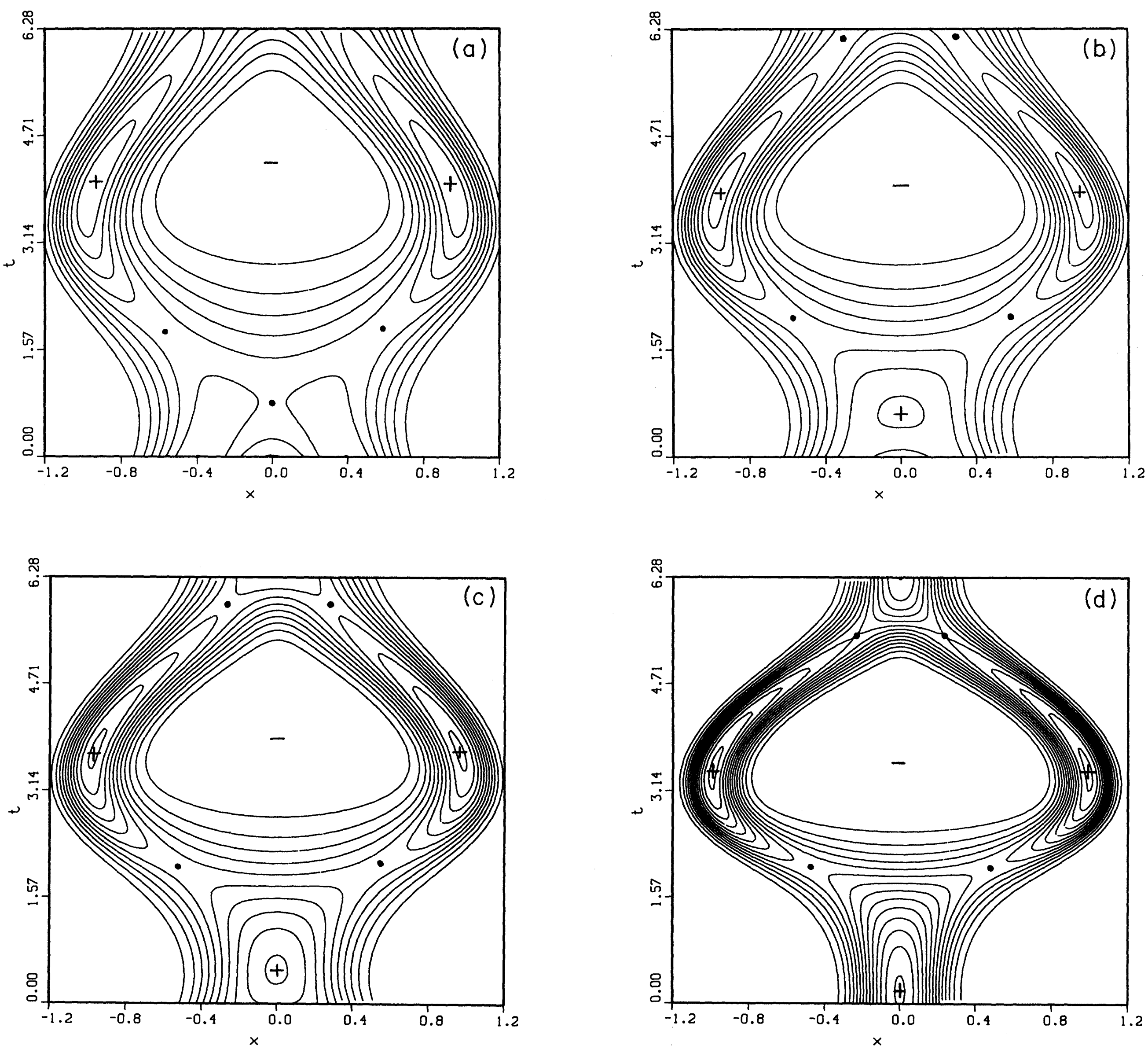

FIG. 2. Altitude charts of $\ln[P_{as}(x,t)+1]$ for noise intensity $\epsilon=0.1$ and $A=-1$. Different values of B $(=1.1,1.5,2,4)$ are plotted in (a), (b), (c), and (d), respectively. The solid dots indicate saddle points and the + (−) signs denote the regions of high (low) probability. The contour lines are equidistant with $\Delta=0.1$.

mean value and variance 1. Such random numbers are easily generated with the Box-Mueller formula

$$w_i = (-2\ln\lambda_{i1})^{1/2}\cos(2\pi\lambda_{i2}) , \tag{4.9}$$

where λ_{i1} and λ_{i2} are independent random numbers which are uniformly distributed between 0 and 1.

In order to get a histogram of the ATPD, $P_{as}(x,t)$, we have calculated one long trajectory $x(\tau)$ $(0 \le \tau \le NT)$, and used the periodicity and mixing property of the process (1.6). For a fixed time t between 0 and T one gets a good simulation of $P_{as}(x,t)$ by making a histogram of the position of this trajectory at the times $t+nT$ $(0 \le n \le N)$, if N is big enough. To obtain the results shown in Fig. 4 we have chosen the time step $h = T/1000$ and $N = 500\,000$. We conclude that for the noise intensity $\epsilon = 0.1$, as in Figs. 1 and 2, the agreement between the SDA (the path-integral approach) and the Monte Carlo simulation is very good. If we use smaller ϵ-noise intensity the agreement is also very good for a wide region in the space of parameters A and B.

We have checked the histogram of $P_{as}(x,t)$ for different starting points, $x(0)$, in the Monte Carlo simula-

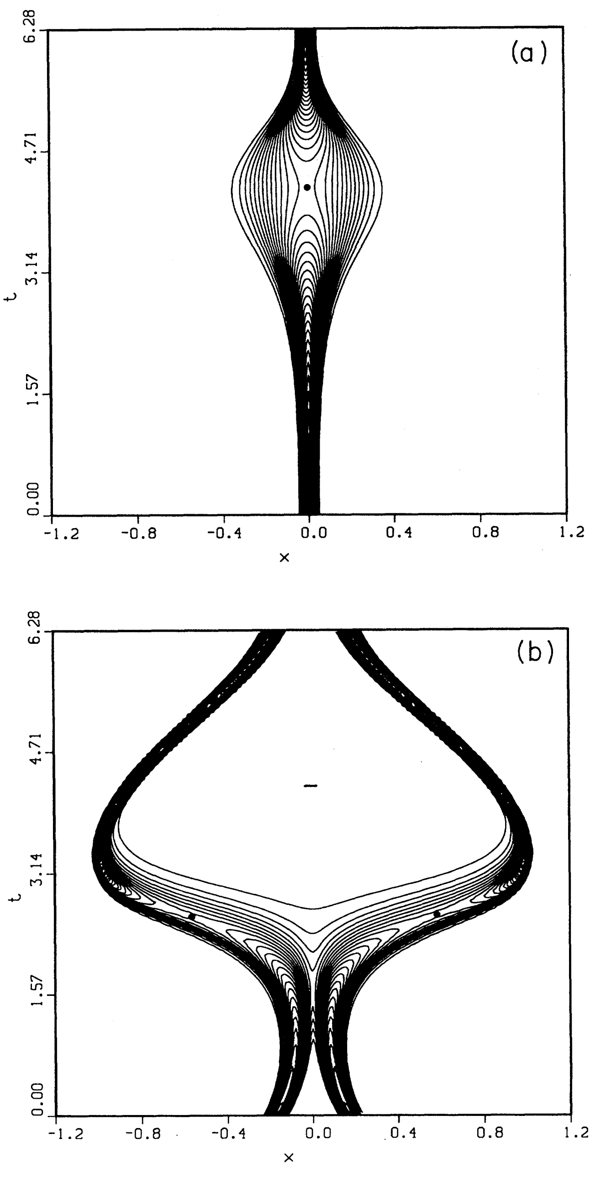

FIG. 3. Altitude charts of $\ln[P_{as}(x,t)+1]$ for noise intensity $\epsilon = 0.001$. The contour lines are equidistant with $\Delta = 0.1$. In Figs. 3(a) and 3(b) the parameters A and B are the same as in Figs. 1(c) and 2(c), respectively. The corresponding destruction of the periodically pulsating pattern formation is obtained by reduction of the noise intensity ϵ. (a) shows a time-dependent monomodality and (b) shows a time-dependent bimodality.

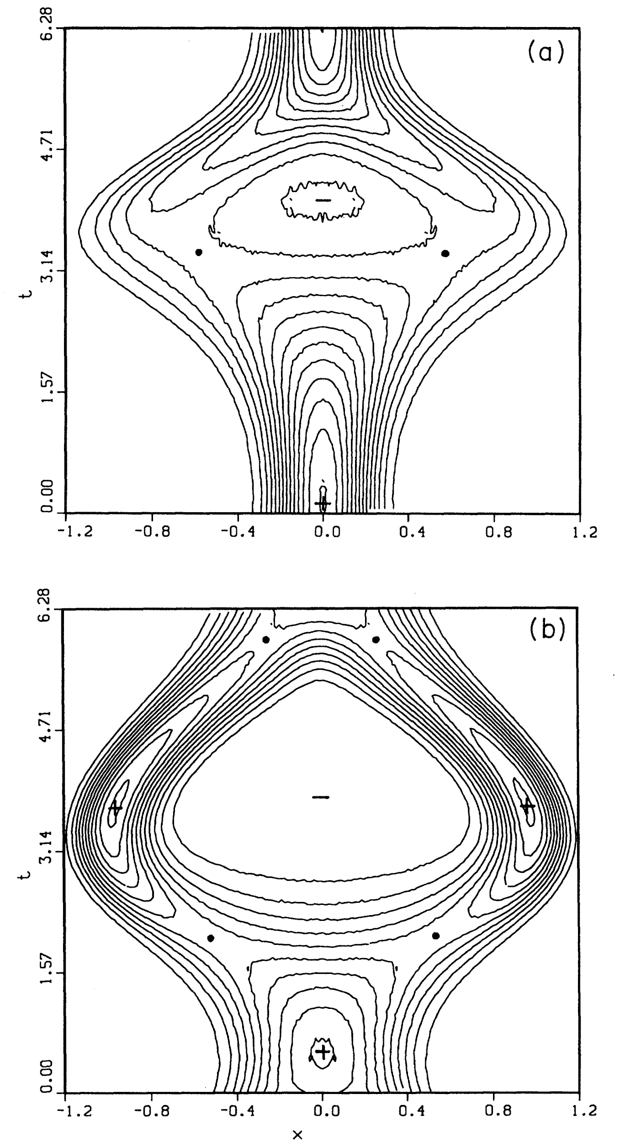

FIG. 4. Altitude charts for the Monte Carlo simulation of $\ln[P_{as}(x,t)+1]$ for noise intensity $\epsilon = 0.1$. The contour lines are equidistant with $\Delta = 0.1$. In (a) and (b) the parameters A and B are the same as in Figs. 1(c) and 2(c), respectively.

tion. The ATPD is always the same independently of $x(0)$.

ACKNOWLEDGMENTS

One of us (L.K.) wishes to thank Professor P. C. Hohenberg and Professor J. Swift for discussions on the subject of this paper during a stay at the Aspen Center for Physics. We have benefited from the help of the Deutsche Forschungsgemeinschaft (especially Grant No. SFB 213). M.O.C. acknowledges partial support from Consejo Nacional de Investigaciones Cientificas y Tecnicas, Argentina (Grant No. 1857/88).

APPENDIX

We here consider the weak-noise limit of the stationary two-dimensional Fokker-Plank equation associated with the SDE (1.1). We are going to study a perturbative expansion of the weak-noise Hamilton-Jacobi equation in the low- and high-frequency limits.

1. Adiabatic limit ($\Omega \to 0$)

If we use the time scaling $t=at'$ and the "space" scaling as in Sec. I B [i.e., $x=\sqrt{g/(b-a)}x'$] in (1.1) we can write the following two-dimensional SDE:

$$\begin{aligned} \dot{x} &= -(1+\beta\cos y)x-(\beta-1)x^3+\sqrt{\Theta_1}\,\xi\ , \\ \dot{y} &= \eta_1\ , \end{aligned} \tag{A1}$$

where $\beta \equiv b/a$, $\eta_1 \equiv \Omega/a$, and the noise intensity is $\Theta_1 \equiv dg/a(b-a)$. From (A1) we can immediately write the associated two-dimensional Fokker-Planck operator

$$\vartheta_{\mathrm{FP}}(x,y,\partial_x,\partial_y) = -\sum_{\nu=x,y}\partial_\nu K^\nu(x,y)+\Theta_{1/2}\partial_x^2\ , \tag{A2}$$

where

$$\begin{aligned} K^x(x,y) &= -(1+\beta\cos y)x-(\beta-1)x^3 \\ &\equiv -[A(y)+\gamma x^2]x\ , \\ K^y(x,y) &= \eta_1 = \mathrm{const}\ . \end{aligned} \tag{A3}$$

Here $A(y) \equiv (1+\beta\cos y)$ and $\gamma \equiv (\beta-1)$. Note that this Fokker-Planck operator does not satisfy *detailed balance*[14] unless $\eta_1=0$.

If we insert the ansatz $P_{\mathrm{st}}(x,y) \cong \exp[-\Phi(x,y)/\Theta_1]$ into the stationary equation

$$\vartheta_{\mathrm{FP}}(x,y,\partial_x,\partial_y)P_{\mathrm{st}}(x,y)=0\ ,$$

we will get, in the zeroth-power order noise 0, a Hamilton-Jacobi equation for the nonequilibrium potential $\Phi(x,y)$:

$$\tfrac{1}{2}(\partial_x\Phi)^2+K^x\partial_x\Phi+K^y\partial_y\Phi=0\ . \tag{A4}$$

From (A4) an associated *Hamiltonian system* can be written.[11] This Hamiltonian can also be inferred from a Lagrangian version in the limit of zero noise [see, for example, Eq. (3.2) for the one-dimensional Fokker-Planck Lagrangian].

In the limit of small frequency, i.e., $\eta_1 \equiv \Omega/a \ll 1$, the solution of Eq. (A4) can be written as an expansion in the small parameter η_1:

$$\Phi(x,y)=\sum_{m=0}^{\infty}\eta_1^m\Phi_m(x,y)\ . \tag{A5}$$

Here $\Phi_0(x,y)$ is the *adiabatic* approximation. So the invariant measure $P_{\mathrm{st}}(x,y)$ can in principle be studied using a perturbation method. But this perturbation scheme breaks down because $\Phi_1(x,y)$ is a nondifferentiable function. On the other hand, it is possible to define a *regular* and an *irregular* contribution to the nonequilibrium potential $\Phi(x,y)$, but this split cannot be seen as an improvement because the regular contribution leads to a non-normalized distribution. So, as we will show below, even in the small-frequency limit a simple perturbative expansion of the Hamilton-Jacobi equation (A4) is not well defined.

Due to the fact that the invariant measure $P_{\mathrm{st}}(x,y)$ can be connected with the ATPD , $P_{\mathrm{as}}(x,t)$, by the formula

$$\begin{aligned} P_{\mathrm{as}}(x,t) &= P_{\mathrm{st}}(x,y\,|\,y=\eta_1 t) \\ &= P_{\mathrm{st}}(x,y)\Big/\int P_{\mathrm{st}}(x,y)dx\,\Big|_{y=\eta_1 t}\ , \end{aligned} \tag{A6}$$

all the constants and possible y functions in the Φ expansion cancel out in (A6). We are only interested in the expansion of $\Phi(x,y=\eta_1 t)$ satisfying the symmetry conditions:

$$\partial_x\Phi(x,y=\eta_1 t)|_{x=0}=0\ , \tag{A7}$$

i.e., the ATPD $P_{\mathrm{as}}(x,t)$ must have the same symmetry as the time-dependent potential $U(x,t)$ [see Eq. (3.6)], and

$$\Phi(x,y=\eta_1 t)=\Phi(x,y=\eta_1 t+2\pi)\ , \tag{A8}$$

i.e., the ATPD must be time periodic.

Inserting (A5) into (A4) we get a hierarchy in powers of η_1, which can be written in the form

$$\begin{aligned} \partial_x\Phi_0 &= -2K^x\ , \\ \partial_x\Phi_1 &= (1/K^x)(\partial_y\Phi_0)\ , \\ \partial_x\Phi_2 &= (1/K^x)[\tfrac{1}{2}(\partial_x\Phi_1)^2+\partial_y\Phi_1]\ . \end{aligned} \tag{A9}$$

Due to the structure of $K^x(x,y)$, see (A3), the perturbation expansion (A9) is singular and the $\partial_x\Phi_m(x,y)$ with $m \geq 1$ have a simple pole at the instantaneous position of the attractor: $x_\pm(y)^2=A(y)/\gamma$. The contribution $\Phi_0(x,y)$ gives the *adiabatic* approximation

$$\Phi_0(x,y)=A(y)x^2+(\gamma/2)x^4+\mathrm{const}\ . \tag{A10}$$

The correction $\Phi_1(x,y)$ is non-differentiable inside the time interval where the potential $U(x,t)$ is bistable (i.e., $y\in[y_1,y_2]$ where $\cos(y_1)=-1/\beta$, $y_2=2\pi-y_1$). Up to this order the ATPD is

$$P(x,t=y/\eta_1) \cong e^{-\Phi_0(x,y)/\Theta_1}|A(y)/\gamma+x^2|^{-\beta\eta_1\sin(y)/2\gamma\Theta_1}\ . \tag{A11}$$

This expression can be physically interpreted as follows. The first factor is the adiabatic approximation and the correction from $\Phi_1(x,y=\eta_1 t)$ narrows the adiabatic distribution (in the "space" variable x) during the time interval $t\in[0,t_1]$ because the partial cannot follow the instantaneous change of the time-dependent potential $U(x,t)$. The same conclusion holds for the time interval $t\in[t_2,2\pi\eta_1]$, but now the correction is obviously in the reverse and the adiabatic approximation is widened (in "space" x). Inside the time interval $[t_1,t_2]$ it is not possible to give a physical interpretation to $\Phi_1(x,y)$ because the perturbation is singular there. So, owing to the structure of $\Phi_1(x,y)$, we see from (A9) that the next correction has *regular* and *irregular* parts. For example, for $\Phi_2(x,t)$ we can choose as the irregular part the contribution which comes from $\partial_y\Phi_1$ [note that Φ_1 is a nondifferentiable function if $\cos(y)\le -1/\beta$]. The regular part of $\Phi_n(x,y)$ can, in general, be written in the form

$$\Phi_m^{\rm reg}(x,y)=(-1)^{2-m}\frac{\beta^m \sin^m(y)}{2\gamma^{2m-1}}\times\frac{C_m}{2(2m-2)[x^2+A(y)/\gamma]^{2m-1}}\ , \qquad (A12)$$

where the C_m ($m\ge 2$) are constant which can be obtained diagrammatically (for example, $C_2=C_3=1$, $C_4=1+\frac{1}{2}^2$, $C_5=1+\frac{1}{2}^2+\frac{1}{2}$, etc.). From the contributions for odd m we see that it is not possible to get a normalized distribution. Therefore, even keeping only a suitable regular part, a perturbative expansion solution of (A4), in the small parameter η_1, is not possible.

2. High-frequency limit ($\omega\to\infty$)

Using the same techniques, as in the adiabatic case, we can study the high-frequency limit. To do this we scale the time as $t=\Omega t'$ and the *space* as before. Then we get a two-dimensional SDE similar to (A1), but now the functions $K^\nu(x,t)$ of the associated two-dimensional Fokker-Planck equation are defined by

$$K^x(x,y)=-\eta_2[A(y)+\gamma x^2]x\ ,\quad K^y(x,y)=1\ , \qquad (A13)$$

where $A(y)\equiv(1+\beta\cos y)$ and $\gamma\equiv(\beta-1)$ as before, but now the noise amplitude is $\Theta_2\equiv dg/\Omega(b-a)$ and $\eta_2\equiv a/\Omega$. So, in principle, we can perform a perturbation expansion in the small parameter η_2 (the limit of high frequency) to get the solution of the corresponding Hamilton-Jacobi equation. Setting

$$\Phi(x,y)=\sum_{m=0}^{\infty}\eta_2^m\Phi_m(x,y) \qquad (A14)$$

in the Hamilton-Jacobi equation (A4) [now $K^\nu(x,y)$ are given by (A13)] and looking, as before, for the solutions $\Phi(x,y)$ which satisfy the symmetry conditions (A7) and (A8), we get, up to order η_2^3, after integration

$$\begin{aligned}
&\Phi_0(x,y)=\text{const}\ ,\\
&\Phi_1(x,y)=(x^2+\gamma/2x^4)\ ,\\
&\Phi_2(x,y)=2\beta\sin(y)(x^2+\gamma x^4)\ ,\\
&\Phi_3(x,y)=-\beta^2(x^2+2\gamma x^4)\cos(2y)\\
&\qquad +4\beta(x+2\gamma x^3)(x+\gamma x^3)\cos(y)+G_2\ .
\end{aligned} \qquad (A15)$$

Here $G_2(x)=2\beta^2\gamma x^4$.

We see that the first correction $\Phi_1(x,y)$ is an average potential. This means that at very high frequency the particles cannot "see" anymore the time-dependent behavior of the potential $U(x,t)$. This contribution will be a monostable or bistable "potential" depending on the sign of the area:

$$I=-\int_0^{2\pi}[a+b\cos(y)]dy\ . \qquad (A16)$$

From (A15) we see that all the corrections $\Phi_m(x,y)$ ($m\ge 2$) lead to a non-normalized distribution. So in the high-frequency limit the weak-noise Hamilton-Jacobi equation cannot be solved by a perturbation expansion in the small parameter η_2.

---

*Permanent address: Centro Atomico Bariloche, 8400 San Carlos de Bariloche, Argentina.

[1](a) G. Ahlers, M. C. Cross, P. C. Hohenberg, and S. Safran, J. Fluid Mech. **110**, 297 (1981); (b) C. W. Meyer, G. Ahlers, and D. S. Cannell, Phys. Rev. Lett. **59**, 1577 (1987). (c) J. J. Niemela and R. J. Donnelly, *ibid.* **59**, 2431 (1987).

[2]H. Riecke, J. D. Crawford, and E. Knobloch, Phys. Rev. Lett. **61**, 1942 (1988); D. Walgraef, Europhys. Lett. **7**, 485 (1988).

[3]E. Bodenschatz, W. Zimmermann, and L. Kramer, J. Phys. (Paris) **49**, 1879 (1988).

[4]F. Moss and P. V. E. McClintock, in *Noise in Nonlinear Dynamical Systems,* edited by F. Moss and P. V. McClintock (Cambridge University Press, Cambridge, 1989); M. San Miguel, E. Hernandez-Garcia, P. Colet, M. O. Caceres, and F. de Pasquale, in *Instabilities and Nonequilibrium Structures III,* edited by E. Tirapegui and W. Zeller (Kluwer, Dordrecht, 1990).

[5]M. O. Caceres and A. Becker, Phys. Rev. A **42**, 696 (1990).

[6]B. McNamara, K. Wiesenfeld, and R. Roy, Phys. Rev. Lett. **60**, 2626 (1990).

[7]Hu Gang, G. Nicolis, and C. Nicolis, Phys. Rev. A **42**, 2030 (1990).

[8]R. F. Fox, Phys. Rev. A **39**, 4148 (1989).

[9]P. Jung and P. Hänggi, Europhys. Lett. **8**, 505 (1989); P. Jung; Z. Phys. B **76**, 521 (1989).

[10]C. Presilla, F. Marchesoni, and L. Gammaitoni, Phys. Rev. A **40**, 2105 (1989).

[11]R. Graham and T. Tel, J. Stat. Phys. **35**, 729 (1984); Phys. Rev. Lett. **52**, 4 (1984); Phys. Rev. A **31**, 1109 (1985); **33**, 1322 (1985); **35**, 1328 (1987).

[12]R. Graham, in *Noise in Nonlinear Dynamical Systems* (Ref. 4), p. 225.

[13]H. R. Jauslin, J. Stat. Phys. **42**, 573 (1986).
[14]H. Risken, in *The Fokker-Planck Equation,* Springer Series in Synergetics Vol. 18 (Springer-Verlag, Berlin, 1984).
[15](a) M. de la Torre Juarez, Ph.D. thesis, Univesidad Nacional de Educación a Distancia, Madrid, 1990; (b) M. de la Torre Juarez and I. Rehberg; Phys. Rev. A **42**, 2096 (1990).
[16]A. Becker, M. O. Caceres, and L. Kramer (unpublished).
[17]J. L. Lebowitz, and P. G. Bergman, Ann. Phys. (Leipzig) **1**, 361 (1957).
[18]F. Langouche, D. Roekaerts, and E. Tirapegui, in *Functional Integration and Semiclassical Expansions* (Reidel, Dordrecht, 1982).
[19]R. L. Stratonovich, in *Noise in Nonlinear Dynamical Systems* (Ref. 4), p. 16.
[20]F. Langouche, D. Roekaerts, and E. Tirapegui, J. Phys. A **11**, L263 (1978).
[21]F. Langouche, D. Roekaerts, and E. Tirapegui, Physica A **97**, 195 (1979); B. Caroli, C. Caroli, and R. Roulet, J. Stat. Phys. **26**, 83 (1981).
[22]M. Suzuki, Adv. Chem. Phys. **46**, 195 (1980); Prog. Theor. Phys. **56**, 77 (1976).
[23]J. B. Swift and P. C. Hohenberg, Phys. Rev. Lett. **60**, 75 (1988).
[24]J. Honerkamp, *Stochastische Dynamische Systeme* (VCH Verlagsgesellschaft, Weinhein, 1990), Chap. 8.