Stochastic Landau equation with time-dependent drift

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The stochastic differential equation $\tau_0\partial_t A=\epsilon(t)A-g_3A^3+\bar{f}(t)$, where $\bar{f}(t)$ is Gaussian white noise, is studied for arbitrary time dependence of $\epsilon(t)$. In particular, cases are considered where $\epsilon(t)$ goes through the bifurcation of the deterministic system, which occurs at $\epsilon=0$. In the limit of weak noise an approximate analytic expression generalizing earlier work of Suzuki [Phys. Lett. A **67**, 339 (1978); Prog. Theor. Phys. (Kyoto) Suppl. **64**, 402 (1978)] is obtained for the time-dependent distribution function $P(A,t)$. The results compare favorably with a numerical simulation of the stochastic equation for the case of a linear ramp (both increasing and decreasing) and for a periodic time dependence of $\epsilon(t)$. The procedure can be generalized to an arbitrary deterministic part $\partial_t A=D(A,t)+\bar{f}(t)$, but the deterministic equation may then have to be solved numerically.

I. INTRODUCTION

Many physical systems are subjected to a combination of deterministic as well as stochastic forces, the latter representing the effects of large numbers of degrees of freedom whose action on the system is random and only described in statistical terms.[1] An important example is the influence of molecular degrees of freedom (thermal noise) acting on a macroscopic variable in a system at or near equilibrium. Such a situation is typically modeled by a stochastic partial differential equation. An even simpler example is the stochastic *ordinary* differential equation

$$\tau_0\partial_t A(t)=\epsilon A-g_3A^3+\bar{f}(t)\ , \tag{1.1}$$

where the deterministic evolution is represented by the first two terms on the right-hand side (the "drift"), and the stochastic force is $\bar{f}(t)$. This force is a random variable defined by its probability distribution, rather than its detailed time dependence. One often chooses for $\bar{f}(t)$ a Gaussian variable with zero average

$$\langle\bar{f}(t)\rangle=0\ , \tag{1.2}$$

and a pair-correlation function

$$\langle\bar{f}(t)\bar{f}(t')\rangle=2\bar{F}\tau_0\delta(t-t') \tag{1.3}$$

representing white noise. The higher correlation functions are, of course, simply calculable in this case. The evolution of $A(t)$ is then only known statistically, in terms of its probability distribution $P(A,t)$, which in general depends on time, as well as on the initial distribution $P(A,t_0)$ at time $t=t_0$. Even when the drift is independent of time the function P is difficult to calculate, but the long-time behavior can be obtained exactly. When the linear drift is an arbitrary given function of time $\epsilon(t)$, the probability distribution cannot be calculated analytically, and a number of approximate methods have been developed for its evaluation. A particularly interesting case is the evolution of P when the drift changes from $\epsilon<0$ to $\epsilon>0$, i.e., when it goes through the bifurcation of the deterministic equation.

In the limit of weak noise ($\bar{F}\ll1$), Suzuki[2] has developed a well-known approximation for the case of a jump in ϵ,

$$\epsilon(t)=\begin{cases}-\epsilon_0<0, & t<t_0 \qquad (1.4a)\\ \epsilon_1>0, & t>t_0\ . \qquad (1.4b)\end{cases}$$

This approximation relies on the fact that for $t<t_0$ and for $t>t_0$ but $t-t_0\ll\tau_0/\epsilon_1$, the quantity $A^2(t)$ remains small and the stochastic equation may be linearized. For long times ($t-t_0\gg\tau_0/\epsilon_1$), on the other hand, the stochastic force is a small perturbation, and the deterministic equation ($\bar{f}=0$) is a good approximation. The effect of the stochastic force is merely to provide an ensemble of initial conditions for the deterministic evolution. A matching procedure interpolates at intermediate times ($t\sim\tau_0/\epsilon_1$), thus providing a complete approximate representation for $P(A,t)$.

A generalization of Suzuki's approximation to an arbitrary time dependence of $\epsilon(t)$ was presented by Ahlers, Cross, Hohenberg, and Safran[3] (ACHS), and successfully tested against a numerical solution of the model for the case of a ramp,

$$\epsilon(t)=\begin{cases}-\epsilon_0, & t<-\epsilon_0/\beta \qquad (1.5a)\\ \beta t, & t>-\epsilon_0/\beta\ . \qquad (1.5b)\end{cases}$$

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It turns out, however, that the approximation of ACHS fails for $\epsilon<0$, and the success of the above-mentioned test depended on having applied it primarily for $\epsilon>0$. For later work see the authors cited in Ref. 4.

The present paper seeks to correct this failing of the ACHS approximation, by retaining it only for $\epsilon>0$ and essentially replacing it with the simpler *linear* approximation ($g_3=0$) for $\epsilon<0$. One is then left with a *matching problem* each time ϵ crosses zero, and a number of procedures for carrying out this matching are proposed here and tested in concrete examples. Specifically, we study ascending and descending ramps ($\epsilon=\beta t$, with β both positive and negative), as well as a sinusoidal variation

$$\epsilon(t)=\epsilon_0+\delta\sin\omega t \ . \tag{1.6}$$

This latter case is a particularly stringent test since the system crosses the deterministic threshold repeatedly, and small secular errors must not be allowed to build up. We find that over a considerable range of variation of the parameters β, ϵ_0, δ, and ω our approximation yields results in good (though not perfect) agreement with a numerical evaluation of the stochastic equation (1.1).

We have considered only the case of a symmetric distribution $P(A,t)=P(-A,t)$ for all t, and do not treat asymmetric initial conditions explicitly. This means that we do not calculate the switch-over time from the positive to the negative attractor in steady state ($\epsilon=\epsilon_0>0$), or the equivalent quantity for sinusoidal modulation[5–7] (1.6). This time controls the transition between "ordered" and "disordered" regimes seen in experiments on modulated convection.[8] Nevertheless, even though we cannot calculate the switch-over time explicitly, we can estimate the position of the above-mentioned order-disorder transition by analyzing symmetric distributions, and we find reasonable agreement with a numerical calculation presented earlier.[7] Onuki[5] has also treated the problem of sinusoidal modulation approximately, and although the order-disorder transition we extract from his formulas agrees with ours, the shape of the distribution function he obtains deviates significantly.

In Sec. II the basic approximation is described and the failing of the ACHS procedure for $\epsilon<0$ exhibited. Section III presents an evaluation of $P(A,t)$ for various ramps and sinusoidal modulations, as well as a comparison with the numerical solution of Eq. (1.1). A succinct statement of our approximation, as well as a generalization to an arbitrary nonlinear equation with a time dependent drift, are presented in the Appendix.

II. APPROXIMATE ANALYTIC CALCULATION

We consider Eqs. (1.1)–(1.3) with initial time $t_0=t^+$, assuming first that

$$\epsilon(t)>0 \text{ for } t>t^+ \tag{2.1}$$

and

$$\epsilon(t^+)=0 \ . \tag{2.2}$$

We follow ACHS (Ref. 3) and introduce the deterministic solution of Eq. (1.1) with initial condition

$$A(t^+)=A_0 \ , \tag{2.3}$$

namely,

$$A^2=\frac{R_1^2A_0^2}{1+R_2A_0^2} \ , \tag{2.4}$$

where

$$R_1(t)\equiv\exp\left[\tau_0^{-1}\int_{t^+}^{t}\epsilon(s)ds\right] \ , \tag{2.5}$$

$$R_2(t)\equiv 2g_3\tau_0^{-1}\int_{t^+}^{t}R_1^2(s)ds \ . \tag{2.6}$$

The stochastic variable A_0 defined by (2.4) satisfies the equation

$$\tau_0\partial_t A_0=\bar{f}(t)R_1^{-1}(t)(1+A_0^2R_2)^{3/2} \ . \tag{2.7}$$

Since for $\epsilon>0$, R_1 and R_2 grow exponentially with time, the variable A_0 only affects A for early times (near t^+) when A_0 is usually small, so we linearize Eq. (2.7) to find

$$\tau_0\partial_t A_0=\bar{f}(t)R_1^{-1}(t) \ . \tag{2.8}$$

The ensuing Fokker-Planck equation is[1]

$$\partial_t P_0(A_0,t)=\tau_0^{-1}\bar{F}R_1^{-2}\frac{\partial^2P_0}{\partial A_0^2} \ , \tag{2.9}$$

and it can be solved exactly when the initial distribution is known. Let us choose

$$P_0(A_0,t^+)=\tilde{P}_+(A,t^+)=\frac{1}{(2\pi\tilde{A}_0^2)^{1/2}}e^{-(A_0-A_1)^2/2\tilde{A}_0^2} \ , \tag{2.10}$$

with $\tilde{A}_0$ and A_1 arbitrary constants. Then the solution of Eqs. (2.9) and (2.10) is

$$P_0(A_0,t)=\frac{1}{(2\pi\tilde{A}^2)^{1/2}}e^{-(A_0-A_1)^2/2\tilde{A}^2} \ , \tag{2.11}$$

where

$$\tilde{A}^2=\tilde{A}_0^2+R_3 \ , \tag{2.12}$$

$$R_3(t)\equiv 2\bar{F}\tau_0^{-1}\int_{t^+}^{t}R_1^{-2}(s)ds \ . \tag{2.13}$$

Then the probability distribution for A is

$$\tilde{P}_+(A,t)=P_0(A_0,t)\frac{dA_0}{dA} \ , \tag{2.14}$$

$$\tilde{P}_+(A,t)=\frac{1}{(2\pi\tilde{A}^2)^{1/2}}(A_0/A)^3R_1^2\times e^{-(A_0-A_1)^2/2\tilde{A}^2} \ , \tag{2.15}$$

where from (2.4)

$$A_0^2(A)=\frac{A^2}{R_1^2-R_2A^2} \ . \tag{2.16}$$

The above derivation assumes $A_1>0$. For a symmetric initial distribution the solution has the form

$$P(A,t)=\tfrac{1}{2}[\tilde{P}(A,t)+\tilde{P}(-A,t)] \ . \tag{2.17}$$

From Eqs. (2.15)–(2.17) it follows that the average $\langle A^{2n} \rangle$ is given by

$$\langle A^{2n}(t) \rangle = (2\pi)^{-1/2} \int_{-\infty}^{\infty} d\xi\, e^{-(\xi-\xi_1)^2/2} \times \left[\frac{A_L^2 \xi^2}{1+\tau\xi^2} \right]^n , \tag{2.18}$$

$$A_L^2 = R_1^2 \tilde{A}^2 , \tag{2.19}$$

$$\tau = A_L^2 R_2 / R_1^2 = R_2 \tilde{A}^2 , \tag{2.20}$$

$$\xi_1 = R_1 A_1 / A_L . \tag{2.21}$$

As the notation suggests, A_L^2 is equal to the average obtained from the *linear* stochastic equation

$$\tau_0 \partial_t A = \epsilon(t) A + \bar{f}(t) , \tag{2.22}$$

with initial distribution

$$P_L(A, t^+) = (2\pi \tilde{A}_0^2)^{-1/2} e^{-A^2/2\tilde{A}_0^2} , \tag{2.23}$$

namely,

$$P_L(A,t) = (2\pi A_L^2)^{-1/2} e^{-A^2/2A_L^2} , \tag{2.24}$$

with A_L given by (2.19) and $\tilde{A}^2$ by (2.12). At early times (t near t^+) we have $R_2 \ll 1$ and hence $\tau \ll 1$ [see Eq. (2.20)], so by (2.18)

$$\langle A^2(t) \rangle \simeq A_L^2 , \tag{2.25}$$

i.e., the linear equation is a good approximation. For late times $[t - t^+ \gg \tau_0/\epsilon(t)]$ Eqs. (2.20), (2.6), and (2.13) imply that $\tau \to \infty$, so we find from (2.18)

$$\langle A^{2n}(t) \rangle = (R_1^2/R_2)^n , \tag{2.26}$$

which agrees with the deterministic answer (2.4) at late times where the latter is independent of initial conditions.

The foregoing approximation thus interpolates between the linear behavior near the instability and the deterministic behavior in the strongly nonlinear region. It is precisely the generalized Suzuki approximation introduced by ACHS,[3] except that here we have specified that $\epsilon > 0$, and we take as our time origin in the integrations in Eqs. (2.5), (2.6), and (2.13) the point t^+ where $\epsilon = 0$. This was the procedure followed by ACHS in their numerical example, but it was not realized at the time that the condition $\epsilon > 0$ was *necessary* to obtain sensible results. Indeed, suppose we try to apply the above approximation to the case of a constant $\epsilon = -\epsilon_0 < 0$, for which we know that the linear answer (2.25) should be an excellent approximation, i.e.,

$$\langle A^2 \rangle = A_L^2 = \bar{F}/\epsilon_0 . \tag{2.27}$$

Carrying out the integrals in (2.18) we find instead, at long times,

$$\langle A^2(t) \rangle = (\epsilon_0/g_3) \exp(-2\epsilon_0 t/\tau_0) , \tag{2.28}$$

which is clearly incorrect.

To treat the case $\epsilon(t) < 0$ correctly, we assume

$$\epsilon(t^-) = 0, \quad \epsilon(t) < 0, \quad t > t^- , \tag{2.29}$$

and parametrize the initial distribution by

$$\tilde{P}(A, t^-) = \frac{1}{(2\pi \tilde{A}_0^2)^{1/2}} e^{-(A-A_1)^2/2\tilde{A}_0^2} , \tag{2.30}$$

as in (2.10), but with *different* constants $\tilde{A}_0$ and A_1. We first suppose simply that A satisfies the linear equation (2.22), whence the probability distribution $\tilde{P}_L(A)$ satisfies the Fokker-Planck equation[1]

$$\tau_0 \partial_t \tilde{P}_L(A,t) = -\partial_A(\epsilon A \tilde{P}_L) + \bar{F} \partial_A^2 \tilde{P}_L . \tag{2.31}$$

The solution of (2.31), with initial condition (2.30) is

$$\tilde{P}_L(A,t) = \frac{1}{(2\pi A_L^2)^{1/2}} e^{-(A-\tilde{A}_1)^2/2A_L^2} , \tag{2.32}$$

where A_L^2 is given by (2.19),

$$\tilde{A}_1 \equiv A_1 R_1 , \tag{2.33}$$

and the integrations in R_1 and R_3, Eqs. (2.12) and (2.13), now start at $t = t^-$.

In order to improve the above approximation slightly, we wish to take the nonlinear term in Eq. (1.1) into account approximately, but not in the same way as for $\epsilon > 0$, since (2.15) leads to large errors. Let us introduce an integrating factor for Eq. (1.1) and define the stochastic variable

$$B = A R_1^{-1} , \tag{2.34}$$

which satisfies the equation without linear drift

$$\tau_0 \partial_t B = -g_3 R_1^2 B^3 + R_1^{-1} \bar{f} , \tag{2.35}$$

and leads to the Fokker-Planck equation

$$\tau_0 \partial_t P_B(B,t) = \partial_B(g_3 R_1^2 B^3 P_B) + \bar{F} R_1^{-2} \partial_B^2 P_B . \tag{2.36}$$

An approximate solution of this equation is obtained by replacing the quantity B^3 in the first term on the rhs by B_D^3, where B_D is the solution of the *deterministic* equation [(2.35) with $\bar{f} = 0$] with initial value $B_D(t^-) = A_1$, namely,

$$B_D = A_1 (1 + R_2 A_1^2)^{-1/2} . \tag{2.37}$$

When this replacement is made, Eq. (2.36) can be solved to yield

$$P_B(B,t) = \frac{1}{(2\pi \tilde{A}^2)^{1/2}} e^{-(B-B_D)^2/2\tilde{A}^2} , \tag{2.38}$$

where $\tilde{A}^2(t)$ is given by (2.12). The corresponding distribution for A has the form

$$\tilde{P}_-(A,t) = \frac{1}{(2\pi A_L^2)^{1/2}} e^{-(A-A_D)^2/2A_L^2} , \tag{2.39}$$

with A_L given by (2.19), and

$$A_D = R_1 B_D = R_1 A_1 (1 + R_2 A_1^2)^{-1/2} . \tag{2.40}$$

(In all of the above expressions pertaining to $\epsilon < 0$, the integrations in the R_i begin at t^-.) Thus the approximate treatment of the nonlinear term in Eq. (2.36) has resulted

in a nonlinear correction to the quantity $\tilde{A}_1$ appearing in (2.32), so that $\tilde{A}_1 \to A_D$. The full distribution $P(A)$ is again obtained by symmetrization, according to (2.17).

The above discussion determines the distribution function $P(A)$ for either $\epsilon > 0$ [Eq. (2.15)] or $\epsilon < 0$ [Eq. (2.39)], once the distribution is known at the crossing points $t^{\pm}$. To find the latter in each case we shall fit to the distribution function in the preceding interval, i.e., we apply the conditions

$$\tilde{P}_-(A,t^-) = \lim_{\eta \to 0} \tilde{P}_+(A,t^- - \eta) , \tag{2.41a}$$

$$\tilde{P}_+(A,t^+) = \lim_{\eta \to 0} \tilde{P}_-(A,t^+ - \eta) , \tag{2.41b}$$

to fit $\tilde{A}_0$ and A_1 appearing in (2.15) and (2.39). In practice these matching conditions are often satisfied by making the following choices.

At $t = t^+$:

$$\tilde{A}_0 = A_L(t^+), \quad A_1 = 0 . \tag{2.42a}$$

At $t = t^-$:

$$\tilde{A}_0 = R_1(t^-)[3\sqrt{6} R_2^{3/2}(t^-)\tilde{A}^2(t^-)]^{-1} ,$$
$$A_1 = R_1(t^-)/R_2^{1/2}(t^-) . \tag{2.42b}$$

In addition, the initial distribution at $t = t_0$ must be known, but in practice t_0 can be taken usually at sufficiently early times so that $P(A,t)$ is independent of $P(A,t_0)$. In view of the somewhat complicated nature of our approximation we have restated the final result in the form of a general recipe in the Appendix. The essential point is that the time axis is broken up into intervals where $\epsilon(t)$ has a definite sign and the equation is integrated in each interval with appropriate initial conditions at the crossing points $t_i^{\pm}$.

In the foregoing we imagine starting the dynamics with a symmetric initial condition, which ensures a symmetric distribution for all time. A more realistic but more difficult case to treat is that of an asymmetric initial distribution [Eq. (2.10), say]. For periodic modulation (1.6) one could then attempt to determine the time necessary to achieve a symmetric periodic distribution $P(A,t)$. Although a precise calculation of this time, which from our point of view is part of the "transient" dynamics, is rather difficult we can say generally that if the asymmetric $\tilde{P}(A,t)$ has little weight at $A = 0$ the time will be long, and if $\tilde{P}(0,t)$ is large the time will be short. In analogy to the procedure employed by Onuki,[5] we will therefore define the ratio

$$\rho = \tilde{P}(0,t^+)/\tilde{P}(A_1,t^+) \tag{2.43}$$

as a measure of how long it will take for the system to make a transition from $A \approx A_1$ to $A \approx -A_1$, and also of the time necessary to achieve a symmetric distribution (2.17). We will pick a threshold value ρ_c and distinguish between "ordered" (asymmetric) distributions for $\rho < \rho_c$ and "disordered" (symmetric) distributions for $\rho > \rho_c$. We may thus evaluate an order-disorder transition line $\epsilon_{0c}(\delta)$ in the (ϵ_0, δ) plane at fixed ω, as was done experimentally by Meyer, Ahlers, and Cannell[8] and numerically by two of us.[7] The results for $\omega = 1$ are shown in Table I for various reasonable choices of ρ_c, and compared with the values obtained numerically,[7] and with those of Onuki's[5] Eq. (B.10), which corresponds to $\rho_c = 0.1$. It is seen that the different results agree to within 25%, which is probably all that can be expected from such crude estimates. Another comparison is possible between our approximation and that of Onuki,[5] namely, the value of the order parameter

$$Z = A_1/\tilde{A} , \tag{2.44}$$

in the ordered phase at $t = t^+$ for $\epsilon_0 > \epsilon_{0c}(\delta)$. Taking Onuki's values from his Eqs. (B.6) and (B.10) we find that Z is a nonmonotonic function of ϵ at fixed δ in the ordered phase, surely an unphysical feature of the approximation. Moreover, Onuki's values for Z differ from ours by factors of 2–3 in the range $0.2 < \delta < 0.4$, $1 < \epsilon_0/\epsilon_{0c} \lesssim 1.3$.

III. COMPARISON WITH NUMERICAL SOLUTION

In order to test our approximate analytic solution we have integrated the stochastic equation (1.1) numerically. We used the function gasdev ( ) of Press *et al.*[9] to generate Gaussian random numbers from uniform deviates over the interval (0,1). The uniform deviates were generated by the function ran1 ( ) or ran3 ( ) of Press *et al.* To test the validity of the numerical procedures we studied the case $\epsilon = -0.1$, using $g_3 = 0$ in Eq. (1.1) so that the linear result given in Eq. (2.27) applies, and found agreement within the statistical errors.[10]

The specific calculations we undertook involve upward and downward ramps, $\epsilon = \beta t$ with $\beta = \pm 0.27, \pm 5.0$, and sinusoidal variation $\epsilon = \epsilon_0 + \delta \sin \omega t$, with various values of ϵ_0, δ, ω. The noise strength $\bar{F}$ was taken as $\bar{F} = 5 \times 10^{-7}$, and the other parameters in Eq. (1.1) were $\tau_0 = 0.055$ and $g_3 = 0.85$. These values are representative of those in the convection experiment of Meyer, Ahlers, and Cannell.[8] The resulting distributions $P(A)$ are

TABLE I. Order-disorder transition ($\omega = 1$).

		ϵ_{0c}		
δ	ρ_c	Present work	SH (Ref. 7)	Onuki (Ref. 5)
0.2	0.1	0.08	0.06	
	0.5	0.07		
	0.1			0.06
0.3	0.1	0.16	0.14	
	0.5	0.14		
	0.1			0.13
0.4	0.1	0.24	0.23	
	0.5	0.22		
	0.1			0.22

shown in in Figs. 1–5, from which it is seen that the analytic expressions in general agree very well with the numerical results. In particular, we show $P(A)$ at selected times chosen to illustrate the changes in shape most vividly. For the ramps (Figs. 1–4) the initial distribution was taken to be a δ function at the stationary values of A [$A=0$ for $\epsilon<0$ and $A=\pm(\epsilon/g_3)^{1/2}$ for $\epsilon>0$]. The matching conditions used were those given in Eqs. (2.42a) and (2.42b) for positive and negative β, respectively.

For the sinusoidal variation of $\epsilon(t)$, the initial distribution is irrelevant since in both the theory and the numerical work the calculation was pursued until a periodic distribution was found. In particular, the matching conditions (2.41a) and (2.41b) had to be iterated repeatedly, adjusting A_1 to fit the position of the peak and $\tilde{A}_0$ to fit the peak height or the width at half maximum, until a periodic $P(A,t)$ was found. Although this procedure involves trial and error, it is entirely self-consistent and no external adjustments were necessary. For relatively slow modulation ($\omega=5$, $\omega\tau_0=0.275$) the agreement shown in Fig. 5 is excellent. For $\omega=20$ ($\omega\tau_0=1.1$), on the other hand, the distribution is extremely narrow at most times, so the comparison is somewhat delicate. A sensitive test of the behavior is exhibited in Fig. 6, where we show the large changes in $P(A,t)$ that occur when the average ϵ_0 is changed from $+0.2$ to -0.05 at fixed ω, $\delta \gg |\epsilon_0|$, and fixed time in the period. Although the agreement is not

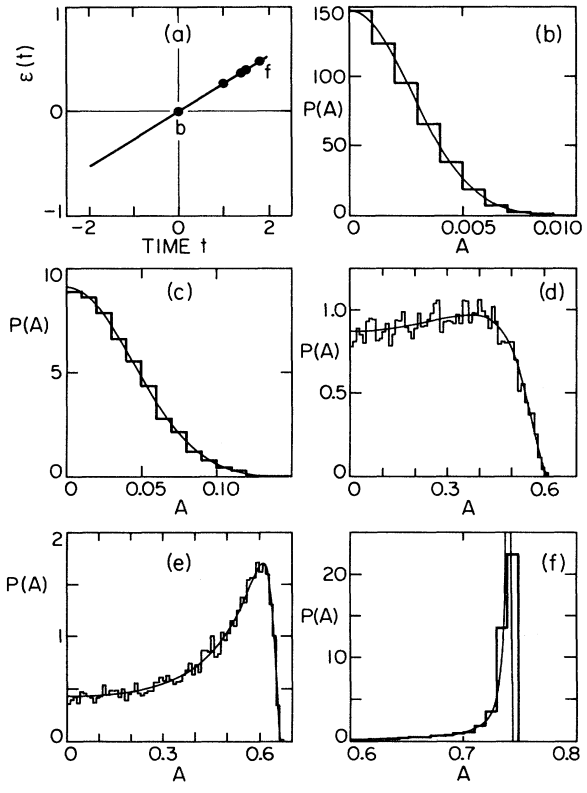

FIG. 1. Probability distribution $P(A)$ vs A at various times for Eqs. (1.1) and (1.3) with $\tau_0=0.055$, $g_3=0.85$, $\bar{F}=5\times10^{-7}$, and $\epsilon(t)$ given by an upward ramp. (a) shows $\epsilon(t)$ vs t for $\beta=0.27$. The five points on the line correspond in order to the times in parts (b)–(f), respectively ($t=0.0$, 1.0, 1.4, 1.5, and 1.8). The smooth line in (b)–(f) is the analytic approximation discussed in the text, and the jagged line is a histogram representing a numerical evaluation of the stochastic equation. Note the vastly different scales in parts (b)–(f).

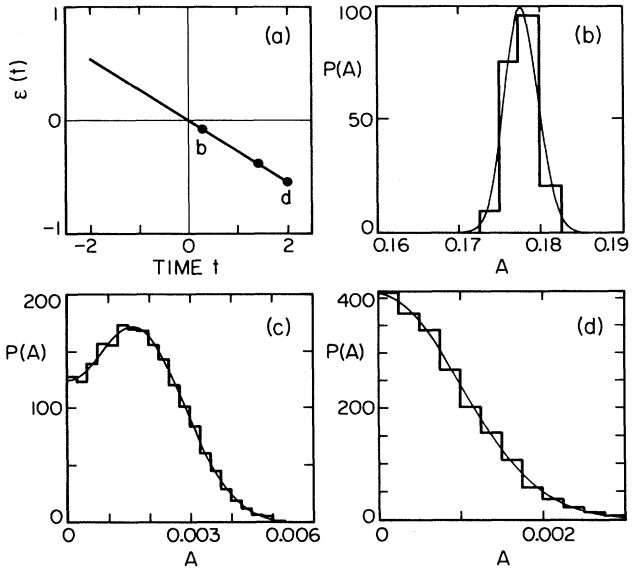

FIG. 2. Similar to Fig. 1, but for a downward ramp, $\epsilon(t)=\beta t$ for $\beta=-0.27$. The times in (b)–(d) are $t=0.3$, 1.4, and 2.0, respectively.

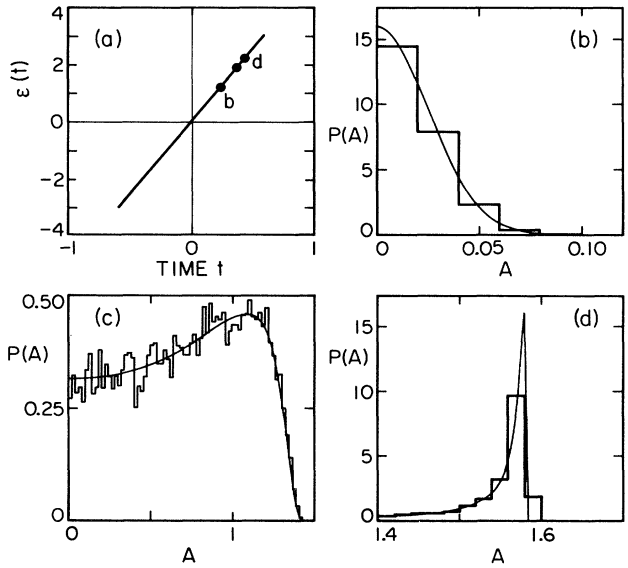

FIG. 3. Similar to Fig. 1, for $\beta=5$. The times in (b)–(d) are $t=0.24$, 0.38, and 0.44, respectively.

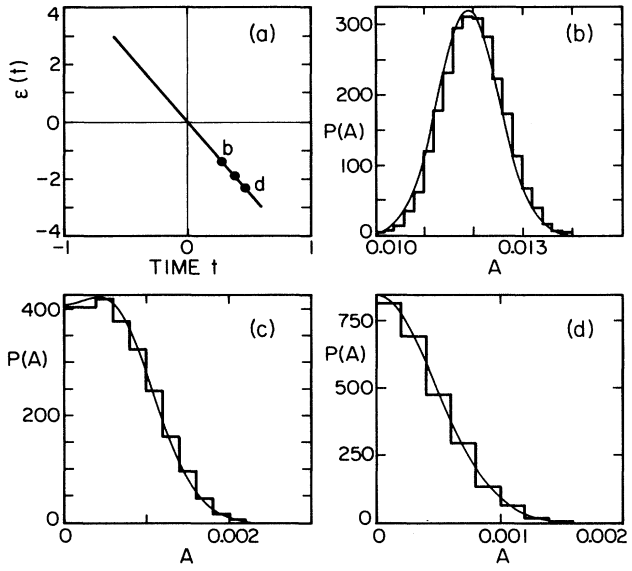

FIG. 4. Similar to Fig. 1, for $\beta=-5$. The times in (b)–(d) are $t=0.28$, 0.38, and 0.46, respectively.

perfect the overall trend is well represented.

In Fig. 7 we illustrate the variation with noise strength $\bar{F}$ of the onset time t_{on} for the bifurcation, by showing how the distribution function $P(A,t)$ at fixed time $t=1.5$ changes its shape. For large noise strength (curve *a*) we have $t_{on}<t$ and $P(A)$ is peaked at $A\simeq 0.65$, whereas for low noise strength (curve *d*), $t_{on}>t$ and $P(A)$ is peaked at $A=0$. From the intermediate cases shown in *c* and *d*, we conclude that $t_{on}=t=1.5$ for $\bar{F}\simeq 10^{-8}$.

Finally, in Fig. 8 we show the generalization of our procedure, as described in the Appendix, Eqs. (A17)–(A22), to a slightly different model, the Maxwell-Bloch equation[11]

$$\tau_0 \partial_t A = A\left[\frac{\epsilon(t)-g_3 A^2}{1+g_3 A^2}\right]+\bar{f}(t)\ . \tag{3.1}$$

The example studied numerically is the same as the one shown in Fig. 1(b), i.e., a ramp $\epsilon=\beta t$ with $\beta=0.27$, $t=1.5$. For this case it is seen that the distribution functions are almost identical in the two models. For further references to the laser literature see Broggi *et al.*[11] and Ciofini, Meucci, and Arecchi.[12]

In conclusion, we have obtained a useful analytic approximation to represent the dynamics of an imperfect pitchfork bifurcation in a single-variable stochastic equation. By making detailed comparisons between the analytic and numerical simulations we have shown that the time-dependent distribution function can be evaluated accurately in the symmetric case.

Note added in proof. After submission of this paper we received a copy of an interesting paper by Caceres, Becker, and Kramer,[13] who treat the case of periodic modulation using a steepest-descent method, and calculate the long-time behavior of the probability distribution.

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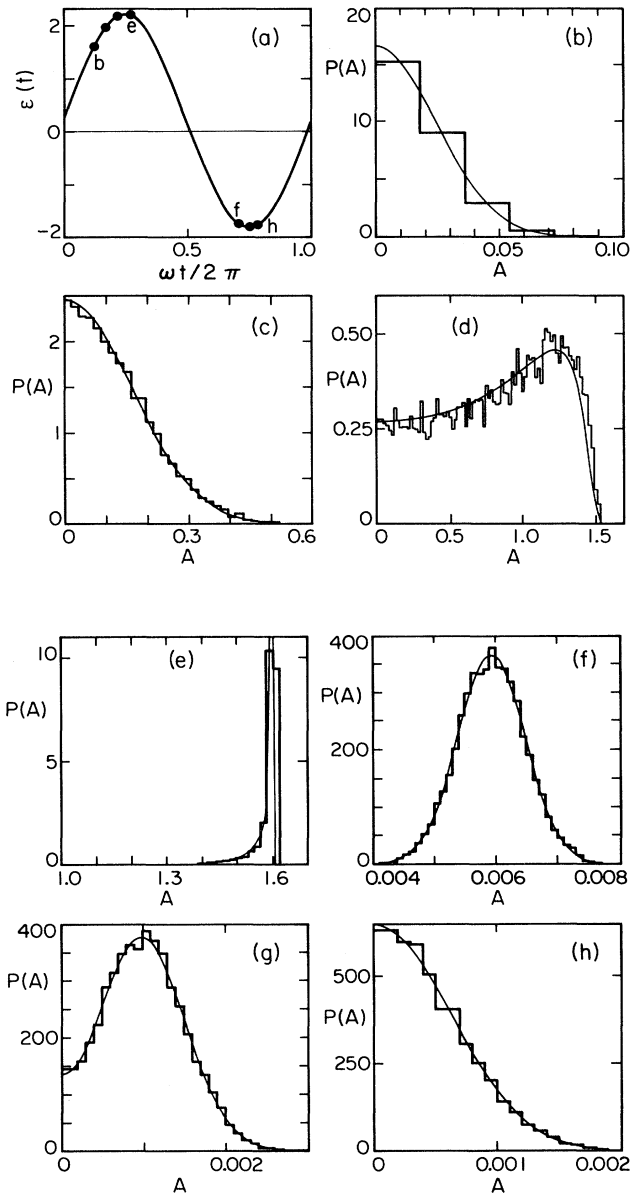

FIG. 5. Similar to Fig. 1, for a sinusoidal modulation $\epsilon(t)=\epsilon_0+\delta\sin\omega t$, with $\epsilon_0=0.2$, $\omega=5$, and $\delta=2$. The times in (b)–(h) in units of the period are $\omega t/2\pi=0.1250$, 0.1719, 0.2188, 0.2734, 0.7075, 0.7520, and 0.7822, respectively.

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APPENDIX: ALGORITHM FOR APPROXIMATE SOLUTION

Our approximate solution of the stochastic equation (1.1) involves the following steps.

(i) For a general function $\epsilon(t)$, $t > t_0$, we denote by t_i^+ and t_i^- the ith zero crossings of $\epsilon(t)$, with $d\epsilon/dt > 0$ at t_i^+ and $d\epsilon/dt < 0$ at t_i^-, as shown schematically in Fig. 9.

(ii) The distribution function is obtained from

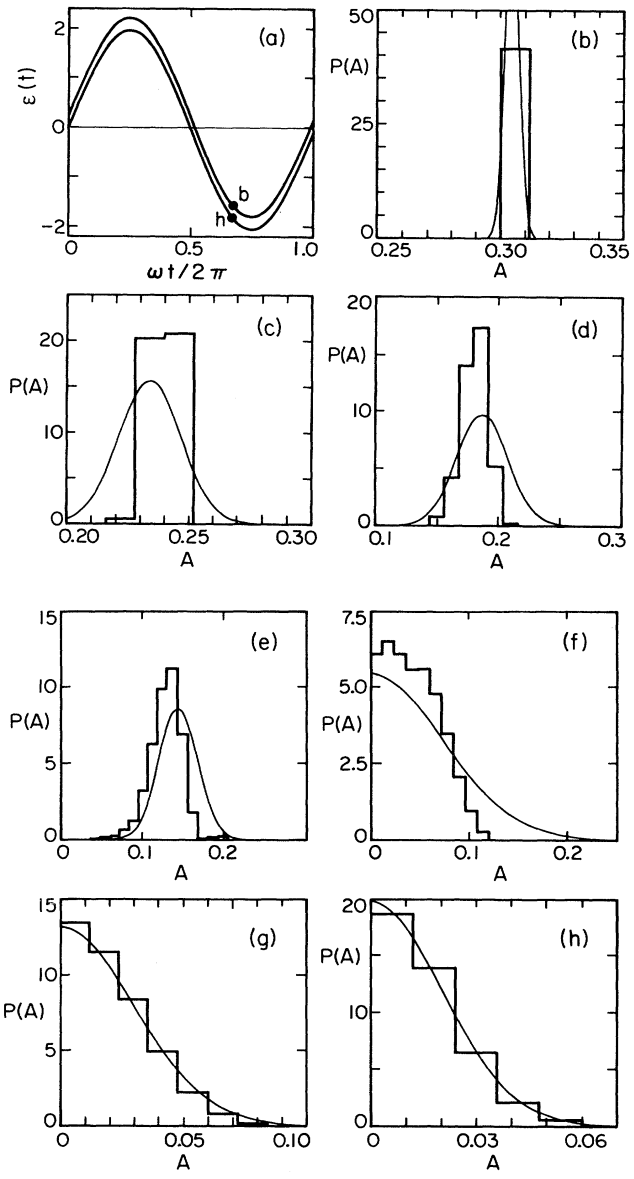

FIG. 6. In (a) $\epsilon(t)=\epsilon_0+\delta\sin\omega t$ is displayed for $\epsilon_0=0.2$ (upper curve) and $\epsilon_0=-0.05$ (lower curve) with $\delta=2$ and $\omega=20$ vs $\omega t/2\pi$. The two points in (a) are at $\omega t/2\pi=0.673$. (b)–(h) give $P(A)$ vs A at $\omega t/2\pi=0.673$ for $\epsilon_0=0.2$, 0.1, 0.05, 0.025, 0.0, −0.025, and −0.05, respectively.

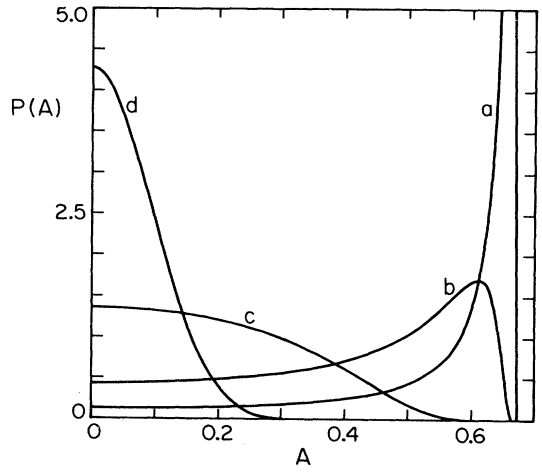

FIG. 7. Probability distribution $P(A)$ vs A at $t=1.5$, as calculated from the approximation discussed in the text for an upward ramp $\epsilon=\beta t$, $\beta=0.27$, as in Fig. 1, for various values of the noise strength $\bar{F}=5\times10^{-6}$ (curve a), 5×10^{-7} (curve b), 5×10^{-8} (curve c), and 5×10^{-9} (curve d).

$$P(A,t)=\tfrac{1}{2}[\tilde{P}(A)+\tilde{P}(-A)]\ , \tag{A1}$$

where

$$\tilde{P}(A)=\tilde{P}_+(A)=(2\pi\tilde{A}^2)^{-1/2}\left|\frac{\partial A_0(A)}{\partial A}\right| \times\exp\{-[A_0(A)-A_1]^2/2\tilde{A}^2\}\ , \tag{A2}$$

for $\epsilon(t)>0$, and

$$\tilde{P}(A)=\tilde{P}_-(A)=(2\pi A_L^2)^{-1/2} \times\exp\{-[A-A_D(A_1)]^2/2A_L^2\}\ , \tag{A3}$$

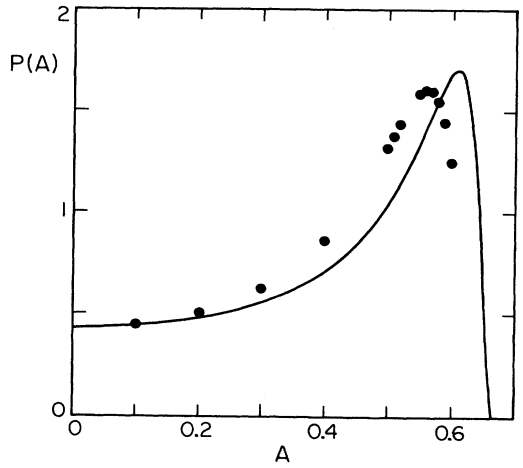

FIG. 8. Comparison of the stochastic Landau equation (1.1) (solid curve) with the stochastic Maxwell-Bloch equation (3.1) (points). The parameters are the same as in Fig. 1(e), i.e., an upward ramp $\epsilon(t)=\beta t$ with $\beta=0.27$ at $t=1.5$.

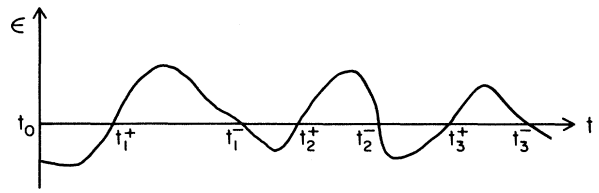

FIG. 9. Schematic representation of the time dependent linear drift $\epsilon(t)$ for $t > t_0$, with the definitions of the ascending (t_i^+) and descending (t_i^-) crossing points.

for $\epsilon(t) < 0$. In the above formulas the quantities are defined as

$$A_0(A) = A(R_1^2 - R_2 A^2)^{-1/2} , \tag{A4}$$

$$\tilde{A}^2 = \tilde{A}_0^2 + R_3 , \tag{A5}$$

$$A_L = \tilde{A} R_1 , \tag{A6}$$

$$A_D(A_1) = R_1 A_1 (1 + R_2 A_1^2)^{-1/2} , \tag{A7}$$

$$R_1 = \exp\left[\tau_0^{-1} \int_{t_i}^{t} \epsilon(s) ds\right] , \tag{A8}$$

$$R_2 = 2 g_3 \tau_0^{-1} \int_{t_i}^{t} R_1^2(s) ds , \tag{A9}$$

$$R_3 = 2 \bar{F} \tau_0^{-1} \int_{t_i}^{t} R_1^{-2}(s) ds . \tag{A10}$$

The quantity t_i in (A8)–(A10) denotes the preceding crossing point (i.e., t_i^- for $t_i^- \le t \le t_{i+1}^+$, $\epsilon < 0$, and t_i^+ for $t_i^+ \le t \le t_i^-$, $\epsilon > 0$). An examination of the above formulas shows that they determine $P(A)$ in each interval in terms of only two constants, $\tilde{A}_0$ [Eq. (A5)] and A_1 [Eq. (A7)], which characterize the initial distribution for that interval.

(iii) To determine the parameters $\tilde{A}_0(t_i)$ and $A_1(t_i)$ at the ith crossing point the distribution function is represented in the form

$$\tilde{P}(A, t_i) = (2\pi \tilde{A}_0^2)^{-1/2} \exp[-(A - A_1)^2 / 2\tilde{A}_0^2] , \tag{A11}$$

and is fitted to the distribution in the preceding interval,

$$\tilde{P}_+(A, t_i^+) = \lim_{\eta \to 0} \tilde{P}_-(A, t_i^+ - \eta) , \tag{A12}$$

$$\tilde{P}_-(A, t_i^-) = \lim_{\eta \to 0} \tilde{P}_+(A, t_i^- - \eta) . \tag{A13}$$

In this way $P(A)$ can be evaluated in succeeding intervals once it is assumed to be known at the earliest time $t = t_0$. A set of simplified matching conditions that are often sufficient is given in Eqs. (2.42a) and (2.42b).

(iv) The expressions for the distribution functions in Eqs. (A1)–(A10) imply the following formula for the moments:

$$\langle A^{2n}(t) \rangle = (2\pi)^{-1/2} \int_{-\infty}^{\infty} d\xi \, e^{-(\xi - \xi_1)^2/2} \left[\frac{A_L^2 \xi^2}{1 + \bar{\tau} \xi^2}\right]^n , \tag{A14}$$

where

$$\bar{\tau} = \begin{cases} R_2 \tilde{A}^2, & \epsilon > 0 \\ 0, & \epsilon < 0 \end{cases} \tag{A15}$$

and

$$\xi_1 = \begin{cases} R_1 A_1 / A_L, & \epsilon > 0 \\ A_D / A_L, & \epsilon < 0 . \end{cases} \tag{A16}$$

(v) The above formulas may be generalized to an equation with an arbitrary deterministic part,

$$\tau_0 \partial_t A = D(A, t) + \bar{f}(t) . \tag{A17}$$

Let the linear drift be

$$\epsilon(t) \equiv \left.\frac{\partial D}{\partial A}\right|_{A=0} , \tag{A18}$$

and define the solution of the deterministic equation with initial value $A = A_0$ as

$$A(t) = A_D(t, A_0) , \tag{A19}$$

and the inverse function

$$A_0(t, A) \equiv A_D^{-1}(t, A) , \tag{A20}$$

which we assume to be unique. (The functions A_D and A_D^{-1} must, in general, be evaluated numerically.) Then the above formulas for $P(A)$ still hold, except that (A4) and (A7) are replaced by (A20) and (A19), respectively, and $\epsilon(t)$ is defined by (A18). In order to evaluate the derivative $\partial A_0 / \partial A$ appearing in Eq. (a2) we take a variation of Eq. (A17) (with $\bar{f} = 0$),

$$\tau_0 \partial_t \delta A(t) = \left[\frac{\partial D}{\partial A}\right] \delta A(t) , \tag{A21}$$

and integrate *backwards* along the trajectory $A_0(t, A) = A_D^{-1}(t, A)$ to find

$$\delta A(t_0) = \delta A(t) \exp\left[\tau_0^{-1} \int_t^{t_0} ds \left[\frac{\partial D(A', s)}{\partial A'}\right]_{A' = A_D^{-1}(s, A)}\right] . \tag{A22}$$

Then the derivative $\partial A_0 / \partial A$ is the exponential factor multiplying $\delta A(t)$ on the rhs of (A22), which is given as a function of A and t.

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