Fabry-Pérot resonators with oscillating mirrors

S. Solimeno, F. Barone, C. de Lisio, L. Di Fiore, L. Milano, and G. Russo
Dipartimento Scienze Fisiche, Università degli Studi di Napoli, Mostra d'Oltremare, Padiglione 19, I-80125 Napoli, Italy
(Received 28 January 1991)

In this paper we derive analytic expressions of the field on the mirror surfaces of a pendular and misaligned Fabry-Pérot resonator by taking into account longitudinal and transverse optical modes. In particular, we obtain analytic expressions of the spectral components of the power reflected by these devices for oscillating mirrors and modulated laser beams. In addition, we discuss the equations of motion of the mirror holders and analyze the onset of instabilities induced by the radiation pressure, by accounting for transverse optical modes and torsional oscillations of the multiple pendula to which the mirrors are suspended.

I. INTRODUCTION

Fabry-Pérot resonators are playing a major role in long baseline interferometers used for detecting gravitational waves (GW). Following the idea first illustrated in a paper by Gertsenshtein and Pustovoit[1] in 1963 and analyzed by Weiss[2] in 1972, Drever[3] built in 1980 a GW detector by placing two Fabry-Pérot (FP) cavities in the arms of a Michelson interferometer. Presently, two prototype Fabry-Pérot interferometers are working at the University of Glasgow, Scotland and at California Institute of Technology, Pasadena, California. In order to reduce the seismic noise, the mirrors of projected GW antennas[4] will be fixed to suitably suspended masses. Consequently, typical features of these detectors will be lengths of about 3 km. and mirrors attached to mechanical pendula.[4]

The mirrors of these long FP resonators must be kept aligned to high precision with the laser beam in order to achieve a projected accuracy of 10^{-21} in the measure of the relative variation of the distance of the test masses. Proper alignment of the input laser beam means that it couples completely to the fundamental spatial mode of the cavity and not at all to the higher-order ones. Transverse displacement and mismatch of the beam with respect to the cavity axis and waist size give rise to in-phase coupling to the first- and second higher-order modes of the cavity.[5] On the other hand, angular misalignment and waist translations couple these modes in quadrature. All these effects reduce the detection sensitivity of GW signals.

The solution to the alignment problem of FP cavities is provided by servo systems using alignment and matching error signals obtained by modulating the laser beam and/or the mirror positions at suitable frequencies and monitoring with coherent detection techniques the intensity of the reflected and/or transmitted beams.[6]

A systematic analysis of the excitation of the cavity modes can be carried out by representing the field inside the cavity and that reflected from it as a combination of Gauss-Hermite modes.[7] The beams incident on and reflected from or transmitted through the FP cavity are represented by vectors (**E**), whose components give the amplitudes of the above modes. In this framework, the FP cavity is represented by the matrices $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$, which transform the incident vector (**e**) into the vectors $(\mathbf{E}_1^{(-)},\mathbf{E}_2^{(+)})$ relative to the beams incident on mirrors M_1 and M_2 respectively. The components of $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ can be calculated by means of scalar diffraction theory, which takes into account the deviation of the mirror surfaces from the ideal profiles, their finite sizes, and misalignments.

The radiation pressure in a Fabry-Pérot cavity provides a spring action, observed experimentally by Dorsel *et al.*[8] and discussed by Meystre *et al.*,[9] which either acts against or reinforces any perturbation. A motion of the mirrors produces not only a phase change on the light emerging from it, but also an intensity change inside the cavity. The resultant change in radiation pressure will act back on the mirrors. Braginsky and Manukin first pointed out that the radiation pressure in a cavity that is not perfectly resonant will provide a spring action that acts against any perturbation, while the changing part of it tends to destabilize the system. Instability will result if this dominates the damping effect of the mirror suspension.

The analysis of the dynamical consequences of radiation-pressure changes has been carried out by Tourrenc, Aguirregabiria, Deruelle, Bel, and Boulanger[10–13] by considering a simple model of cavity with plane mirrors. They have shown that there exists a threshold power of the laser beam above which the cavity becomes unstable. In particular, Bel *et al.*[12,13] have associated a nonlinear differential equation to the retarded system, whose solution approximates asymptotically the exact one. More recently, Meers and MacDonald[14] have analyzed this problem by taking into account the stabilizing effects of the electronic system controlling the mirror position.

All these authors have treated the cavity modes as simple plane waves. Now, the question arises, to what extent do their analyses remain valid for cavities with spherical mirrors? The present paper addresses this problem by considering both longitudinal and tilting displacements of the mirror holders. For the sake of simplicity we consider only the misalignment of the FP input mirror M_1.

When the mirrors are tilted, a mechanical momentum arises as a result of the asymmetry of the radiation pressure. A rotation of the mirrors produces a change in the radiation-pressure momentum which may tend to destabilize the system. In this paper we analyze this type of instability by establishing a relation among the moment of inertia of the mass of the mirror holder, the natural period of the torsional oscillation, its damping times, and the power threshold for torsional oscillation.

The paper is organized in eight section. In Sec. II we discuss the response of a FP resonator in terms of Gauss-Laguerre modes. In particular, we show that the matrix $\mathbf{\Sigma}_1$ can be expressed as a product $\mathbf{\Sigma}^{(0)}$, representing the response of a perfectly aligned cavity, by the alignment matrix $\mathbf{A}$, which describes the mirror misalignment (tilting and axial displacement). In Sec. III we show that in case of mirror tilting, $\mathbf{A}$ can be represented by means of the displacement operator used in quantum mechanics for describing coherent states of harmonic oscillators. Section IV is dedicated to the evaluation of the alignment matrix for mirrors undergoing torsional and/or axial oscillations with generally complex frequencies $\tilde{\omega}$. The series representing the matrix $\mathbf{\Sigma}_1$ relative to longitudinal oscillations is summed up by introducing an operator

$$O(\Lambda)=[R(\Lambda)-\Lambda]/(\Lambda-1)$$

obtained from the scaling operator R $[R(\Lambda)f(x)=f(\Lambda x)]$. The parameter $\Lambda=\exp(-i\tilde{\omega}r)$ represents the dephasing of a signal at frequency $\tilde{\omega}$ during the round trip r of a photon in cavity. Analogously, the series relative to tilting oscillations is also summed up by introducing a displacement operator whose argument is proportional to $O(\Lambda e^{i2\phi})$. The presence of the factor $e^{i2\phi}$ marks the difference with the longitudinal oscillations. The quantity 2ϕ represents the dephasing undergone by the fundamental mode during a round-trip, with respect to a plane wave. Consequently, the tilting effects, contrary to the longitudinal ones, depend on the cavity geometry through the parameter 2ϕ. Section V is dedicated to the discussion of a longitudinal detuning described by a generic function of time. The case in which the cavity is excited by a TEM_{00} Gaussian mode is discussed in Sec. VI. In Sec. VII we consider different alignment errors obtained by either imposing mechanical oscillations on the mirrors or modulating the laser beam at some suitable frequencies. The radiation pressure and the relative momentum with respect to the suspension axis of the mirrors are analyzed in Sec. VIII. Section IX is dedicated to the analysis of the torsional and pendular oscillations. The work is completed by four appendices dedicated, respectively, to the properties of the operator $O(\lambda)$ (Appendix A), to the extension of the so-called Jacobi formula to an exponential function of $O(\lambda)$ (Appendix B), to the representation of $\exp[\beta R(\lambda)]\Sigma^{(0)}$ by means of degenerate hypergeometric function (Appendix C), and, finally, to the pupil of an element of an array detector (Appendix D).

II. RESPONSE OF A FP RESONATOR TO AN EXTERNAL LASER BEAM

Let us consider a linear FP resonator consisting of two concave mirrors M_1 and M_2 with curvature radii R_1 and R_2, separated by a distance d much greater than the mirror dimensions (see Fig. 1). The linewidth of the laser beam used in GW detectors is so small that we may assume the cavity to be excited by a monochromatic field of frequency ω, entering the resonator through the backside of mirror M_1. In addition, we assume that M_1 and M_2 move very slowly with respect to ω^{-1}. Under the latter assumption, the field inside the cavity can be represented by the superposition of two opposite traveling waves, whose complex amplitudes are denoted by $u^{(+)}(\mathbf{r},t)e^{i\omega t}$ and $u^{(-)}(\mathbf{r},t)e^{i\omega t}$, respectively, for those traveling forward and backward from mirror M_1 to M_2 and vice versa.

Solving Maxwell equations for a field confined between moving boundaries is a formidable task, which admits exact analytic solutions only in a few cases. Approximate solutions can be found when the walls change their positions by much less than a wavelength during the photon lifetime. In this case recourse can be made to the slowly varying approximation (SVA). If we indicate by Π_1 and Π_2 the surfaces of M_1 and M_2 located at the respective reference positions, in the SVA the fields relative to Π_1 and Π_2 are represented, respectively, in the form

$$u_{1,2}^{(\pm)}(\mathbf{r}_t,t)e^{i\omega t}\equiv u^{\pm}(\mathbf{r}_{1,2},t)e^{i\omega t}$$

($\mathbf{r}_t$ being the transverse ray vector), where $u^{(\pm)}$ are slowly varying functions of time, such that we neglect their

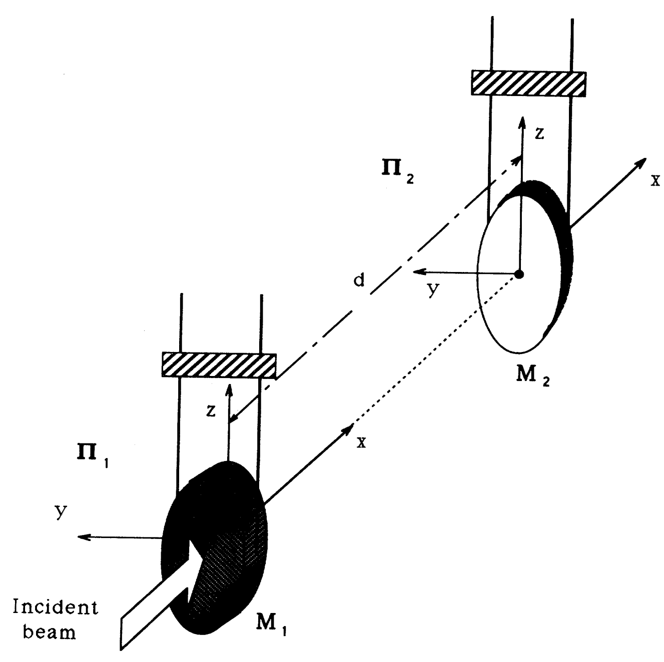

FIG. 1. Schematic of a Fabry-Pérot interferometer with the mirrors supported by suitable multiple pendula for attenuating the seismic noise.

second time derivatives, when needed. Under this assumption, the $u^{(\pm)}$ are related to the input $u_{\rm in}(\mathbf{r}_1)e^{i\omega t}$ (relative to Π_1) by the string of equations[7]

$$u_1^{(+)}(t)=r_1 e^{2ikW_1(t)}u_1^{(-)}(t)e^{i2k\delta x_1(t)}+t_1 e^{-ikT_1}u_{\rm in} ,$$
$$u_2^{(-)}(t)=r_2 e^{i2kW_2(t)-i2k\delta x_2(t)}u_2^{(+)}(t) , \quad (1)$$

being $k=\omega/c$, together with the Fox-Li integral equations

$$u_1^{(-)}(t)=e^{-ikd+ik\delta x^{(1)}(t)}\Delta(r/2)e^{-ik\delta x^{(2)}(t)} \times \int u_2^{(-)}(t)K_{12}dS_2' ,$$
$$u_2^{(+)}(t)=e^{-ikd-ik\delta x^{(2)}(t)}\Delta(r/2)e^{ik\delta x^{(1)}(t)} \times \int u_1^{(+)}(t)K_{21}dS_1' , \quad (2)$$

having omitted the dependence of the fields and the aberration functions on the transverse coordinates y and z. $\delta x^{(1,2)}$ represent the axial displacements of $M_{1,2}$ from $\Pi_{1,2}$. The aberration functions $W_{1,2}$ of $M_{1,2}$ measure the deviations of the surfaces of $M_{1,2}$ from those of perfectly aligned mirrors (see Fig. 2). $\exp(-ikT_1)$ represents the dephasing undergone by the input beam in passing through the mirror 1. t_1,r_1,r_2 stand for the generally complex transmission and reflection coefficients of M_1 and M_2. $\Delta(r/2)$ is a shift operator which transforms $f(t)$ into the time-delayed function $f(t-r/2)$, while $r=2d/c$ is the round-trip time interval. In particular,

$$u_{\rm in}=\frac{1}{\sqrt{2Z_0\mathcal{P}}}\mathcal{E}_{\rm in}e^{ikW_1} , \quad (3)$$

$\mathcal{E}_{\rm in}$ representing the electric field of the input beam measured on the front surface of a perfectly aligned and aberration-free mirror M_1. $\mathcal{P}$ and Z_0 stand for instantaneous intensity (in watts) of the laser beam and the free-space impedance ($120\pi\ \Omega$). $\sqrt{2\mathcal{P}Z_0}$ is a normalization factor such that the power carried by $u_{\rm in}$ is equal to unity. The integrals on the right side of (2) are extended to the whole plane (yz). As a consequence of the simple form of these integrals, we pay the penalty of neglecting the diffraction effects due to the finite mirror sizes.

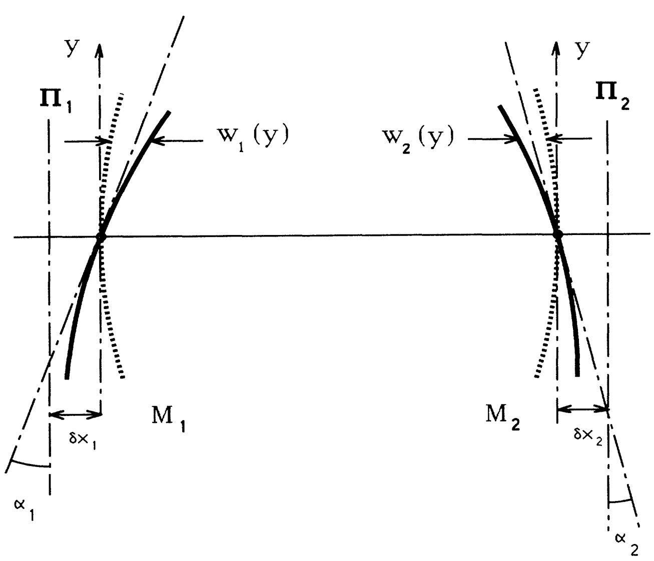

FIG. 2. Geometry of the mirrors. The field traveling from mirror 1 to mirror 2 is indicated by $u^{(+)}$, while $u^{(-)}$ represents the wave propagating in the opposite direction.

The kernel K_{12} of the system of Fox-Li integral equations (2) is given by

$$K_{12}(y_1,z_1;y_2,z_2)=K(y_1,y_2)K(z_1,z_2) , \quad (4)$$

with

$$K(u,v)=\left[\frac{i}{\lambda d}\right]^{1/2}\exp\left[-i\frac{k}{2d}(g_1u^2+g_2v^2-2uv)\right] , \quad (5)$$

$g_i=1-d/R_i$ being the so-called g parameters of the cavity.

Before discussing the significance and utility of the above vector relations, we comment on their generality. They have been derived by using the SVA. Accordingly, they hold true when the coherence time of the laser field is much longer than the photon lifetime τ_p in the cavity. In Eq. (2) the assumption that the bandwidth B of the field $u^{(\pm)}$ is so small that the error introduced by using a propagator relative to a perfectly monochromatic field is negligible is implicit. This amounts to assuming $Bd/c\leq 1$. We can estimate B by observing that a wave of frequency ω undergoes a Doppler shift $\delta\omega=\omega v/c$ when it is reflected by a wall moving at speed v. Since a wave bounces the FP mirrors back and forth a number of times equal to the finesse $\mathcal{F}$ of the resonator, then B is approximately given by $B=\omega v\mathcal{F}/c$. Accordingly, the SVA holds true subject to the inequality

$$r\mathcal{F}kv=\tau_p kv \ll \pi , \quad (6)$$

$\tau_p=r\mathcal{F}$ being the photon lifetime. In particular, for a sinusoidal motion of the walls of amplitude δx and frequency $\tilde{\omega}$, the SVA can be used only if the mirror displacement during an interval τ_p is much smaller than a wavelength.

A. Modal representation of the field

A simple way to solve the system (1) and (2) makes use of the field representation in terms of the orthonormal functions u_{lm},

$$u_{lm}(y,z)=\frac{1}{w}\left[\frac{1}{\pi 2^{l+m-1}l!m!}\right]^{1/2}H_l\left[\frac{\sqrt{2}y}{w}\right] \times H_m\left[\frac{\sqrt{2}z}{w}\right]\exp\left[-\frac{y^2+z^2}{w^2}\right] \equiv u_l(y)u_m(z) , \quad (7)$$

with $H_{l,m}$ the Hermite polynomial, and w a generic parameter. Accordingly, we represent the fields relative to M_1 and M_2 as superpositions of modes $u_{lm}^{(1)},u_{lm}^{(2)}$ obtained by setting w equal to the spot sizes relative to the two mirrors respectively.[7] To this end, we introduce the vectors $\mathbf{E}$ defined by the series expansions

$$u_1^{(+)}=\sum_{l,m=0}^{\infty} E_1^{(+)lm}u_{lm}^{(1)} , \quad (8)$$

$$u_2^{(+)}=\sum_{l,m=0}^{\infty} E_2^{(+)lm}u_{lm}^{(2)} , \quad (9)$$

$$u_{\text{in}}=\sum_{l,m=0}^{\infty} e^{lm}u_{lm}^{(1)} , \quad (10)$$

and the matrices $\mathbf{U}_1,\mathbf{U}_2,\mathbf{T}_1$,

$$e^{i2kW_1}u_{lm}^{(1)}=\sum_{l',m'} U_{1,lm}^{l'm'}u_{l'm'}^{(1)} ,$$
$$e^{i2kW_2}u_{lm}^{(2)}=\sum_{l',m'} U_{2,lm}^{l'm'}u_{l'm'}^{(2)} , \quad (11)$$
$$e^{-ikT_1}u_{lm}^{(1)}=\sum_{l',m'} T_{1,lm}^{l'm'}u_{l'm'}^{(1)} .$$

Now, exploiting the integral identities

$$u_{lm}^{(1)}=e^{i(l+m+1)\phi}\int u_{lm}^{(2)}K_{12}dS_2 ,$$
$$u_{lm}^{(2)}=e^{i(l+m+1)\phi}\int u_{lm}^{(1)}K_{21}dS_1 , \quad (12)$$

where

$$\phi=\cos^{-1}\sqrt{g_1g_2} , \quad (13)$$

we recast the system (1) and (2) in the matrix form

$$\mathbf{E}_1^{(+)}(t;\omega)=r_1\mathbf{U}_1(t;\omega)\cdot\mathbf{E}_1^{(-)}(t;\omega)+t_1\mathbf{T}_1(\omega)\cdot\mathbf{e} ,$$
$$\mathbf{E}_2^{(-)}(t;\omega)=r_2\mathbf{U}_2(t;\omega)\cdot\mathbf{E}_2^{(+)}(t;\omega) ,$$
$$\mathbf{E}_1^{(-)}(t;\omega)=\frac{1}{\sqrt{r_1r_2}}\mathbf{\Phi}(\omega)e^{ik\delta x^{(1)}(t)}\times\Delta e^{-ik\delta x^{(2)}(t)}\cdot\mathbf{E}_2^{(-)}(t;\omega) ,$$
$$\mathbf{E}_2^{(+)}(t;\omega)=\frac{1}{\sqrt{r_1r_2}}\mathbf{\Phi}(\omega)e^{-ik\delta x^{(2)}(t)}\times\Delta e^{ik\delta x^{(1)}(t)}\cdot\mathbf{E}_1^{(+)}(t;\omega) , \quad (14)$$

where

$$\Phi_{lm}^{l'm'}(\omega)=\sqrt{|\rho|}e^{i(l+m)\phi+i\chi/2}\delta_{ll'}\delta_{mm'} , \quad (15)$$

while $|\rho|=|r_1r_2|$, the phase of ρ is $\chi=2\phi-\omega r+2\psi$, and the phase of r_1r_2 is 2ψ. In the following we will find it worthwhile to express the matrix $\mathbf{\Phi}^2$ and related quantities in the form

$$\mathbf{\Phi}^2=\mathbf{\Psi}\rho , \quad (16)$$

with $\mathbf{\Psi}$ the diagonal matrix

$$\Psi_{lm}^{l'm'}=R(e^{i2(l+m)\phi})\delta_{ll'}\delta_{mm'}$$

and $R(\lambda)$ the scaling operator transforming a function $f(\rho)$ into $f(\lambda\rho)$.

Next, solving with respect to $\mathbf{E}_1^{(-)}$ yields

$$\mathbf{E}_1^{(-)}(t;\omega)=\frac{t_1}{r_1}\mathbf{\Sigma}_1(t;\omega)\cdot\mathbf{T}_1(\omega)\cdot\mathbf{e} , \quad (17)$$

where

$$\mathbf{\Sigma}_1(t;\omega)=\sum_{n=1}^{\infty}\mathbf{B}_1^n(t;\omega)\cdot\mathbf{U}_1^*(t;\omega) ,$$
$$\mathbf{B}_1(t;\omega)=\mathbf{\Phi}(\omega)e^{ik\delta x^{(1)}(t)}\Delta e^{-i2k\delta x^{(2)}(t)}\cdot\mathbf{U}_2(t;\omega)\cdot\mathbf{\Phi}(\omega)\Delta e^{ik\delta x^{(1)}(t)}\cdot\mathbf{U}_1(t;\omega) . \quad (18)$$

On the other hand, it is easy to show that the field $\mathbf{E}_1$ reflected from the entrance mirror M_1 is given by

$$\mathbf{E}_1(t;\omega)=\frac{t_1^2}{r_1}\mathbf{T}_1(\omega)\cdot\left[\mathbf{\Sigma}_1(t;\omega)+\frac{r_1^2}{t_1^2}e^{-i2k\delta x^{(1)}(t)}\right]\cdot\mathbf{T}_1(\omega)\cdot\mathbf{e} . \quad (19)$$

Similar expressions can be derived from (14) for $\mathbf{E}_2^{(+)}$ and $\mathbf{\Sigma}_2$.

$\mathbf{\Sigma}_1(t;\omega)$ depends parametrically on the frequency of the incident beam and on time due to the presence in (18b) of the phase factors $\exp[ik\delta x^{(1,2)}(t)]$ and the matrices $\mathbf{U}_{1,2}$, which represent generally time-dependent mirror misalignments. When the mirrors are at rest, the delay effects disappear and Δ reduces to unity.

In general, $\mathbf{\Sigma}_1$ satisfies the delay equation

$$\mathbf{\Sigma}_1(r)\cdot\mathbf{U}_1(t)=\mathcal{B}(t)[1+\mathbf{\Sigma}_1(t-r)\cdot\mathbf{U}_1(t-r)] , \quad (20)$$

where

$$\mathcal{B}(t)=e^{ik[\delta x^{(1)}(t)-2\delta x^{(2)}(t-r/2)+\delta x^{(1)}(t-r)]}\mathbf{\Phi}\cdot\mathbf{U}_2(t-r/2)\cdot\mathbf{\Phi}\cdot\mathbf{U}_1(t-r) .$$

For a perfectly aligned cavity ($\mathbf{U}_{1,2}=1$, $\Delta=1$, $\delta x^{(1,2)}=0$), (18a) reduces to

$$\mathbf{\Sigma}_1=\sum_{n=1}^{\infty}\mathbf{\Phi}^2=(1-\mathbf{\Phi}^2)^{-1}\cdot\mathbf{\Phi}^2\equiv\mathbf{\Sigma}^{(0)} . \quad (21)$$

In some cases it is worth expressing $\mathbf{\Sigma}^{(0)}$ by means of either the matrix $\mathbf{\Psi}$ or the projectors $\mathbf{P}_{lm}$ over the modes lm,

$$\mathbf{\Sigma}^{(0)}=\mathbf{\Psi}\Sigma^{(0)}=-\frac{i}{r}\sum_{lm}\mathbf{P}_{lm}\sum_{n=-\infty}^{\infty}\frac{1}{\omega-\omega_{nlm}^{(0)}} , \quad (22)$$

where $\Sigma^{(0)}=\rho/(1-\rho)$ and [see (15)]

$$r\omega_{nlm}^{(0)}=i\frac{\pi}{\mathcal{F}}\sqrt{|r_1r_2|}+2\pi n+2(l+m+1)\phi+2\psi \; , \tag{23}$$

with

$$\mathcal{F}=\pi\sqrt{|r_1r_2|}/(1-|r_1r_2|)$$

the finesse of the resonator (see Ref. 15, p. 172).

III. ABERRATION MATRIX U FOR MIRROR TILTING

In the paraxial optics limit a generic mirror misalignment can be decomposed in the superposition of a rotation by an angle α around an axis tangent to the mirror vertex and a displacement δx along the optic axis of the cavity. For example, a rotation by an angle β around an axis crossing the optic axis at a distance l from the mirror vertex can be decomposed into a rotation by the angle $\alpha=(1+l/R)\beta$ plus a displacement $\delta x=(l/R-1)l\beta^2/2$, R being the curvature radius of the mirror. Similarly, a transverse displacement d_t along a horizontal direction perpendicular to the optic axis can be obtained by superimposing a rotation around the vertical axis by the angle $\alpha=d_t/R$ and a longitudinal displacement $\delta x=d_t^2/2R$.

In view of the above property we will limit ourselves to considering the contribution to the aberration function due to the rotation around an axis tangent to the mirror. To this end, we introduce a coordinate system with the x axis coincident with the optic axis of the cavity and pointing from M_1 toward M_2. The z axis is upward oriented parallel to the suspension axis, while the y axis is horizontal.

With the above choice of coordinates, for a rotation by an angle α around a vertical axis ($\alpha>0$ corresponds to a counterclockwise rotation for an observer looking at the mirror from the top), the aberration function is given by $W(y,z,t)=\alpha(t)y$. If we consider the Hermite-Gauss modes (7) as the eigenfunctions of a harmonic oscillator, we can conveniently represent y as the sum of the creation and annihilation operators $a^\dagger$ and a, respectively, whose matrix elements are given by

$$\begin{aligned}(a^\dagger)_l^{l'}&=\int_{-\infty}^{\infty}u_{l'}^{(1)}(y)a^\dagger u_l^{(1)}(y)dy=\sqrt{l+1}\delta_{l',l+1}\; ,\\ a_l^{l'}&=\sqrt{l}\,\delta_{l',l-1}\; .\end{aligned} \tag{24}$$

Accordingly, we put

$$W(y,z,t)=\alpha(t)y=w\alpha(t)y/w=w\alpha(t)(a+a^\dagger)/2\; .$$

Eventually, for a rotation of M_1 the matrix $\mathbf{U}_1$ reads

$$\mathbf{U}_1=e^{-i\epsilon(\mathbf{a}+\mathbf{a}^\dagger)}=\mathbf{1}-i\epsilon(\mathbf{a}+\mathbf{a}^\dagger)-\frac{\epsilon^2}{2}(\mathbf{a}+\mathbf{a}^\dagger)^2+\cdots\; , \tag{25}$$

where

$$\epsilon(t)\equiv\alpha^{(1)}(t)kw^{(1)}\; . \tag{26}$$

To be consistent with the SVA, ϵ must satisfy the condition $|\epsilon|\tilde{\omega}r\mathcal{F}\ll1$. For typical values of r (20 μsec), $\mathcal{F}$ (30), and $\tilde{\omega}$ (<10 kHz), the SVA holds true for $|\epsilon|\ll0.2$. In the interferometers used as GW antennas, α is in general less than 6×10^{-8} (see Ref. 4). Consequently, $|\epsilon|$ is smaller than 10^{-2}, a value consistent with the SVA.

If we consider the modes u_l as the eigenfunctions $|l\rangle$ of a harmonic oscillator, the matrix $\mathbf{U}_1$ will represent the displacement operator $D(-i\epsilon)$ introduced by Glauber[16] for generating a coherent state of generally complex amplitude α starting from the ground state $|0\rangle$

$$\mathbf{U}_1=D(-i\epsilon)\; . \tag{27}$$

In general, $D(\alpha)$ is defined as $D(\alpha)=e^{\alpha\mathbf{a}^\dagger-\alpha^*\mathbf{a}}$. When applied to $|0\rangle$, it gives

$$D(\alpha)|0\rangle\equiv|\alpha\rangle=e^{-(1/2)|\alpha|^2}\sum_n\frac{\alpha^n}{\sqrt{n!}}|n\rangle\; . \tag{28}$$

For typical operating conditions ϵ is so small that $D(-i\epsilon)$ coincides with $1-i\epsilon(a^\dagger+a)$. However, for the sake of generality we will develop the theory for arbitrary values of ϵ.

IV. ALIGNMENT MATRIX A

The most general mirror motion can be represented as a superposition of oscillations at different frequencies. In particular, the damping or the amplification of these perturbations are accounted for by adding imaginary components to these frequencies. In the following, we will show that $\mathbf{\Sigma}_1$ factorizes into the product $\mathbf{A}\cdot\mathbf{\Sigma}^{(0)}$ of the alignment matrix $\mathbf{A}$ by $\mathbf{\Sigma}^{(0)}$ (relative to the perfectly aligned cavity). Accordingly, all the information about the cavity misalignment (longitudinal detuning and tilting) is contained in $\mathbf{A}$. In addition, it turns out that $\mathbf{A}$ can be represented as the product $\mathbf{A}_t A_l$ of the matrix operator $\mathbf{A}_t$ relative to the tilting by the scalar operator A_l representing the longitudinal detuning. The combination of tilting and detuning at several frequencies leads to an alignment matrix represented by a suitable product of the matrix and scalar operators relative to the single frequencies.

A. Tilting oscillations

When mirror 1 undergoes tilting oscillations around the vertical axis, $\delta x^{(1)}=\delta x^{(2)}=W_2=0$, $\mathbf{U}_2=1$, and in view of (24), (18a) reduces to

$$\mathbf{\Sigma}_1=\sum_{n=1}^{\infty}[\mathbf{\Phi}^2\Delta(r)D(-i\epsilon)]^nD(i\epsilon)\; . \tag{29}$$

Since

$$\mathbf{\Phi}\cdot\mathbf{a}^\dagger=e^{i\phi}\mathbf{a}^\dagger\cdot\mathbf{\Phi},\quad \mathbf{\Phi}\cdot\mathbf{a}=e^{-i\phi}\mathbf{a}\cdot\mathbf{\Phi}\; , \tag{30}$$

we can displace each factor $\mathbf{\Phi}^2$, occurring in (27), to the far right by multiplying the argument of each operator D on its right by the phase factor $e^{i2\phi}$, thus getting

$$\mathbf{B}^n\cdot\mathbf{U}_1^*=D(\alpha_1)D(\alpha_2)\cdots D(\alpha_{n-1})\mathbf{\Phi}^{2n}\; , \tag{31}$$

where

$$\begin{aligned}\alpha_q(t)&=-i\epsilon_q(t)e^{i2q\phi}\; ,\\ \epsilon_q(t)&=\Delta^{2q}\epsilon(t)=\epsilon(t-qr)\; .\end{aligned}$$

Since the commutator $[a,a^\dagger]=1$ is a c number, applying iteratively the Baker-Hausdorff theorem yields

$$D(\alpha_1)D(\alpha_2)\cdots D(\alpha_{n-1})=D(\alpha_1+\alpha_2+\cdots+\alpha_{n-1})\exp\left[\tfrac{1}{2}\sum_{\substack{i,j\\ j>i}}\alpha_i\alpha_j^*-\text{c.c.}\right]$$

$$=\exp\left[\sum_i \alpha_i a^\dagger\right]\exp\left[-\sum_i \alpha_i^* a\right]\exp\left[-\tfrac{1}{2}\left|\sum_i \alpha_i\right|^2+\tfrac{1}{2}\sum_{\substack{i,j\\ j>i}}(\alpha_i\alpha_j^*-\alpha_i^*\alpha_j)\right] . \tag{32}$$

In particular, when $\epsilon\propto e^{i\bar{\omega}t}$, Δ^2, and ϵ_q reduce, respectively, to $\Lambda=e^{-i\bar{\omega}r}$ and $\Lambda^q\epsilon$, and

$$\sum_{i=1}^{n-1}\alpha_i=-i\epsilon O_n(\Lambda_+) ,$$

$$\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\alpha_i\alpha_j^*=\epsilon^2\frac{O_n(\Lambda_+\Lambda_-)-\Lambda_+O_n(\Lambda_-)}{\Lambda_+-1} ,$$

$$\sum_{\substack{i,j\\ j>i}}(\alpha_i\alpha_j^*-\text{c.c})-\left|\sum_i\alpha_i\right|^2=\sum_i|\alpha_i|^2-2\sum_{\substack{i,j\\ i\ge j}}\alpha_i\alpha_j^*=\epsilon^2\left[-\frac{\Lambda_+\Lambda_-+\Lambda_+-\Lambda_--1}{(\Lambda_+-1)(\Lambda_--1)}O_n(\Lambda_+\Lambda_-)+2\frac{\Lambda_-}{\Lambda_--1}O_n(\Lambda_+)\right] , \tag{33}$$

where

$$O_n(\lambda)=\frac{\lambda^n-\lambda}{\lambda-1},\quad \Lambda_\pm=\Lambda e^{\pm i2\phi} . \tag{34}$$

We shall adopt throughout this paper the convention of leaving unchanged the functions $e^{i\omega t}$ and $e^{-i\omega r}$ under complex conjugation. Accordingly, we will use the symbol c.c. for indicating the complex conjugate obtained by leaving these quantities unchanged. Finally, by inserting (33) into (32), and relying on (A5), we obtain

$$\mathbf{\Sigma}_1=\sum_{n=1}^{\infty}D(-i\epsilon O_n(\Lambda_+))\exp\left[\frac{\epsilon^2}{2}\left[\frac{O_n(\Lambda_+\Lambda_-)-\Lambda_+O_n(\Lambda_-)}{\Lambda_+-1}-\text{c.c.}\right]\right]\mathbf{\Phi}^{2n}\equiv\mathbf{A}_t\cdot\mathbf{\Sigma}^{(0)}=\mathbf{A}_t\cdot\mathbf{\Psi}\mathbf{\Sigma}^{(0)} , \tag{35}$$

where the tilting contribution $\mathbf{A}_t$ to the alignment matrix can be represented in the form

$$\mathbf{A}_t(t)=D(-i\epsilon(t)O(\Lambda_+))\exp\left[\frac{\epsilon^2(t)}{2}\left[\frac{O(\Lambda^2)-\Lambda_+O(\Lambda_-)}{\Lambda_+-1}-\text{c.c.}\right]\right]$$

$$=\exp[-i\epsilon O(\Lambda_+)\mathbf{a}^\dagger]\exp[-i\epsilon O(\Lambda_-)\mathbf{a}]\exp\left[-\frac{\epsilon^2}{2}\frac{\Lambda_+\Lambda_-+\Lambda_+-\Lambda_--1}{(\Lambda_+-1)(\Lambda_--1)}O(\Lambda^2)+\epsilon^2\frac{\Lambda_-}{\Lambda_--1}O(\Lambda_-)\right] . \tag{36}$$

Now, using the scaling operator $R(\lambda)$, we can split $D(-i\epsilon O(\Lambda_+))$ in the product of two displacement operators, thus representing $\mathbf{A}_t$ as

$$\mathbf{A}_t=D\left[-i\epsilon\frac{R(\Lambda_+)}{\Lambda_+-1}\right]D\left[i\epsilon\frac{\Lambda_+}{\Lambda_+-1}\right]\exp\left[\epsilon^2\frac{O(\Lambda^2)(\Lambda_--\Lambda_+)/2+\Lambda_-R(\Lambda_+)-\Lambda_+R(\Lambda_-)}{(\Lambda_+-1)(\Lambda_--1)}\right] . \tag{37}$$

The above representation of $\mathbf{A}_t$ contain exponential functions of the scaling operator $R(\lambda)$, which can be dealt with by exploiting the frequency representation (22) of $\mathbf{\Sigma}^{(0)}$ together with Eq. (B1) based on the confluent hypergeometric function. In several cases, ϵ is so small that we can expand the displacement operator and the exponentials occurring in the above equations in power series, thus obtaining, with the help of (A3),

$$\mathbf{\Sigma}_1=\{1-i\epsilon[O(\Lambda_+)\mathbf{a}^\dagger+O(\Lambda_-)\mathbf{a}]+O(\epsilon^2)\}\cdot\mathbf{\Sigma}^{(0)}$$

$$=\mathbf{\Psi}\mathbf{\Sigma}^{(0)}(\rho)-i\epsilon[\mathbf{a}^\dagger\cdot\mathbf{\Psi}\mathbf{\Sigma}^{(0)}(\rho)\mathbf{\Sigma}^{(0)}(\rho\Lambda_+)+\mathbf{a}\cdot\mathbf{\Psi}\mathbf{\Sigma}^{(0)}(\rho)\mathbf{\Sigma}^{(0)}(\rho\Lambda_-)]+O(\epsilon^2) . \tag{38}$$

B. Longitudinal oscillations

When $\delta x^{(2)}=W_1=W_2=0$, $\mathbf{B}_1^n$ reduces to

$$\mathbf{B}_1^n=\exp\left[i\frac{\delta\chi}{2}+i\sum_{q=1}^{n-1}\delta\chi_q+i\frac{\delta\chi_n}{2}\right]\mathbf{\Phi}^{2n} , \tag{39}$$

where $\delta\chi=2k\,\delta x^{(1)}(t)$ and

$$\delta\chi_q=2k\,\delta x^{(1)}(t-rq) . \tag{40}$$

In interferometers locked on a given fringe, the detuning $\delta\chi$ is subject to the condition $|\delta\chi| \leq 2\pi/\mathcal{F}$. In view of this, in Ref. 13 the quantity $\delta\chi$ is replaced by $y = \delta\chi\mathcal{F}/2\pi$. In addition, for a detuning oscillating with angular frequency $\tilde{\omega}$, the use of the SVA is conditioned by the inequality $r\tilde{\omega} \ll 1$ [see Eq. (6)].

In particular, for $\delta\chi \propto e^{i\tilde{\omega}t}$, $\mathbf{\Sigma}_1$ reads

$$\mathbf{\Sigma}_1 = \sum_{n=1}^{\infty} \exp\left[i\frac{\delta\chi}{2}(\Lambda+1)[1+O_n(\Lambda)] \right] \mathbf{\Phi}^{2n} \equiv A_l \mathbf{\Sigma}^{(0)} = \mathbf{\Psi} A_l \mathbf{\Sigma}^{(0)} , \tag{41}$$

which is

$$A_l = \exp\left[i\frac{\delta\chi}{2}(\Lambda+1)[1+O(\Lambda)] \right] = \exp\left[-\delta\chi \frac{R(\Lambda)-1}{2\tan(r\tilde{\omega}/2)} \right] . \tag{42}$$

The above equations generalize those obtained in Refs. 10–14 to a cavity with spherical mirrors.

C. Combination of rotation and displacement at several frequencies

When ϵ and $\delta\chi$ correspond to the superposition of oscillations at different frequencies,

$$\epsilon(t) = \sum_{k=1}^{K} \epsilon^{(k)} e^{i\tilde{\omega}(k)t} ,$$

$\delta\chi(t) = \sum_{q=1}^{Q} \delta\chi^{(q)} e^{i\tilde{\omega}(q)t}$, we must replace α_q in (31)–(33) with $\sum_k \alpha_q^{(k)}$. A rather long but straightforward calculation of the alignment matrix yields $\mathbf{A} = \mathbf{A}_t A_l$ with

$$\begin{aligned} \mathbf{A}_t &= \mathbf{A}_t^{(1)} \cdot \mathbf{A}_t^{(2)} \cdots \mathbf{A}_t^{(K)} \exp\left[\sum_{\substack{i,j \\ j>i}} \epsilon^{(i)}\epsilon^{(j)} O^{(i,j)} \right] \\ &= D\left[-i \sum_{k=1}^{K} \epsilon^{(k)} O(\Lambda_+^{(k)}) \right] \exp\left[\tfrac{1}{2} \sum_{i,j=1}^{K} \epsilon^{(i)}\epsilon^{(j)} \left[\frac{O(\Lambda^{(i)}\Lambda^{(j)}) - \Lambda_+^{(i)} O(\Lambda_0 -^{(i)})}{\Lambda_+^{(i)}-1} - \text{c.c.} \right] \right] , \\ A_l &= A_l^{(1)} \cdots A_l^{(Q)} = \exp\left[-\sum_{q=1}^{Q} \delta\chi^{(q)} \frac{R(\Lambda^{(q)})-1}{2\tan(r\tilde{\omega}^{(q)}/2)} \right] , \end{aligned} \tag{43}$$

where

$$2O^{(i,j)} = (1+\mathfrak{P})\{[O(\Lambda^{(i)}\Lambda^{(j)}) - O(\Lambda_-^{(i)})]\Lambda_+^{(i)}/(\Lambda_+^{(i)}-1)\} + O(\Lambda_-^{(i)})O(\Lambda_+^{(j)}) - \text{c.c.} ,$$

$\mathfrak{P}$ being the index exchange operator.

V. LONGITUDINAL DETUNING DESCRIBED BY A GENERIC FUNCTION OF TIME

Using the Fourier transform

$$\delta\chi(t) = \int_{-\infty}^{\infty} \delta\chi_{\tilde{\omega}} e^{i\tilde{\omega}t} d\tilde{\omega} ,$$

(43b) can be extended to a generic function $\delta\chi(t)$ of the longitudinal detuning,

$$A_l = \exp\left[-\int \delta\chi_{\tilde{\omega}} e^{i\tilde{\omega}t} \frac{R(e^{-i\tilde{\omega}r})-1}{2\tan(r\tilde{\omega}/2)} d\tilde{\omega} \right] \equiv R(e^{-i\delta\bar{\chi}(t)}) , \tag{44}$$

where the function $\delta\bar{\chi}(t)$, introduced by Deruelle and Tourrenc,[10] satisfies the delay equation (20), namely,

$$R(e^{-i\delta\bar{\chi}(t)})\mathbf{\Sigma}^{(0)}(\rho) = e^{(i/2)[\delta\chi(t)+\delta\chi(t-r)]}[1+R(e^{-i\delta\bar{\chi}(t-r)})]\mathbf{\Sigma}^{(0)}(\rho) . \tag{45}$$

For $\delta\chi$ sufficiently small we can approximate the exponential with the first two terms of its series expansion,

$$A_l \mathbf{\Sigma}^{(0)} \approx \mathbf{\Sigma}^{(0)} \left[1 + i\int \delta\chi_{\tilde{\omega}} \frac{1+e^{-i\tilde{\omega}r}}{1-\rho e^{-i\tilde{\omega}r}} e^{i\tilde{\omega}t} d\tilde{\omega} \right] . \tag{46}$$

On the other hand, when the spectral range of $\delta\chi$ is smaller than the distance $2\pi/r\mathcal{F}$ of ω from the closest pole of $\mathbf{\Sigma}^{(0)}$, we can expand $R(e^{-i\tilde{\omega}r})$ in power series of $\hat{D} = -\rho\partial/\partial\rho = ir^{-1}\partial/\partial\omega$,

$$A_l = \exp\left[-i\sum_{n=1}^{\infty} \frac{\eta^{(n-1)}}{n!} \hat{D}^n \right] , \tag{47}$$

where

$$\eta = \int \delta\chi_{\tilde{\omega}} e^{i\tilde{\omega}t}[\tilde{\omega}r/2\tan(\tilde{\omega}r/2)]d\tilde{\omega}$$

and $\eta^{(n)} = r^n d^n \eta/dt^n$. Next, expressing η as a series in $\chi^{(n)} = r^n d^n \delta\chi/dt^n$,

$$\eta = \sum_{q=0}^{\infty} C_{2q}\chi^{(2q)} = \delta\chi + \frac{\chi^{(2)}}{12} - \frac{\chi^{(4)}}{720} + \cdots , \tag{48}$$

we obtain,

$$\begin{aligned} A_l &= R(e^{-i\delta\chi})\exp\left[-i\sum_{m=1}^{\infty}\chi^{(m)}\sum_{q=0}^{[(m+1)/2]}\frac{C_{2q}}{(m+1-2q)!}\hat{D}^{\,m+1-2q}\right] \\ &= R(e^{-i\delta\chi})\exp\left[-\frac{i}{2}\chi^{(1)}(\hat{D}^{\,2}+\tfrac{1}{6})-\frac{i}{6}\chi^{(2)}\left[\hat{D}^{\,3}+\frac{\hat{D}}{2}\right]+\cdots\right] \\ &\equiv R(e^{-i\delta\chi})R(e^{-i(\delta\chi-\delta\bar{\chi})}) , \end{aligned} \tag{49}$$

having indicated by $[a]$ the largest integer no greater than a. Since $\hat{D}^{\,n}\Sigma^{(0)}$ is independent of r, the operator at exponent of the term on the right-hand side (r.h.s.) of the above equation represents a series in the cavity round-trip delay r. In particular, at the zeroth order in r the response of the cavity follows adiabatically the evolution of the resonator length, as assumed in the work by Meystre *et al.*[9] The delay effects appear with the first-order correction with a term proportional to the time derivative of the cavity length,

$$A_l \approx R(e^{-i\delta\chi})\left[1-\frac{i}{2}\chi^{(1)}\left[\frac{1+\rho}{(1-\rho)^2}+\tfrac{1}{6}\right]\right] , \tag{50}$$

in view of the relation

$$\hat{D}^{\,2}\Sigma^{(0)} = (1+\rho)/(1-\rho)^2\Sigma^{(0)} .$$

VI. FUNDAMENTAL MODE

When the beam entering the cavity coincides with the fundamental mode, i.e., $\mathbf{T}_1\cdot\mathbf{e}$ is proportional to the vector $\mathbf{e}_{00}$ $[=(1,0,\ldots)]$, and M_1 undergoes oscillations, we have

$$\mathbf{\Sigma}_1(t)\cdot\mathbf{e}_{00} = \sum_l C_l(t)\mathbf{e}_{10} , \tag{51}$$

where

$$C_l(t) = \exp\left[\frac{\epsilon^2}{2}\left[\frac{1-\Lambda^2+\Lambda_- - \Lambda_+}{(\Lambda_+ - 1)(\Lambda_- - 1)}O(\Lambda^2) + 2\frac{\Lambda_-}{\Lambda_- - 1}O(\Lambda_+)\right]\right]\frac{(-i\epsilon)^l O(\Lambda_+)^l}{\sqrt{l!}}\Sigma^{(0)} . \tag{52}$$

In general the function

$$\exp\{\epsilon^2[\Lambda_-/(\Lambda_- - 1)]O(\Lambda_+)\}\Sigma^{(0)}$$

can be calculated by using the expansion of Appendix C. However, for small tilting angles it is worth expanding the exponential in series.

In particular, for a steady-state tilting the relative amplitude C_0 of the fundamental mode reads

$$\begin{aligned} C_0 &= \exp\left[-i\frac{\epsilon^2}{2}\left[\frac{O(1)}{\tan\phi}+\frac{e^{-i\phi}}{\sin\phi}O(e^{i2\phi})\right]\right]\Sigma^{(0)} \\ &\approx \exp\left[-i\frac{\epsilon^2 O(1)}{2\tan\phi}\right]\left[1-i\frac{\epsilon^2 e^{-i\phi}}{2\sin\phi}\Sigma^{(0)}(\rho e^{i2\phi})\right]\Sigma^{(0)}(\rho) \approx \left[1+\frac{\epsilon^2}{4\sin^2\phi}\right]\frac{\rho}{1-\rho e^{-i(\epsilon^2/2\tan\phi)}} . \end{aligned} \tag{53}$$

Expressing ϵ and ϕ by means of the g parameters, we have

$$\epsilon^2/2\tan\phi = \alpha^2 k d g_2(1-g_1g_2)^{-1} ,$$

$$\epsilon w^{(1)}/2\tan\phi = \alpha d g_2(1-g_1g_2)^{-1} ,$$

$$\epsilon = \alpha(2kd)^{1/2}g_2^{1/4}g_1^{-1/4}(1-g_1g_2)^{-1/4} .$$

For typical cavities used in GW interferometers, mirror 1 is plane ($g_1=1$), $d=3$ km, $\lambda=0.5\ \mu$m, and

$$\epsilon = 6\times10^5\alpha(R_2/d-1)^{3/4} .$$

Accordingly, if we compare the phase of C_0 with the dephasing due to a longitudinal displacement, we see that the tilting is equivalent to a detuning $\delta\chi=\epsilon^2/2\tan\phi$, which in turn corresponds to a factor

$$h \equiv \delta d/d = \alpha^2 g_1/(1-g_1 g_2) .$$

Then, for achieving an antenna sensitivity h, the misalignment α must be less than $\sqrt{hd/(R_2-d)}$.

If we represent $\mathbf{A}_t$ by Eq. (37) and neglect the round-trip delay ($\Lambda=1$), with the help of the Jacobi expansion (B1) we obtain

$$\boldsymbol{\Sigma}_1\cdot\mathbf{e}_{00}=D\left[-\epsilon\frac{e^{-i\phi}}{2\sin\phi}R(e^{i2\phi})\right]D\left[i\frac{\epsilon}{2}\right]D\left[\frac{\epsilon}{2\sin\phi}\right]\mathbf{e}_{00}\Sigma ,$$

$$\Sigma \equiv e^{i\epsilon^2/4\tan\phi}\sum_{s=-\infty}^{\infty} i^s I_s\left[\frac{\epsilon^2}{2\sin^2\phi}\right] \times e^{-i2s\phi}R(e^{i2s\phi-(i\epsilon^2/2\tan\phi)})\boldsymbol{\Sigma}^{(0)} . \quad (54)$$

$D(\epsilon/2\tan\phi)\mathbf{e}_{00}$ represents the fundamental Gaussian mode displaced by $\epsilon w^{(1)}/2\tan\phi$ $[=adg_2(1-g_1g_2)^{-1}]$ from the mirror vertex. A simple geometric analysis shows that this displacement coincides with the distance of the vertex of M_1 (see Fig. 3) from the optic axis of the cavity. On the other hand, $D(i\epsilon/2)$ represents a rotation of the beam by the angle α [see (27)]. Accordingly, we can drop this operator from the above equation by referring the beam to a plane forming an angle α with Π_1. For what concerns $D(-\epsilon e^{-i\phi}R(e^{i2\phi})/2\sin\phi)$, we rely on the relations $a|\alpha\rangle=\alpha|\alpha\rangle$, $a^\dagger|\alpha\rangle=(\partial/\partial\alpha)|\alpha\rangle$ for writing

$$\boldsymbol{\Sigma}_1\cdot\mathbf{e}_{00}=\exp\left[-\epsilon\frac{e^{-i\phi}}{2\sin\phi}R(e^{i2\phi})\left[\beta+\frac{\partial}{\partial\beta}\right]\right] \times u_0\left[\frac{y'}{w^{(1)}}-\frac{\epsilon}{2\tan\phi}\right]\Sigma , \quad (55)$$

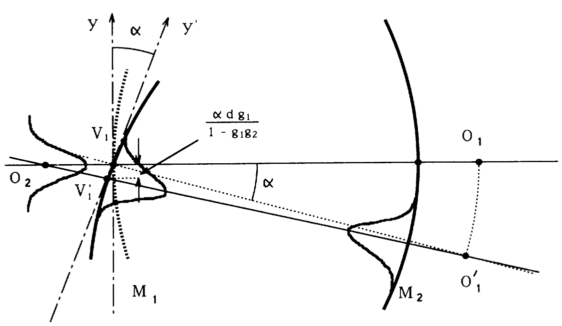

FIG. 3. Variation of the optic axis of the cavity induced by a tilting of M_1. As a result of the tilting, the center of curvature of M_1 moves from O_1 to O_1' while the y axis transforms into y'. The cavity field is still represented with good approximation by a Gaussian centered on the intersection V_1' of the mirror with the optic axis.

with $\beta=\epsilon/2\tan\phi$. Since

$$|\boldsymbol{\Sigma}^{(0)}/R(e^{\pm i2\phi})\boldsymbol{\Sigma}^{(0)}|\approx 2\mathcal{F}|\sin\phi|/\pi ,$$

for $\mathcal{F}\sin\phi \gg \pi$ for F sufficiently large, we can ignore the operator $R(e^{i2s\phi})$ in the above equations. In conclusion, we observe that when the mirror is tilted, the field inside a cavity with infinite finesse is a Gaussian centered on the intersection with the optic axis. In other words, the beam follows adiabatically the intersection with the optic axis. For a finite finesse higher-order modes appear as a result of the presence of the scaling operator $R(e^{i2\phi})$.

VII. ALIGNMENT ERROR

In order to design a closed-loop system for maintaining alignment of the FP resonator it is necessary to measure the deviation of the actual configuration of the cavity from the ideal one. For example, the error can be provided by some spectral components of the signal obtained by combining in a suitable way the outputs of a detector array. This multielement signal is given by

$$s(t)=\frac{\mathbf{e}^\dagger\cdot\mathbf{T}_1^\dagger\cdot\boldsymbol{\Sigma}_1^\dagger(t)\cdot\mathbf{P}\cdot\boldsymbol{\Sigma}_1(t)\cdot\mathbf{T}_1\cdot\mathbf{e}}{|\boldsymbol{\Sigma}^{(0)}\cdot\mathbf{e}|^2} , \quad (56)$$

where $\mathbf{P}=\sum_k d_k\mathbf{P}_k\cdot\mathbf{P}_k$ represents the pupil matrix of the kth element of the polyelement detector (see Appendix D) and d_k a set of coefficients. The adjoint of $\boldsymbol{\Sigma}_1$ is calculated by treating the parameters $\Lambda^{(i)}$ and $e^{i\tilde{\omega}^{(k)}t}$ as real quantities with respect to complex conjugation. The signal s depends on (i) the modes present in the optical beam (vector $\mathbf{e}$) and (ii) on the mirror alignment (matrix $\boldsymbol{\Sigma}_1$). Consequently, a suitable set of coefficients d_k and pupils $\mathbf{P}_k$ can provide useful information about the mismatching of the laser beam and the misalignment of the cavity mirrors.

In particular, for a single element detector with a unit pupil matrix and a TEM_{00} mode, the signal s reduces to

$$s(t)=|\boldsymbol{\Sigma}_1(t)\cdot\mathbf{e}_{00}|^2/|\boldsymbol{\Sigma}^{(0)}\cdot\mathbf{e}_{00}|^2 .$$

In some current prototype interferometers the alignment error is sensed by modulating the relative alignment and coherently detecting the intensity change. For a cavity, such a modulation is achieved either by varying the orientation of the cavity mirrors or the position and angle of the input laser beam. Using this control technique on their 10-m prototype GW antenna, Ward[17] succeeded in keeping the cavities accurately aligned over periods of hours.

In the above situation, when M_1 undergoes oscillations of amplitude $\epsilon_{\tilde{\omega}}$ at frequency $\tilde{\omega}$, $s(t)$ is a periodic function containing only even harmonics of $\tilde{\omega}$. In case a steady-state tilting ϵ is overimposed, odd harmonics of $\tilde{\omega}$ appear in the spectrum of s. In fact, from (43a) we have for $\epsilon_{\tilde{\omega}}$ sufficiently small,

$$\Sigma_1\cdot \mathbf{e}_{00}=\Sigma^{(0)}\mathbf{e}_{00}-i[\epsilon O(e^{i2\phi})+\epsilon_{\tilde\omega}e^{i\tilde\omega t}O(\Lambda_+)]\Sigma^{(0)}\mathbf{e}_{10}$$
$$-\tfrac{1}{2}\epsilon\epsilon_{\tilde\omega}e^{i\tilde\omega t}\left[[O(e^{-i2\phi})O(\Lambda_+)+\text{c.c.}]+\left[\frac{O(\Lambda)-e^{i2\phi}O(e^{-i2\phi})}{e^{i2\phi}-1}+\frac{O(\Lambda)-\Lambda_+O(\Lambda_-)}{\Lambda_+-1}-\text{c.c.}\right]\right]\Sigma^{(0)}\mathbf{e}_{00}+\cdots \; . \tag{57}$$

Consequently, for the superposition $\epsilon(t)=\epsilon+2\epsilon_{\tilde\omega}\cos(\tilde\omega t)$ of a steady-state tilting ϵ and an oscillation of amplitude $\epsilon_{\tilde\omega}$ at frequency $\tilde\omega$, the signal s obtained by using a single-element detector and a unit pupil matrix contains an harmonic at frequency $\tilde\omega$ of amplitude $\epsilon\epsilon_{\tilde\omega}e^{i\tilde\omega t}s_{\tilde\omega}$ where for real ρ,

$$s_{\tilde\omega}=\Sigma^{(0)}(\rho e^{-i2\phi})\Sigma^{(0)}(\rho\Lambda_+)+\frac{e^{-i2\phi}}{e^{-i2\phi}-1}\Sigma^{(0)}(\rho\Lambda_+)$$
$$+\frac{\Lambda_+}{\Lambda_+-1}\Sigma^{(0)}(\rho e^{-i2\phi})-\frac{\Lambda-1}{(e^{-i2\phi}-1)(\Lambda_+-1)}\Sigma^{(0)}(\rho\Lambda)+\text{c.c.}+O(\epsilon_{\tilde\omega}^3)\; . \tag{58}$$

A noteworthy feature of the above expression is the dependence of the error signal on the mechanical frequency, as shown for a typical cavity in Fig. 4.

Earlier work[6] centered on a system that used auxiliary phase modulation to monitor the amplitude of higher-order modes in the cavity present due to misalignment. When the laser beam is modulated at frequency Ω, $s(t)$ contains two harmonics s_Ω and $s_{2\Omega}$, namely,

$$s(t)\equiv s_0+\frac{m}{2}\text{Re}\left[e^{i\Omega t}s_\Omega+\frac{m}{2}e^{2i\Omega t}s_{2\Omega}\right]\,, \tag{59}$$

where

$$s_\Omega=\frac{\mathbf{e}^\dagger\cdot\Sigma_1^\dagger(\rho)\cdot\mathbf{P}\cdot\Sigma_1(\rho e^{-i\Omega r})\cdot\mathbf{e}+\mathbf{e}^\dagger\cdot\Sigma_1^\dagger(\rho e^{i\Omega r})\cdot\mathbf{P}\cdot\Sigma_1(\rho)\cdot\mathbf{e}}{|\Sigma^{(0)}\cdot\mathbf{e}|^2}=s_\Omega^{(0)}-i\epsilon s_\Omega^{(1)}+O(\epsilon^2)\; . \tag{60}$$

In particular,

$$\Sigma_1\cdot\mathbf{e}_{00}=\Sigma^{(0)}(\rho)\mathbf{e}_{00}-i\epsilon\Sigma^{(0)}\rho\Sigma^{(0)}(\rho e^{i2\phi})\mathbf{e}_{10}+O(\epsilon^2)\,, \tag{61}$$

so that, for real ρ,

$$s_\Omega^{(0)}=2\frac{\Sigma^{(0)}(\rho e^{-i\Omega r})}{\Sigma^{(0)}(\rho)}\mathbf{e}_{00}\cdot\mathbf{P}\cdot\mathbf{e}_{00}\,,$$
$$s_\Omega^{(1)}=\frac{\Sigma^{(0)}(\rho e^{-i\Omega r})}{\Sigma^{(0)}(\rho)}[\Sigma^{(0)}(\rho e^{i2\phi})-\Sigma^{(0)}(\rho e^{-i2\phi})+\Sigma^{(0)}(\rho e^{i2\phi-i\Omega r})-\Sigma^{(0)}(\rho e^{-i2\phi-i2\Omega r})]\mathbf{e}_{00}\cdot\mathbf{P}\cdot\mathbf{e}_{10}\; . \tag{62}$$

By combining in a suitable way the outputs of the different detectors, we can obtain a signal proportional to $s_\Omega^{(1)}$ and free of the contributions $s_\Omega^{(0)}$. Equation (62b) clearly shows that $s_\Omega^{(1)}$ exhibits a peak for $\Omega=2\phi/r$. Then, by imposing sidebands at the appropriate frequency $(2\phi/r)$ onto a resonant nominally aligned input beam, one can measure the in-phase and in-quadrature components of the error function by measuring either the reflected or transmitted beams. In this way, the alignment and mode-matching errors are obtained in real time and may be inserted into a closed-loop control system designed to maintain alignment. Anderson[6] was the first to suggest this approach for monitoring the departure of the mirrors from their ideal positions.

VIII. RADIATION PRESSURE

As a result of the radiation pressure produced by a laser beam of intensity $\mathcal{P}(t)$, mirror 1 is acted upon by a force $f(t)$ given by

$$f(t)=\frac{2\mathcal{P}_c(t)}{c}|r_1|^2 s(t;\omega)\,, \tag{63}$$

where

$$\mathcal{P}_c=\mathcal{P}|\Sigma^{(0)}\cdot\mathbf{e}|^2|t_1/r_1|^2$$

represents the power circulating in a perfectly aligned cavity, while s is defined by Eq. (56) with a unity pupil matrix. When the mirrors are at rest and the cavity field coincides with the fundamental mode, s reduces to unity. Since at equilibrium $\partial f/\partial d$ must be negative, then the phase χ of ρ [see (15)] must satisfy the condition $0>\chi>-\pi$ (mod2π). On the other hand, $\Sigma^{(0)}$ is different from zero for $|\chi|<\pi/\mathcal{F}$. Hence, eventually, combining these two conditions,

$$0>\chi>-\frac{\pi}{\mathcal{F}} \quad (\mathrm{mod}2\pi) \ . \tag{64}$$

In case M_1 changes its position in time, Eq. (50) yields

$$f(t)=\frac{2\mathcal{P}}{c}|t_1|^2\{|\Sigma^{(0)}(\rho e^{-i\delta\chi})\cdot\mathbf{e}|^2+r\delta\dot{\chi}\,\mathrm{Re}[\mathbf{e}^\dagger\cdot\Sigma^{(0)\dagger}(\rho e^{-i\delta\chi})\cdot\Upsilon(\rho e^{-i\delta\chi})\cdot\mathbf{e}]\}$$
$$\equiv f^{(0)}(\delta\chi)+r\delta\dot{\chi}f^{(1)}(\delta\chi) \ , \tag{65}$$

where

$$\Upsilon(\rho)=\Psi(1+\rho)/(1-\rho)^2\Sigma^{(0)}(\rho) \ .$$

When M_1 is tilted, its illumination becomes slightly asymmetric, thus producing a mechanical torque N,

$$N(t)=\frac{2\mathcal{P}}{c}|r_1|^2\int\int_{\infty}^{\infty}dy\,dz\,y|u_1^{(-)}|^2=\frac{\mathcal{P}}{c}|r_1|^2w^{(1)}\mathbf{E}_1^{(-)*}\cdot(\mathbf{a}+\mathbf{a}^\dagger)\cdot\mathbf{E}_1^{(-)}$$
$$\equiv\frac{\mathcal{P}_c}{c}w^{(1)}n(t) \ , \tag{66}$$

where

$$n(t)=(\Sigma_1\cdot\mathbf{T}_1\cdot\mathbf{e})^\dagger\cdot(\mathbf{a}+\mathbf{a}^\dagger)\cdot\Sigma_1\cdot\mathbf{T}_1\cdot\mathbf{e}/|\Sigma^{(0)}\cdot\mathbf{e}|^2 \ .$$

For $\mathbf{T}_1\cdot\mathbf{e}$ coinciding with the fundamental mode and $\mathbf{A}_t\approx 1-i\epsilon\mathbf{A}_t^{(1)}$, (66) specializes into

$$n(t)\approx\epsilon n^{(1)}\equiv 2\epsilon\frac{\mathrm{Im}[\Sigma^{(0)*}O(\Lambda_+)\Sigma^{(0)}]}{|\Sigma^{(0)}|^2}$$
$$=2\epsilon\frac{\Lambda|\rho|}{|1-\Lambda_+\rho|^2}\sin\beta$$
$$\approx\epsilon(n_0^{(1)}-i\tilde{\omega}rn_1^{(1)})\sin\beta \ , \tag{67}$$

with $\beta=2\phi+\chi$ and

$$n_0^{(1)}=\frac{2|\rho|}{|1-\rho e^{i2\phi}|^2} \ ,$$
$$n_1^{(1)}=\frac{2|\rho|(1-|\rho|^2)}{|1-\rho e^{i2\phi}|^4} \ . \tag{68}$$

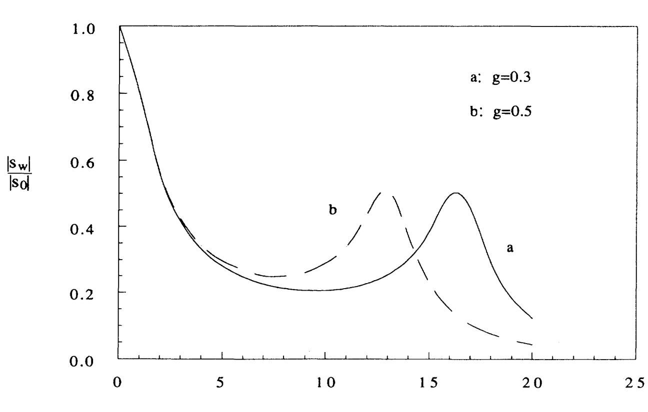

FIG. 4. Plot of $|s_\omega|/|s_0|$ vs the mechanical frequency [see Eq. (58)] for $g_1=1$, $r=20$ μsec, and $F=40$.

IX. MIRROR INSTABILITIES

When the lifetime of the photons in a plane-parallel cavity becomes comparable with the periods of oscillations of the mirror distance, Deruelle, Bel, and Tourrenc *et al.*[10–14] have shown that for a laser power exceeding a threshold value (depending on the masses of the suspended mirrors and the damping times of these oscillations), these oscillations increase and the system is unstable. All these authors have represented the cavity modes by simple plane waves by ignoring the finite cross sections of the actual fields. Using the formulas developed above, we can reexamine this problem for a cavity with spherical mirrors. In so doing, we discover a different type of instability connected with the tilting of the mirrors which occurs when the resonance peak of the fundamental mode overlaps that relative to the first excited one. In the following we will discuss first the conditions for the onset of the latter type of instability and, subsequently, we will reexamine the case of longitudinal instabilities.

A. Torsional oscillations

Regarding torsion around the vertical axis, the motion is determined by the combined action of the mechanical restoring torque and the radiation force torque, i.e.,

$$\ddot{\epsilon}+\frac{\dot{\epsilon}}{\tau}+\Omega_t^2\epsilon=\frac{kw^{(1)2}}{c}\frac{\mathcal{P}_c}{I}n(t) \ , \tag{69}$$

I being the moment of inertia of the mass holding the mirror and Ω_t the frequency of torsional oscillations, while (see Ref. 15, p. 475)

$$\frac{k(w^{(1)})^2}{c}=r\left[\frac{g_2}{g_1(1-g_1g_2)}\right]^{1/2} \ . \tag{70}$$

Approximating n with $\epsilon n^{(1)}$, the above equation is satisfied by $\epsilon\propto e^{-i\tilde{\omega}t}$ with $\tilde{\omega}$ solution of the characteristic equation

$$-\tilde{\omega}^2+i\frac{\tilde{\omega}}{\tau}+\Omega^2=r\left[\frac{g_2}{g_1(1-g_1g_2)}\right]^{1/2}\frac{\mathcal{P}_c}{I}\frac{\Lambda|\rho|\sin\beta}{|1-\Lambda_+\rho|^2} . \tag{71}$$

For $\tilde{\omega}r$ sufficiently small, we can use the approximate expression (68b) of $n^{(1)}$,

$$-\tilde{\omega}^2+i\tilde{\omega}\left[\frac{1}{\tau}+r^2\left[\frac{g_2}{g_1(1-g_1g_2)}\right]^{1/2}\frac{\mathcal{P}_c}{I}n_1^{(1)}\sin\beta\right]+\Omega_t^2-r\left[\frac{g_2}{g_1(1-g_1g_2)}\right]\frac{\mathcal{P}_c}{I}n_0^{(1)}\sin\beta=0 . \tag{72}$$

A solution of the above equation with $\mathrm{Im}\tilde{\omega}>0$ corresponds to an unstable equilibrium point. When $\sin\beta<0$, the term proportional to $\tilde{\omega}$ vanishes when the laser power reaches the threshold value $\mathcal{P}_{\mathrm{th}}$,

$$\mathcal{P}_{\mathrm{th}}=-\frac{I}{r^2\tau}\left[\frac{g_1}{g_2}(1-g_1g_2)\right]^{1/2}\times\left|\frac{(1-\rho)(1+e^{i2\phi}\rho)}{t_1\rho}\right|^2\frac{1}{|\rho|(1-|\rho|^2)}\frac{1}{\sin\beta} . \tag{73}$$

For $\mathcal{P}>\mathcal{P}_{\mathrm{th}}$ the torsional oscillations of the resonator mirror become unstable.

When the laser frequency is within a resonance peak of the fundamental mode, in view of the condition $\beta<0$, a necessary conditions for observing this instability is that $\pi>2\phi\mathcal{F}$, that is, the peaks relative to the fundamental and first excited modes must overlap.

For ϕ sufficiently small ($\phi\approx\sqrt{1-g_1g_2}$), the threshold power reads

$$\mathcal{P}_{\mathrm{th}}\approx\frac{I}{4r^2\tau}\frac{1}{|t_1|^2}\left[\frac{\pi}{\mathcal{F}}\right]^5(1+y^2)\times[1+(y+\phi')^2]^2\frac{\phi'}{y+\phi'} , \tag{74}$$

having put $y=\chi\mathcal{F}/\pi$, $\phi'=2\phi\mathcal{F}/\pi$, with χ ($|\chi|\le\pi$) the phase of ρ. The factor $I/r^2\tau$ represents the power dissipated by the mirror mass when it rotates at the angular speed $1/r$.

B. Longitudinal oscillations

In order to reduce the seismic noise, the mirrors are sustained by multiple pendula.[4] In a simple case they are made of equal masses M suspended to each other at a distance l, except for the last one whose mass is a fraction h of M. If we indicate by x_j the transverse displacement of each mass, we have[18]

$$\ddot{x}_7+\frac{\dot{x}_7}{\tau}+\Omega^2(11+2h)x_7-\Omega^2(5+h)x_6=0 ,$$
$$\ddot{x}_j+\frac{\dot{x}_j}{\tau}+\Omega^2(-3+2h+2j)x_j-\Omega^2(j+h-1)x_{j+1}-(j+h-2)x_{j-1}=0 , \tag{75}$$
$$\ddot{x}_1+\frac{\dot{x}_1}{\tau}+\Omega^2hx_1-\Omega^2hx_2=f(t)/M ,$$

where the r.h.s. of the last equation represents the radiation pressure due to the laser field stored in the cavity. The dots represent derivatives with respect to time with $\Omega=\sqrt{g/l}$, l being the distance between two consecutive masses of the composite pendulum. If we approximate $f(t)$ with the r.h.s. of (65), we can recast the last equation of the above system in the form

$$\ddot{x}_1+\left[\frac{1}{\tau}-2kr\frac{f^{(1)}}{M}(2kx_1)\right]\dot{x}_1+\Omega^2hx_1-\Omega^2hx_2=f^{(0)}(2kx_1)/M . \tag{76}$$

Now, if we approximate $f^{(0)}(2kx_1)$ with $f^{(0)}(0)+2kf^{(0)\prime}x_1$ and $f^{(1)}(2kx_1)$ with $f^{(1)}(0)$, we obtain a linear system which admits solutions of the form $x_j(t)=X_je^{i\tilde{\omega}t}$. In particular, we have $X_2\equiv X_1H(\tilde{\omega})$, with $H(\tilde{\omega})$ a rational function of $\tilde{\omega}$. Accordingly, we have for the last mass

$$\tilde{\omega}^2-i\tilde{\omega}\left[\frac{1}{\tau}-\frac{2kr}{M}f^{(1)}(0)\right]+h\Omega^2-h\Omega^2H(\tilde{\omega})-\frac{2k}{M}[f^{(0)}(0)]'=0 . \tag{77}$$

The last equation can be immediately generalized by including the effects of an electronic feedback system with the addition of a rational function of $\tilde{\omega}$, which represents the response of the feedback network.

The roots of Eq. (77) depend parametrically on the laser power through the functions $f^{(0,1)}$. For $\mathcal{P}$ exceeding a certain value, some roots have negative imaginary parts, thus signaling the onset of the instability. For studying the evolution of this instability, the dominant reduction technique can be used, as was done by Bel, Boulanger, and Deruelle,[13] for obtaining an ordinary differential equation whose general solution is conjectured to approach the general solution of the retarded equation of motion of the pendulum asymptotically.

X. CONCLUSIONS

We have discussed the main characteristics of pendular Fabry-Pérot resonators used in GW antennas by focusing attention of the dependence of the fields on the mirror misalignment. For solving the electromagnetic problem with time-dependent boundaries, we resorted to the slowly varying approximation in conjunction with a representation of the fields in terms of Hermite-Gauss modes. By introducing a suitable scaling operator, we have obtained close expressions of the time-dependent field on the surfaces of mirrors undergoing longitudinal and tilting oscillations at several frequencies. Combining this result with Fourier transform, we have obtained an expression of the field for a generic law of motion of the longitudinal detuning. Capitalizing on this result, we have obtained a nonlinear equation of motion for the longitudinal detuning exhibiting the effects of the cavity finite delay. This result extends the analysis of Refs. 10–14 to a cavity with spherical mirrors. In addition, we have also considered the mechanical torque produced by the radiation pressure in case of mirror tilting. We have derived, analogous to the longitudinal motion, an equation of motion for the tilting oscillations by including the effects of radiation pressure. In particular, we have shown that under certain conditions these oscillations can become unstable. Finally, we have obtained the expressions of some error signals used for controlling the cavity alignment.

ACKNOWLEDGMENTS

The present work was performed under the auspices of the Istituto Nazionale di Fisica Nucleare (INFN). L. Di-Fiore wishes to thank the INFN for financial support.

APPENDIX A: OPERATOR $O(\lambda)$

Let us indicate by $\mathbf{F}(\mathbf{M})$ an entire function of an infinite order matrix $\mathbf{M}$,

$$\mathbf{F}(\mathbf{M})=\sum_n F_n\mathbf{M}^n \tag{A1}$$

and by $O(\lambda)$ an operator acting on $\mathbf{F}$ as follows:

$$O(\lambda)\mathbf{F}(\mathbf{M})=\frac{\mathbf{F}(\lambda\mathbf{M})-\lambda\mathbf{F}(\mathbf{M})}{\lambda-1}\equiv\frac{R(\lambda)-1}{\lambda-1}\mathbf{F}(\mathbf{M}) , \tag{A2}$$

$R(\lambda)$ being the scaling operator $[R(\lambda)f(\mathbf{M})=f(\lambda\mathbf{M})]$. In particular, when $\mathbf{F}(\mathbf{M})$ reduces to the scalar function $\Sigma^{(0)}=\rho/(1-\rho)$,

$$O(\lambda)\Sigma^{(0)}=\Sigma^{(0)}R(\lambda)\Sigma^{(0)} . \tag{A3}$$

Accordingly,

$$O(\lambda)\mathbf{\Sigma}^{(0)}=\mathbf{\Psi}\mathbf{\Sigma}^{(0)}R(\lambda)\mathbf{\Sigma}^{(0)} . \tag{A4}$$

Combining two operators,

$$O(\lambda_1)O(\lambda_2)\mathbf{F}(\mathbf{M}) =\frac{\mathbf{F}(\lambda_1\lambda_2\mathbf{M})-\lambda_1\mathbf{F}(\lambda_2\mathbf{M})-\lambda_1\mathbf{F}(\lambda_1\mathbf{M})+\lambda_1\lambda_2\mathbf{F}(\mathbf{M})}{(\lambda_1-1)(\lambda_2-1)} , \tag{A5}$$

we verify that they commute with respect to the product operation

$$O(\lambda_1)O(\lambda_2)=O(\lambda_2)O(\lambda_1) =\frac{\lambda_1\lambda_2-1}{(\lambda_1-1)(\lambda_2-1)}O(\lambda_1\lambda_2) -\frac{\lambda_1}{\lambda_1-1}O(\lambda_2)-\frac{\lambda_2}{\lambda_2-1}O(\lambda_1) . \tag{A6}$$

The introduction of the operator $O(\lambda)$ allows us to establish for the matrix $\mathbf{\Sigma}^{(0)}$ [see (21)] the following relation:

$$\sum_{n=1}^{\infty} O_n(\lambda_1)\cdots O_n(\lambda_q)\Phi^{2n}=O(\lambda_1)\cdots O(\lambda_q)\mathbf{\Sigma}^{(0)} , \tag{A7}$$

$O_n(\lambda)$ being defined by (34).

If we consider $\rho\propto e^{-i\omega r}$ as a function of the frequency ω, it is immediately shown that

$$O(\lambda)=\frac{\Delta[(i/r)\ln\lambda]-\lambda}{\lambda-1} , \tag{A8}$$

Δ being the shift operator. Accordingly, $O(\Lambda)$ and $O(\Lambda_\pm)$ can be rewritten as

$$O(\Lambda_\pm)=\frac{\Delta(-\tilde{\omega}\pm 2\phi/r)-\Lambda_\pm}{\Lambda_\pm-1} ,$$
$$O(\Lambda)=\frac{\Delta(-\tilde{\omega})-\Lambda}{\Lambda-1} =-1+\rho\frac{\partial}{\partial\rho}+\frac{i}{2}r\tilde{\omega}\left[1-\rho\frac{\partial}{\partial\rho}\right]\rho\frac{\partial}{\partial\rho}+O((\tilde{\omega}r)^2) . \tag{A9}$$

Consequently,

$$e^{\beta O(\Lambda)}=e^{-\beta}\Delta(-i\beta/r) \times\left[1+i\frac{\beta}{2}r\tilde{\omega}\left[1-\rho\frac{\partial}{\partial\rho}\right]\rho\frac{\partial}{\partial\rho}+\cdots\right] . \tag{A10}$$

APPENDIX B: JACOBI EXPANSION

In analogy with the Jacobi expansion of the function $\exp(iz\sin\phi)$ in terms of phase factors $\exp(is\phi)$, we can prove by series expansion the relation

$$\exp\left[\beta\frac{e^{-i2\phi}R(e^{i2\phi})-e^{i2\phi}R(e^{-i2\phi})}{2i}\right] =\sum_{s=-\infty}^{\infty}J_s(\beta)e^{-i2s\phi}R(e^{i2s\phi}) , \tag{B1}$$

with J_s the Bessel function of order s. In particular,

$$\exp\left[\epsilon^2\frac{\Lambda_-R(\Lambda_+)-\Lambda_+R(\Lambda_-)}{(\Lambda_+-1)(\Lambda_--1)}\right] =\sum_{s=-\infty}^{\infty}J_s\left[2i\epsilon^2\frac{\Lambda R(\Lambda)}{(\Lambda_+-1)(\Lambda_--1)}\right] \times e^{-i2s\phi}R(e^{i2s\phi}) . \tag{B2}$$

APPENDIX C: REPRESENTATION OF $\exp[\beta R(e^{-i\Omega r})]\Sigma^{(0)}$ BY MEANS OF THE DEGENERATE HYPERGEOMETRIC FUNCTION Φ

If we represent $\Sigma^{(0)}$ in the form suggested by Eq. (22), we have (see Ref. 19, Eqs. 8.354 and 8.351)

$$\exp[\beta R(e^{-i\Omega r})]\Sigma^{(0)}=\frac{i}{r}\sum_n\sum_q\frac{\beta^q}{q!}\frac{1}{\omega+q\Omega-\omega_{n00}^{(0)}}$$
$$=-\frac{i}{r\Omega}e^{\beta}\sum_n\frac{1}{\gamma_n}\Phi(1,1+\gamma_n;-\beta)\ , \quad \text{(C1)}$$

where $\gamma_n=(\omega-\omega_{n00}^{(0)})/\Omega$, while Φ is the degenerate hypergeometric function

$$\Phi(1,1+\gamma;-\beta)=1-\frac{1}{\gamma+1}\beta+\frac{1}{(\gamma+2)(\gamma+1)}\beta^2+\cdots\ . \quad \text{(C2)}$$

APPENDIX D: PUPIL MATRIX OF AN ELEMENT OF AN ARRAY DETECTOR

A single element of a multielement detector can be represented by a squarish pupil extending along the y and z axes from $a_{y,z}$ to $b_{y,z}$. The relative pupil matrix factorizes into the product of a matrix $\mathbf{P}_y$, coupling the modes $u_l(y)$ times a matrix $\mathbf{P}_z$ relative to the modes $u_m(z)$. Both matrices can be calculated by introducing the generating functions of the Hermite polynomials. After lengthy calculations, we obtain

$$(\pi 2^{l+l'}l!l'!)^{1/2}P_{l'}^{l}=\sum_k\frac{2^k l!l'!}{(l-k)!(l'-k)!k!}\left[H_{l+l'-2k-1}\left[\frac{\sqrt{2}a}{w}\right]e^{-2a^2/w^2}-H_{l+l'-2k-1}\left[\frac{\sqrt{2}b}{w}\right]e^{-2b^2/w^2}\right]$$
$$+\left[\frac{2^{l+l'-2}}{\pi}\right]^{1/2}\frac{l!l!'}{(l-k)!(l'-k)!k!}\delta_{l+l',2k}\left[\operatorname{erf}\left[\frac{\sqrt{2}b}{w}\right]-\operatorname{erf}\left[\frac{\sqrt{2}a}{w}\right]\right]. \quad \text{(D1)}$$

[1]M. E. Gertsenshtein and V. I. Pustovoit, Zh. Eksp. Teor. Fiz. **43**, 605 (1962) [Sov. Phys.—JETP **16**, 433 (1963)].

[2]R. Weiss, Quarterly Progress Report, Research Laboratory of Electronics, MIT (1972), Vol. 105, p. 54.

[3]R. W. P. Drever, G. M. Ford, J. Hough, I. M. Kerr, A. J. Munley, J. R. Pugh, N. A. Robertson, H. Ward, in *Proceedings of the Ninth International Conference on General Relativity and Graviation, Jena, 1980,* edited by E.Schmutzer (Deutscher Verlag der Wissenschften, Berlin, 1983).

[4]A. Giazzotto, Phys. Rep. **182**, 365 (1989).

[5]R. W. P. Drever, J. L. Hall, F. V. Kowalski, J. Hough, G. M. Ford, A. J. Munley, and H. Ward, Appl. Phys. **31**, 97 (1983); H. Ward, J. Hough, G. A. Kerr, N. L. Mackenzie, J. B. Maugan, B. J. Meers, G. P. Newton, D. I. Robertson, N. A. Robertson (unpublished).

[6]See, for example, D. Z. Anderson, Appl. Opt. **23**, 2944 (1984); D. Fattacioli, A. Boulharts, A. Brillet, and C. N. Man, J. Opt. **17**, 115 (1985).

[7]S. Solimeno, F. Barone, L. Di Fiore, L. Milano, and G. Russo, in *General Relativity and Gravitational Physics,* edited by M. Cerdonio, R. Ciani, M. Francaviglia, and M. Toller (World Scientific, Singapore, 1989).

[8]A. Dorsel, J. D. McCullen, P. Meystre, E. Vignes, and H. Walther, Phys. Rev. Lett. **51**, 1550 (1983).

[9]P. Meystre, E. M. Wright, J. D. McCullen, and E. Vignes, J. Opt. Soc. Am. B **2**, 1830 (1985).

[10]N. Deruelle and P. Tourrenc, in Gravitation, Geometry and Relativistic Physics (Springer Verlag, Berlin, 1984).

[11]P. Tourrenc and N. Deruelle, Ann. Phys. (Paris) **10**, 241 (1985).

[12]J. M. Aguirregabiria and L. Bel, Phys. Rev. A **36**, 3768 (1987).

[13]L. Bel. J. L. Boulanger, and N. Deruelle, Phys. Rev. A **37**, 1563 (1988).

[14]B. Meers and N. MacDonald, Phys. Rev. A **40**, 3754 (1989).

[15]S. Solimeno, B. Crosignani, and P. Di Porto, *Guiding, Diffraction, Confinement of Optical Radiation* (Academic, Orlando, 1986).

[16]R. J. Glauber, Phys. Rev. **131**, 2766 (1963).

[17]H. Ward (private communication).

[18]C. Bradaschia, R. Del Fabbro, L. Di Fiore, A. Di Virgilio, A. Giazotto, H. Kautzky, V. Montelatici, and D. Passuello, Phys. Lett. **137A**, 329 (1989).

[19]I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic, New York, 1966).