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Strange attractor in the reflectivity of a phase-conjugate mirror

M. R. Belić and D. Timotijević
Institute of Physics, P.O. Box 57, 11001 Belgrade, Yugoslavia
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Wave equations describing optical phase conjugation in photorefractive crystals are solved using a technique that transforms a boundary-value problem into an initial-value problem. Boundary conditions are satisfied using an iterative map constructed in the parameter space. Within the iterative procedure unstable situations arise, leading to a chaotic response of the crystal. A strange attractor is discovered in the reflectivity of the crystal, existing in the multiparameter space of the single-interaction-region four-wave-mixing process.

Instabilities in optical phase conjugation (OPC) have become a subject of considerable interest lately. This is only natural in view of its potential applicability.[1] However, no one can make a dependable device before understanding and eliminating eventual unstable modes of operation. By the same token, any novel unstable or possibly chaotic system presents an interesting investigation subject *per se*.

Initial reports on chaos and turbulence in OPC (Ref. 2) were concerned primarily with resonator configurations, in which one mirror is a phase-conjugating element. In such geometries the presence of all necessary ingredients for dissipative chaos is easily ensured:[3] nonlinearity is provided by the medium, driving is enabled by laser pumps, and feedback comes from the normal mirror. The situation is not so simple if one considers the phase-conjugate mirror by itself, especially in the experimentally interesting self-pumped geometry of multiple interacting regions and multiple gratings in photorefractive crystals. One should contrast the ease of obtaining phase-conjugate beams in such geometries with the difficulties in trying to explain their behavior.[4]

We display in this report the emergence of deterministic chaos in a single interaction region of a photorefractive crystal, assuming multigrating four-wave-mixing (4MW) optical conjugation. No other feedback mechanism is envisaged, such as internal corner reflections in the crystal, or external mirrors. We show that even ordinary 4WM, with optical feedback provided only by the energy transfer between waves is sufficiently nonlinear to produce unstable phase-conjugate output. The results reported are obtained by an alternative integration procedure, in which a boundary-value problem is transformed into an initial-value problem, that can be treated in quadratures. The boundary-value nature of the problem is retained in the two-point dependence of some parameters, which are analyzed by an iterative mapping procedure. Within the iteration procedure unstable situations arise, leading to chaotic outputs and a strange attractor in the parameter plane.

Our starting point is the slowly varying envelope wave equations describing steady-state multigrating phase conjugation in photorefractive media:

$$I A_1' = g(A_T A_4 - A_R A_3) - \gamma |A_2|^2 A_1 , \tag{1a}$$

$$I A_2^{*\prime} = g(A_T A_3^* - A_R A_4^*) - \gamma |A_1|^2 A_2^* , \tag{1b}$$

$$I A_3' = -g(A_T A_2 + A_R^* A_1) , \tag{1c}$$

$$I A_4^{*\prime} = -g(A_T A_1^* + A_R^* A_2^*) , \tag{1d}$$

where A_i are the slowly varying amplitudes of the four electric fields interacting inside the crystal, g and γ are the interaction coefficients (assumed to be real in this case of photorefractives), $A_T = A_1 A_4^* + A_2^* A_3$ and $A_R = A_1 A_3^* + A_2^* A_4$ are the amplitudes of the transmission and the reflection gratings that have been established in the crystal, and $I = \sum_i |A_i|^2$ is the total intensity. The geometry assumed is the standard 4WM setup: counterpropagating pumps A_1 and A_2 impinge on the crystal from the opposite sides, and a signal A_4 enters from the side of the pump A_1, tilted for some small angle. Due to photorefractive interaction of these waves, a fourth wave A_3 is generated in the crystal, which is the counterpropagating phase-conjugate replica of the signal A_4. The prime in Eqs. (1) denotes the derivative along the propagation (z) direction through the crystal, and the asterisk represents complex conjugation.

In arriving at Eqs. (1) from their most general form, as given, for example, in Ref. 5, we assumed that the transmission grating (built by the waves A_1 and A_4) and the reflection grating (built by A_2 and A_4) contribute equally, and that the two-wave contribution, coming from the mixing of the signal A_4 with the phase conjugate A_3, can be neglected. Both of these assumptions are reasonable and frequently occurring in experiments.

The object of the analysis is to solve Eqs. (1) subject to split boundary conditions: the fields A_1 and A_4 are given on the one side of the crystal, and A_2 and A_3 (which is zero) on the other. Many authors have tried to solve these equations, using a variety of methods: from analytic in special cases, to different numerical schemes.[2,5,6] A very systematic numerical study has been performed by Ja,[7] who tried almost all known methods applicable to boundary-value problems: from the shooting method to the finite element method; from first-order algorithms all the way to the seventh order.

The main problem is that these equations allow for multistable solutions, or even chaotic. Part of the instabilities is real, since it is known that the generation of a phase-conjugate beam can be very unstable. However, part of the instabilities could be spurious, and arising from the fact that one is solving steady-state equations. The situation is analogous to computational fluid dynamics, where solving a steady-state potential equation instead of Euler or Naiver-Stokes equations may lead to multiple solutions, which are unacceptable on physical grounds. No systematic study of instabilities or chaotic scenarios in OPC has been offered. Our method appears to be advantageous compared to standard methods, such as shooting or relaxation, based on the fact that it provides a rapid and accurate solution to the boundary-value problem at hand, and that it deals readily with multistable situations.

The solution procedure is as follows. Going over to intensity equations, assuming exact phase conjugation, and changing variables into $u_1=I_2+I_1$, $v_1=I_2-I_1$, $u_2=I_4+I_3$, $v_2=I_4-I_3$, leads to the following set of equations:

$$Iu_1'=2gv_1v_2+\gamma f_1^2 \ , \tag{2a}$$

$$Iv_1'=2gu_1v_2 \ , \tag{2b}$$

$$Iu_2'=2g(u_1u_2+f_1f_2) \ , \tag{2c}$$

$$Iv_2'=2gu_1v_2 \ , \tag{2d}$$

where $I=u_1+u_2$, while $f_1^2=4I_1I_2$ and $f_2^2=4I_3I_4$ obey

$$If_1'=\gamma u_1 f_1 \ , \tag{3a}$$

$$If_2'=2g(u_1f_2+f_1u_2) \ . \tag{3b}$$

Equations (2b) and (3a) are easily integrated in terms of v_2,

$$v_1=v_2+\Delta, \quad f_1=f_{1d}\left[\frac{v_2}{v_{2d}}\right]^{\gamma/2g} \ , \tag{4}$$

where $\Delta=v_{1d}-v_{2d}$ is a constant evaluated at $z=d$, d being the thickness of the crystal. Explicit knowledge of v_1 and f_1 allows evaluation of $u_1=(f_1^2+v_1^2)^{1/2}$. It remains to find u_2 and f_2 in terms of v_2, and then to solve an equation for v_2. This is most easily accomplished by introducing a new variable w:

$$u_2=v_2\cosh w, \quad f_2=v_2\sinh w \ , \tag{5}$$

and by rescaling all variables with respect to v_{2d}; $u=u_1/v_{2d}$, $f=f_1/v_{2d}$, $v=v_2/v_{2d}$. The equations to be solved become

$$iv'=2guv, \quad iw'=2gf \ , \tag{6}$$

where now $i=u+v\cosh w$, $u(v)=[(v+\delta)^2+a^2v^b]^{1/2}$, $f(v)=av^{b/2}$. Parameters $\delta=\Delta/v_{2d}$ and $a=f_{1d}/v_{2d}$, as will become apparent in a moment, are crucial in our solution procedure. Here $b=\gamma/g$.

The system of equations (6) presents an initial-value problem. The values of both unknown variables are known on the $z=d$ face of the crystal: $v_d=1$, $w_d=0$. In this manner the boundary-value nature of the problem is transferred to the parameter space. In addition, this system is integrable. The solution is given in terms of quadratures:

$$\ln v(z)+\int_1^v \frac{\cosh w(x)}{u(x)}dx=2g(z-d) \ , \tag{7a}$$

$$w(v)=\int_1^v \frac{f(x)}{xu(x)}dx \ , \tag{7b}$$

but the integrals indicated cannot be evaluated in closed form for arbitrary b. Their numerical evaluation or tabulation, however, entails little difficulty. Even less troublesome numerically is to integrate Eqs. (6) directly on a computer, as an initial-value problem. In any case there remains the problem of boundary values.

Let us denote the given boundary values of intensities by $I_{10}=C_1$ and $I_{40}=C_4$ on the $z=0$ face of the crystal, and by $I_{2d}=C_2$ and $I_{3d}=0$ on the $z=d$ face. Parameters a and δ are connected with these via

$$a=\frac{2(C_2I_{1d})^{1/2}}{I_{4d}}, \quad \delta=\frac{C_2-I_{1d}}{I_{4d}}-1 \ , \tag{8}$$

where I_{1d} and $I_{4d}=v_{2d}$ are the missing boundary values at $z=d$. The same parameters can also be connected with the missing values I_{20} and I_{30} (or, equivalently, with v_0 and w_0) at $z=0$:

$$\begin{aligned} a^2&=4C_1C_2x\frac{x(C_1+C_2)+v_0-1}{C_1+C_2v_0^b} \ , \\ \delta&=\frac{x(C_2^2v_0^b-C_1^2)-C_1v_0-C_2v_0^b}{C_1+C_2v_0^b} \ . \end{aligned} \tag{9}$$

Here x denotes the inverse of v_{2d}, which also equals $v_0(\cosh w_0+1)/2C_4$.

The problem of boundary values consists in that a and δ are given in terms of v_0 and w_0, and these can only be evaluated after the correct values of a and δ are supplied to Eqs. (6) or (7). Such problems, however, are conveniently addressed by iteration procedures. Starting with some arbitrary initial values $a^{(0)}$ and $\delta^{(0)}$, these are substituted into Eqs. (6) or (7), the equations are integrated, and the values of v_0 and w_0 found. From these the new values of $a^{(1)}$ and $\delta^{(1)}$ are calculated, and the procedure is repeated until the desired accuracy is achieved. In this manner an iterative map in the plane is defined, and the procedure actually presents an evaluation of the fixed points of the map. An interesting question to be asked is whether the map can become unstable, and what happens when it becomes unstable. This question will be addressed in the remainder.

There are four relevant control parameters in the problem: the wave-mixing coefficients g and γ (actually gd and γd, but we keep $d=1$ throughout), and the boundary values C_2 and C_4 (the other pump is used as the intensity unit, and C_1 is fixed at one). We report here only on the case $C_2>C_1>C_4$, but other combinations are also allowed.

We find the procedure to be stable for $g<0$, and for arbitrary other parameters. For $g > 0$ it is stable up to about 2, and then it loses stability. The point where it actually becomes unstable, as well as the form of instabilities and their dynamics, depends strongly on the other parameters. An intriguing question is whether this unstable behavior is a true physical phenomenon, or a property of the model only, or merely a numerical instability. We checked by an alternative method (brute-force shooting method) that these instabilities belong to equations, but how real they are can only be resolved by experiment.

Figure 1 depicts the development of instabilities as g is varied, for $\gamma = 3$. Boundary values are chosen as $C_2 = 3$ and $C_4 = 0.6$. It is seen that roughly in between $g \cong 2.38$ and $g \cong 3.12$ a quasiperiodic egg-shaped region is formed, with many commensurate windows visible. The chaos, however, is not reached through quasiperiodicity, and at the end of the interval a unique, period 1 solution is recovered. This fixed point starts to bifurcate at about $g \cong 3.22$, and after a cascade of pitchfork bifurcations, chaos is reached at $g \cong 3.39$. Thus, through a repeated loss of stability of the fixed points of various powers of the map, an aperiodic state is reached where no stable reflectivity exists.

A different scenario is observed for γ positive. In Fig. 2 the development of the reflectivity is followed for $\gamma = 3$. Now no quasiperiodic region appears, and the system proceeds to chaos via period doubling starting from $g \cong 3.56$. A four-piece strange attractor is formed at $g \cong 3.572$, and at $g \cong 3.573$, through an interior crisis, a one-piece attractor emerges. The chaotic region ends when the system enters an unphysical region of negative a^2.

Figure 3 presents the attractor in the a-δ phase plane, for $g = 3.505$, $\gamma = -1$, $C_2 = 3$, and $C_4 = 0.6$. We measured its correlation dimension using embedding techniques, and found it to be $D_2 \cong 1.11$. The development in the a-δ plane starts with a single fixed point, which bifurcates until a four-piece attractor is reached. Due to the implicit and noninvertible nature of the map, it is difficult to draw stable and unstable manifolds of different fixed points. It should be noted that it makes little difference whether one looks at the chaos in variables a and δ, or I_{1d} and I_{4d}, or I_{20} and I_{30}. The same qualitative behavior is observed.

Generally, the phase diagram of the system shows that the system is becoming more unstable as g and $|\gamma|$ are increasing. Moreover, in the region of large g ("strong-coupling limit") sooner or later a boundary is approached, where unphysical solutions appear (with negative intensities). Chaos is found in the band that separates periodic from unphysical solutions.

However, there exists a special set of unique solutions for $\gamma = 0$ and arbitrarily large g, which is submerged into a sea of unstable solutions. They require a special set of numbers for boundary values. Choosing, say, C_2 to be a

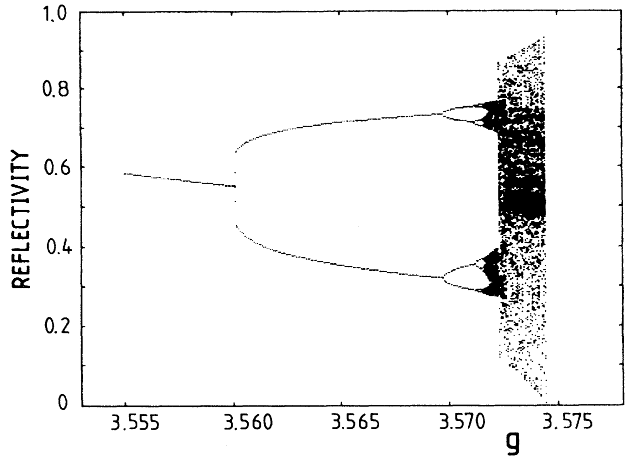

FIG. 2. The same as Fig. 1, but for $\gamma = 3$.

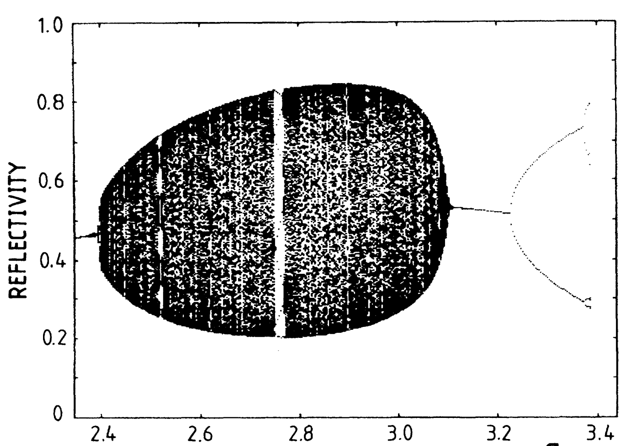

FIG. 1. Bifurcation diagram of the intensity reflectivity for $\gamma = -3$, $C_2 = 3$, $C_4 = 0.6$.

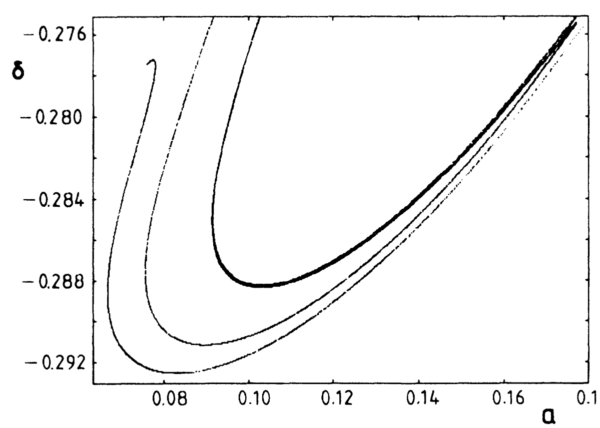

FIG. 3. Strange attractor in the plane of a-δ parameters. Its correlation dimension is $D_2 \cong 1.112$, and the parameters are $g = 3.505$, $\gamma = -1$, $C_4 = 0.6$, and $C_2 = 3$. A very similar attractor is seen in the first return map (R_n, R_{n+1}) of the reflectivity.

rational number n/m ($n>m$), and if C_4 is set equal to $(n-m)/(n+m)$, then the attractor falls onto a rational fixed point $a=2nm/(n^2-m^2)$, $\delta=0$ in the phase plane. If the prescribed boundary values are missed even slightly, the system goes to chaos, or it becomes unphysical. For these solutions $I_1+I_4=I_2+I_3$, i.e., power flux to the left is balanced by the flux to the right. A more complete account of these and other findings will be published elsewhere.

In summary, we have presented an alternative integration scheme for treatment of two-point boundary-value problems, in which the boundary-value problem is transformed into an initial-value problem, and the fitting of boundary conditions is transferred to the parameter space. An iterative mapping is formed in the parameter plane, and its fixed points, and the fixed points of its various composition powers are analyzed. It is found that for g negative there are no instabilities—the iterative procedure rapidly converges to stable fixed points. For g positive and increasing, sooner or later, the instabilities set in, and the system proceeds to chaos following the Feigenbaum period-doubling scenario. A strange attractor is discovered in the intensity reflectivity of the crystal, with the correlation dimension between 1 and 2.

In the end, it should be pointed out that instabilities obtained in this manner do not imply with certainty the existence of experimental instabilities. In the absence of a dynamical picture of the process, it can only be established that for a given set of control parameters, and a given set of boundary values, the original wave equations allow for multiple solutions, possibilly even infinitely many such solutions. The reality of these solutions, however, can only be ascertained by experimental verification.

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