Adiabatic amplification of optical solitons

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We study the adiabatic evolution of the fundamental nonlinear Schrödinger soliton under a general integro-differential perturbation. This perturbation is shown to model a saturable bandwidth-limited amplification of an optical soliton with a nonresonant carrier wave. We use the soliton perturbation theory to calculate the evolution of the amplitude, frequency, group velocity, and phase of the pulse. The perturbative analytical steady-state solution is obtained and its stability is studied using the phase-plane formalism.

I. INTRODUCTION

Optical pulse propagation, for which material dispersion, gain dispersion, and nonlinearity all contribute significantly, is attracting considerable attention. For instance, let us mention the works of Martinez, Fork, and Gordon,[1] Diels *et al.*,[2] and Haus and Silberberg[3] on the effects of group-velocity dispersion and self-phase modulation in mode-locked lasers, those of Blow, Doran, and Wood,[4] Grigor'yan, Maimistou, and Sklyarov,[5] Höök, Anderson, and Lisak,[6] Bélanger, Gagnon, and Paré,[7] Paré, Gagnon, and Bélanger,[8] and Petrov and Rudolph[9] on the evolution and stabilization of solitary waves in an amplified and absorbed nonlinear Schrödinger equation, the experimental works of Gouveia-Neto, Gomes, and Taylor[10] that led to the theoretical analysis of Blow, Doran, and Wood[11] on the suppression of the soliton self-frequency shift by bandwidth-limited amplification, and the works of Ainslie *et al.*,[12] Krushchev *et al.*,[13] Nakazawa *et al.*,[14] and Agrawal[15] on the amplification of very short optical pulses in erbium-doped fiber amplifiers.

Under restrictions to second-order dispersion and assumption of instantaneous variation of the nonlinearity, the mathematical model that describes the normalized field envelope $u(z,t)$ of such systems is

$$iu_z+(a_1-i\gamma_2)u_{tt}+(a_2-i\gamma_n)u|u|^2-i\gamma_0 u=0\ , \tag{1.1}$$

where all parameters are real and $u(z,t)$ is complex. Equation (1.1) is known as the complex Ginzburg-Landau equation and is of major interest in many branches of physics and mathematics. It is continuously a subject of studies in many different contexts. For instance, see Refs. 16–24 and references therein for various discussions on exact solutions, dynamics, and stability.

In this paper, we study a somewhat generalized version of (1.1), that is,

$$iu_z+\tfrac{1}{2}u_{tt}+u|u|^2=(t_c-i\beta_1)u_{ttt}+i\gamma_2 u_{tt}+Au_t+i\gamma_0 u+i\gamma_n u|u|^2+(t_a-i\beta_2)(u|u|^2)_t$$
$$+(t_d-i\beta_3)u(|u|^2)_t+(B+iC)u\int_{-\infty}^{t}|u|^2dt+(D+iE)u_t\int_{-\infty}^{t}|u|^2dt\ , \tag{1.2}$$

which takes into account simultaneous contributions of third-order material and gain dispersions, gain saturation, nonresonant carrier wave, and higher-order nonlinear effects.

In a first analysis, we restrict ourselves to the case where the right-hand side of (1.2) is a perturbation of the nonlinear Schrödinger (NLS) equation and study the adiabatic evolution of the fundamental NLS soliton. In particular, this permits us to apply the results of the perturbation theory of the inverse scattering transform (IST) method[25,26] and obtain analytical expressions.

The paper is organized as follows. In Sec. II, we present the optical model that motivates the present analysis from general considerations about dispersion and amplification. In Sec. III, we calculate the adiabatic evolution of the amplitude, frequency, group velocity, and phase of the fundamental NLS soliton and obtain a perturbative analytical solution for the steady-state pulse. Finally, in Sec. IV, we present some examples of amplitude-frequency phase portraits in order to enlighten the dynamics and stability of the equilibrium solution.

II. THE PHYSICAL MODEL

We propose a physical model of soliton propagation in optical fiber amplifiers[12–15] that is governed, under appropriate assumptions, by Eq. (1.2). The procedure we use is not formal but based on standard arguments.[27]

First, we denote $k=k(\omega,|\mathcal{E}|^2)$ the nonlinear dispersion of the medium in absence of amplification, where

$$\mathcal{E}(\zeta,\tau)=V(\zeta,\tau)\exp(ik_0\zeta-i\omega_0\tau)$$

is the electric field and ζ,τ are the true physical space and time coordinates. As usual, we expand it around the carrier frequency ω_0, the wave number k_0, and zero amplitude[27] to obtain

$$k(\omega,|V|^2)=k_0+\frac{\partial k_0}{\partial\omega}(\omega-\omega_0)+\frac{1}{2}\frac{\partial^2 k_0}{\partial\omega^2}(\omega-\omega_0)^2+\frac{1}{6}\frac{\partial^3 k_0}{\partial\omega^3}(\omega-\omega_0)^3+\frac{\partial k_0}{\partial|V|^2}|V|^2+\frac{\partial^2 k_0}{\partial\omega\partial|V|^2}(\omega-\omega_0)|V|^2\ , \qquad (2.1)$$

where the coefficients of the Taylor expansion are evaluated at $\omega=\omega_0$, $V=0$, and $k_0=k(\omega_0,0)$. For the moment, the nonlinearity in (2.1) is assumed to vary instantaneously with the field (i.e., no explicit time dependence on $|V|^2$). This assumption will be removed later.

Second, let us model the frequency and nonlinearity dependence of the amplification with the standard homogeneously broadened complex Lorentzian line shape,[28] that is,

$$g(\omega,|V|^2)=\frac{g_0-g_2|V|^2}{1+iT_0(\omega-\omega_g)}\ , \qquad (2.2)$$

where $g_0>0$ is the small-signal gain, $T_0=-2/\Delta\omega$, $\Delta\omega$ is the gain bandwidth parameter, and ω_g is the gain-center frequency. For now, we also assume that the gain nonlinearity varies instantaneously, that is, $g_2|V|^2$ has no explicit time dependence. This rather phenomenological assumption will also be removed later.

We now suppose that the carrier frequency ω_0, which becomes the reference frequency of our model, is slightly nonresonant, that is, $\omega_g=\delta\omega+\omega_0$. We follow the same procedure in the obtention of (2.1) and expand (2.2) around ω_0, g_0, and $V=0$ with the assumptions $T_0^2(\omega-\omega_0)^2\ll 1+T_0^2(\delta\omega)^2$ and $T_0^2(\delta\omega)^2\ll 1$. We then obtain

$$g=g_0[1+iT_0\delta\omega-(i-2T_0\delta\omega)T_0(\omega-\omega_0)-(1+3iT_0\delta\omega)T_0^2(\omega-\omega_0)^2+(i-4T_0\delta\omega)T_0^3(\omega-\omega_0)^3]-g_2[1+iT_0\delta\omega-(i-2T_0\delta\omega)T_0(\omega-\omega_0)]|V|^2\ . \qquad (2.3)$$

The simultaneous contribution of material dispersion (2.1) and gain dispersion (2.3) leads to the general dispersion relation

$$K(\omega,|V|^2)=k-ig\ . \qquad (2.4)$$

Using the standard operator equivalences $K-k_0\to -i\partial_\zeta, \omega-\omega_0\to i\partial_\tau$ and operating K on the amplitude $V(\zeta,\tau)$ of the electric field yields the resultant equation

$$-iV_\zeta=g_0(T_0\delta\omega-i)V+i(k_0'-g_0T_0-2ig_0T_0^2\delta\omega)V_\tau-(\tfrac{1}{2}k_0''-3g_0T_0^3\delta\omega+ig_0T_0^2)V_{\tau\tau}-i\left[\frac{1}{6}k_0'''+g_0T_0^3+4ig_0T_0^4\delta\omega\right]V_{\tau\tau\tau}+\left[\frac{\partial k_0}{\partial|V|^2}-g_2T_0\delta\omega+ig_2\right]V|V|^2+i\left[\frac{\partial k_0'}{\partial|V|^2}+g_2T_0+2ig_2T_0^2\delta\omega\right](V|V|^2)_\tau+i\alpha_0V-i\alpha_2V|V|^2\ , \qquad (2.5)$$

where $\alpha_0>0$ and $\alpha_2>0$ have been included to take nonlinear absorption into account.

At this stage, it is particularly convenient to include the noninstantaneous variation of the nonlinearity. First, following Gordon,[29] we assume a quasi-instantaneous Raman response of the medium by making the substitution

$$\frac{\partial k_0}{\partial|V|^2}|V|^2\to\frac{\partial k_0}{\partial|V|^2}[|V|^2-(\mu+i\sigma)(|V|^2)_\tau]\ , \qquad (2.6)$$

where μ and σ are positive constants that represent the part of the nonlinearity with a delayed response.

Second, because of the gain (loss) saturation, the gain (loss) nonlinearity is time dependent with a long relaxation time and approximately governed by the integral of the field intensity.[3,5,15,28] We then make the substitutions

$$g_2|V|^2\to g_2|V|^2+g_3\int_{-\infty}^{\tau}|V|^2d\tau\ , \qquad (2.7a)$$

$$\alpha_2|V|^2\to\alpha_2|V|^2+\alpha_3\int_{-\infty}^{\tau}|V|^2d\tau\ , \qquad (2.7b)$$

with $g_2|V|^2\neq 0$ and $\alpha_2|V|^2\neq 0$ because of their potential interest in other contexts than optics.[16–24]

Finally, we normalize (2.5)–(2.7) in terms of the reduced time

$$t=(\tau-k_0'\zeta+g_0T_0\zeta)T^{-1}\ , \qquad (2.8)$$

where T is a measure of the initial pulse width, and

$$\zeta=|k_0''-6g_0T_0^3\delta\omega|^{-1}T^2z,\quad k_0''-6g_0T_0^3\delta\omega<0\ , \qquad (2.9)$$

$$V=\left[\frac{|k_0''-6g_0T_0^3\delta\omega|}{(\partial k_0/\partial|V|^2)-g_2T_0\delta\omega-2g_3T_0^2\delta\omega}\right]^{1/2}\times T^{-1}u(z,t)\exp(ig_0T_0\delta\omega\zeta)\ , \qquad (2.10)$$

$$\frac{\partial k_0}{\partial|V|^2}-g_2T_0\delta\omega-2g_3T_0^2\delta\omega>0\ .$$

The resulting equation is then (1.2) with

$$\begin{aligned}
\gamma_0 &= \frac{g_0-\alpha_0}{|k_0''-6g_0T_0^3\delta\omega|}T^2 \approx \frac{g_0-\alpha_0}{|k_0''|}T^2, \quad A = -\frac{2g_0T_0^2\delta\omega}{|k_0''-6g_0T_0^3\delta\omega|}T \approx -\frac{2g_0T_0^2\delta\omega}{|k_0''|}T , \\
\gamma_2 &= \frac{g_0T_0^2}{|k_0''-6g_0T_0^3\delta\omega|} \approx \frac{g_0T_0^2}{|k_0''|}, \quad t_c = -\frac{4g_0T_0^4\delta\omega}{|k_0''-6g_0T_0^3\delta\omega|}T^{-1} \approx -\frac{4g_0T_0^4\delta\omega}{|k_0''|}T^{-1} , \\
\beta_1 &= -\frac{\frac{1}{6}k_0''' + g_0T_0^3}{|k_0''-6g_0T_0^3\delta\omega|}T^{-1} \approx -\frac{\frac{1}{6}k_0''' + g_0T_0^3}{|k_0''|}T^{-1} , \\
\gamma_n &= \frac{\alpha_2-g_2-g_3T_0}{(\partial k_0/\partial|V|^2)-g_2T_0\delta\omega-2g_3T_0^2\delta\omega} \approx \frac{\alpha_2-g_2-g_3T_0}{(\partial k_0/\partial|V|^2)} , \\
t_a &= \frac{2g_2T_0^2\delta\omega}{(\partial k_0/\partial|V|^2)-g_2T_0\delta\omega-2g_3T_0^2\delta\omega}T^{-1} \approx \frac{2g_2T_0^2\delta\omega}{(\partial k_0/\partial|V|^2)}T^{-1} , \\
\beta_2 &= \frac{g_2T_0+(\partial k_0'/\partial|V|^2)}{(\partial k_0/\partial|V|^2)-g_2T_0\delta\omega-2g_3T_0^2\delta\omega}T^{-1} \approx \frac{g_2T_0+(\partial k_0'/\partial|V|^2)}{(\partial k_0/\partial|V|^2)}T^{-1} , \\
t_d &= \frac{\mu(\partial k_0/\partial|V|^2)}{(\partial k_0/\partial|V|^2)-g_2T_0\delta\omega-2g_3T_0^2\delta\omega}T^{-1} \approx \mu T^{-1} , \\
\beta_3 &= -\frac{\sigma(\partial k_0/\partial|V|^2)}{(\partial k_0/\partial|V|^2)-g_2T_0\delta\omega-2g_3T_0^2\delta\omega}T^{-1} \approx -\sigma T^{-1} , \\
B &= \frac{g_3T_0\delta\omega}{(\partial k_0/\partial|V|^2)-g_2T_0\delta\omega-2g_3T_0^2\delta\omega}T \approx \frac{g_3T_0\delta\omega}{(\partial k_0/\partial|V|^2)}T , \\
C &= \frac{\alpha_3-g_3}{(\partial k_0/\partial|V|^2)-g_2T_0\delta\omega-2g_3T_0^2\delta\omega}T \approx \frac{\alpha_3-g_3}{(\partial k_0/\partial|V|^2)}T , \\
D &= \frac{2g_3T_0^2\delta\omega}{(\partial k_0/\partial|V|^2)-g_2T_0\delta\omega-2g_3T_0^2\delta\omega} \approx \frac{2g_3T_0^2\delta\omega}{(\partial k_0/\partial|V|^2)} , \\
E &= -\frac{g_3T_0}{(\partial k_0/\partial|V|^2)-g_2T_0\delta\omega-2g_3T_0^2\delta\omega} \approx -\frac{g_3T_0}{(\partial k_0/\partial|V|^2)} .
\end{aligned} \tag{2.11}$$

Nonresonance of the carrier wave is described, at this order, by parameters A, t_c, t_a, B and D. Gain (loss) saturation is described by γ_n, B, C, D, and E.

III. ADIABATIC EFFECTS ON THE FUNDAMENTAL NLS SOLITON

We now concentrate on Eq. (1.2) and consider the right-hand side $h(z,t)$ as a perturbation of the NLS equation. To study this effect on the adiabatic evolution of the fundamental NLS soliton, we consider the general form of the soliton solution with parameters depending on the propagation coordinate z, that is,

$$u_s(z,t)=\eta(z)\operatorname{sech}\{\eta(z)[t+\kappa(z)]\} \times\exp\{-i\omega(z)[t+\kappa(z)]+i\alpha(z)\} , \tag{3.1}$$

where η, ω, κ, and α are the amplitude, frequency, group velocity, and phase of the pulse.

The evolution equations for these parameters are obtained from the perturbation theory of the IST method.[25,26] They are given explicitly by

$$\dot{\eta}=\frac{1}{\eta}\,\mathrm{Im}\left[\int_{-\infty}^{\infty}h(u_s)u_s^*dx\right] , \tag{3.2a}$$

$$\eta\dot{\omega}+\omega\dot{\eta}=\mathrm{Re}\left[\int_{-\infty}^{\infty}h(u_s)u_{sx}^*dx\right] , \tag{3.2b}$$

$$\dot{\kappa}=\omega-\frac{1}{\eta^2}\,\mathrm{Im}\left[\int_{-\infty}^{\infty}xh(u_s)u_s^*dx\right] , \tag{3.2c}$$

$$\dot{\alpha}=\frac{1}{2}(\omega^2+\eta^2)-\frac{1}{\eta^2}\,\mathrm{Re}\left[\int_{-\infty}^{\infty}h(u_s)(u_s^*+xu_{sx}^*)dx\right] -2\frac{\omega}{\eta^3}\mathrm{Im}\left[\int_{-\infty}^{\infty}xh(u_s)u_s^*dx\right] , \tag{3.2d}$$

where $x=\eta(t+\kappa)$ and a dot denotes a derivative with respect to z. The first two equations come from the evolution of the discrete spectrum (a single eigenvalue here) of the spectral problem or, equivalently, from the evolution of the first two conserved quantities of the NLS equation, $\int|u|^2dt$ and $\int i(u_t^*u-u_tu^*)dt$. The last two equations are calculated from the evolution of the proportionality factor between the Jost functions of the discrete spectrum.[25]

Substituting (3.1) in (3.2) and integrating the right-hand sides yields

$$\dot{\eta}=2\eta(t_c\omega^3-\gamma_2\omega^2-A\omega+\gamma_0)+2\eta^2(C-D\omega)+2\eta^3[(t_c-\tfrac{2}{3}t_a)\omega+\tfrac{2}{3}\gamma_n-\tfrac{1}{3}\gamma_2-\tfrac{1}{3}E], \quad (3.3a)$$

$$\dot{\omega}=2\eta^2[t_c\omega^2-\tfrac{2}{3}\gamma_2\omega+\tfrac{1}{3}E\omega+\tfrac{1}{3}(B-A)]-\tfrac{2}{3}\eta^3 D+\tfrac{2}{15}\eta^4(7t_c-6t_a-4t_d), \quad (3.3b)$$

$$\dot{\kappa}=\omega+\eta^2(\beta_1-\beta_2-\tfrac{2}{3}\beta_3)+3\beta_1\omega^2-C+D\omega+E\eta, \quad (3.3c)$$

$$\dot{\alpha}=\tfrac{1}{2}(\omega^2+\eta^2)-2\eta^2\omega(\beta_1+\tfrac{1}{3}\beta_3)+2\beta_1\omega^3-B\eta-C\omega+D\omega^2. \quad (3.3d)$$

Relations (3.3) describe the evolution of the four soliton parameters η, ω, κ, and α as long as the hyperbolic-secant-like amplitude assumption (3.1) remains valid. After a long distance of propagation, the pulse reaches its final steady state where the amplitude deformation can be significant. To take this deformation into account, we calculate the final perturbative steady-state solitary wave by directly solving (1.2) under the assumption

$$u_f(z,t)=\eta_f\,\mathrm{sech}(\eta_f\xi_f)[1+U(\eta_f\xi_f)]\times\exp[-i\omega_f\xi_f+(i/2)(\omega_f^2+\eta_f^2+2\varphi_f)z], \quad (3.4)$$

where $\eta_f,\omega_f,v_f,\varphi_f$ are constants, $\xi_f=t+\omega_f z+v_f z$, and $U(\eta_f\xi_f)$ is the perturbative part. Solving the linear differential equation for U, one obtains

$$u_f=\eta_f\mathrm{sech}(\eta_f\xi_f)\{1+c_1+c_2\tanh(\eta_f\xi_f)\ln[\mathrm{sech}(\eta_f\xi_f)]+ic_3\ln[\mathrm{sech}(\eta_f\xi_f)]+ic_4\tanh(\eta_f\xi_f)\}\times\exp[-i\omega_f\xi_f+(i/2)(\omega_f^2+\eta_f^2+2\varphi_f)z], \quad (3.5)$$

where η_f and ω_f are given by $\eta_z=\omega_z=0$ in (3.3a) and (3.3b) and

$$c_1=\omega_f(3\beta_1-\tfrac{1}{2}\beta_2)+\tfrac{1}{2}D, \quad (3.6a)$$

$$c_2=\tfrac{2}{5}(2t_d+3t_a-6t_c), \quad (3.6b)$$

$$c_3=\tfrac{2}{3}[2\gamma_2-\gamma_n-E+(t_a-6t_c)\omega_f], \quad (3.6c)$$

$$c_4=\eta_f(3\beta_1-\tfrac{3}{2}\beta_2-\beta_3), \quad (3.6d)$$

$$v_f=-\beta_1\eta_f^2+3\beta_1\omega_f^2-C+D\omega_f+E\eta_f, \quad (3.6e)$$

$$\varphi_f=2\beta_1\eta_f^2\omega_f+2\beta_1\omega_f^3-B\eta_f-C\omega_f+D\omega_f^2. \quad (3.6f)$$

Coefficients c_1, c_4, φ_f, and v_f have been calculated first in Ref. 30 for $B=C=D=E=0$ (see also Ref. 31) using an infinite-dimensional extension of the Birkhoff theory of normal form expansion. Coefficients c_2 and c_3 are given here for the first time.

Let us conclude this section with few comments.

(i) The stationary solution (3.5)–(3.6) shows explicitly how the final parameter values of the NLS soliton are affected by the perturbation term in (1.2). It is valid as long as constant solutions η_f and ω_f exist for (3.3a) and (3.3b). The simplest counterexample is the case where all parameters vanish in (1.2) except $t_d\neq 0$. There is, the frequency ω is a linear function of the variable z that leads to a self-frequency shift[29] and to a different steady-state solution.[32]

(ii) A comparison between equations (3.6e), (3.6f) and (3.3c), (3.3d) shows that the former provide a uniform description of the evolution of κ and α only if

$$\beta_3=-\beta_2=-6\beta_1. \quad (3.7)$$

The physical implication of this is not clear. However, it is interesting to note that relation (3.7) is a familiar one in the theory of integrable NLS-type equations. When condition (3.7) is satisfied and all other parameters are zero, Eq. (1.2) becomes the second member in the NLS equation hierarchy and is completely integrable.[33] Then, it has multisoliton solutions and the fundamental one is precisely given by (3.5) with $c_i=0$.

(iii) The relations (3.2) obtained by a perturbation of the exact IST theory is the standard way to calculate the adiabatic evolution equations (3.3) of the soliton parameters η, ω, κ, and α. An apparently different approach is sometime used to obtain these evolution equations. It consists in multiplying (1.2) by u^* and iu_t^* and integrating over time to obtain two complex equations. According to the IST perturbation theory, the imaginary part of the first equation and the real part of the second provide the evolution of the first two conserved quantities of the NLS equation which lead to (3.2a) and (3.2b), respectively. Furthermore, the real part of the first equation and the imaginary part of the second yield evolution equations for κ and α that can be similar to (3.3c) and (3.3d). For example, when $B=C=D=E=0$ we have

$$\dot{\kappa}=\omega+\tfrac{1}{5}\eta^2(7\beta_1-6\beta_2-4\beta_3)+3\beta_1\omega^2, \quad (3.8a)$$

$$\dot{\alpha}=\tfrac{1}{2}(\omega^2+\eta^2)+\tfrac{2}{15}\eta^2\omega(3\beta_1-4\beta_2-6\beta_3)+2\beta_1\omega^3, \quad (3.8b)$$

which coincide with (3.3c) and (3.3d) only when (3.7) is satisfied. This last procedure has been applied with success in Ref. 11 because $\beta_1=\beta_2=\beta_3=B=C=D=E=0$. However, the above observation shows that it does not lead, in general, to the same result as the IST perturbation theory.

IV. AMPLITUDE-FREQUENCY PHASE PORTRAITS

In order to get more information on the solitary wave evolution given by equations (3.3), it is useful to solve them using the phase-plane formalism. This will also permit us to give a simple qualitative picture of the dynamics and stability properties of the steady-state solution (3.4).

We concentrate on the system (3.2a) and (3.2b) which gives the more physically relevant information and present three particular cases. For all of them, we take $\gamma_2=0.056$ and $\gamma_0=0.025$. In the framework of the optical model of Sec. II, these values correspond to $T_0^2/T^2=2.25$, $\alpha_0=0$, $g_0T^2/|k_0''|=0.025$ and were used in Ref. 11 to model experimental results of Ref. 10. In addition, we set $\gamma_n=t_a=B=C=D=E=0$ (that is, $\alpha_2=g_2=\alpha_3=g_3=0$) in order to emphasize the effect of the nonresonant terms. In any case, the gain (loss) saturation terms do not change the main features of the phase portraits and are very small for amplification of the fundamental NLS soliton in optical fibers.[15]

For generality, we will not restrict ourselves to the phase-space trajectory compatible with the normalization introduced in Sec. II, that is, for initial values $\eta_i=1$ and $\omega_i=0$. This will permit us to have a generic representative phase plane for (1.2). In fact, one can always find a set of parameters $\{a,b,c,\ldots,f\}$ that transforms (1.2) into another similar equation under

$$u(z,t)=au'(z',t')\exp(ibt+icz)\ ,\qquad z'=dz,\quad t'=e(t+fz)\ . \tag{4.1}$$

The coefficients of the new equation are generally different from the original ones. In particular, the new equation for $u'(z',t')$ can have nonresonant-type terms that may be absent in the original one. The effect of transformation (4.1) is to move up or down and scale the axis of the phase plane. The initial point $\{\eta_i=a,\ \omega_i=b\}$ in the original system is then transformed into the point $\{1,0\}$ in the new one, and evolves according to similar trajectory. The explicit determination of $\{a,b,c,\ldots f\}$ in (4.1) is straightforward and will not be presented here.

First, let us examine the phase portrait of Fig. 1 for $t_c=A=0$ (the resonant case $\delta\omega=0$) and $t_d=0$ (instantaneous Raman response $\mu=0$). The sketch is symmetric with respect to $\omega=0$ and the nonvanishing equilibrium solution has $\eta_f=1.16$ and $\omega_f=0$. A linear stability analysis shows that it is a stable node of the system (3.3a) and (3.3b) for $\gamma_0>0$.

An interesting feature of Fig. 1 is the limited basin of attraction of the steady-state solution. For example, initial conditions with $\eta_i=0.5$ and $\omega_i=\pm 1$ rather evolve toward the trivial solution $\eta_f=0$ of (3.3a) and (3.3b). For these initial conditions, the nonlinearity is not sufficiently strong to balance dispersion and the pulse disperses away. This differs from the NLS equation which predicts, because of its Galilean symmetry, nonvanishing solitary waves of any frequency. In fact, the NLS equation is no longer Galilean invariant when $\gamma_2\neq 0$.

We did not calculate the limiting curves between the different basins of attraction. Only approximate limits are given by the dashed curves on Fig. 1. However, from a perturbation analysis of (3.3a) and (3.3b) around $\eta=0$, one can show that these curves cross the $\eta=0$ axis at $\omega_c=\pm 0.67$. Thus, for weak initial amplitudes, NLS solitons grow up to $\eta_f=1.16$ if $|\omega_i|<0.67$ and vanish otherwise. The existence of a limited basin of attraction for the nonvanishing stable solitary wave is a characteristic in all calculations we have done.

Phase portrait of Fig. 2 shows the effect of $t_d=0.05$. In view of the optical model of Sec. II, this shows how the gain dispersion (γ_2 term) stabilizes the linear self-frequency shift due to a noninstantaneous Raman response.[11] (The perturbative analytical steady-state solution for $\gamma_2=\gamma_0=0$ can be found in Ref. 32.)

The mirror symmetry $\omega\leftrightarrow-\omega$ is no longer present and the equilibrium solution has $\eta_f=0.99$, $\omega_f=-0.35$, and is a stable sink of (3.3a) and (3.3b). For particular initial conditions (for instance $\eta_i=\omega_i=1$), the pulse amplitude evolves through successive minimum and maximum values before reaching η_f. Still here, the bassin of attraction of the steady-state solution is limited with $\omega_c=\pm 0.67$.

The last phase plane shown on Fig. 3 is the result of the values $t_c=0.034$ and $A=0.007$ (that correspond to a nonresonant carrier wave with $(\delta\omega)^2T_0^2=0.01$). Here, there are three nonvanishing equilibrium solutions $\{\eta_{f1}=0.56,\omega_{f1}=1.15\}$, $\{\eta_{f2}=1.18,\omega_{f2}=0.023\}$, and $\{\eta_{f3}=2.29,\omega_{f3}=0.49\}$. The first is an unstable node, the second a stable node, and the third a saddle point.

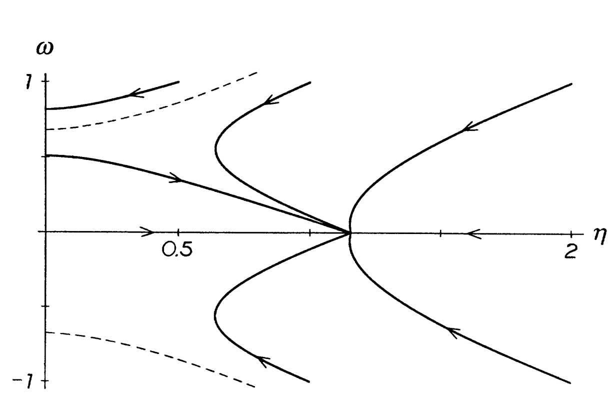

FIG. 1. Phase portrait of (3.3a) and (3.3b) with $\gamma_2=0.056$, $\gamma_0=0.025$, and $t_c=A=\gamma_n=t_a=t_d=B=C=D=E=0$.

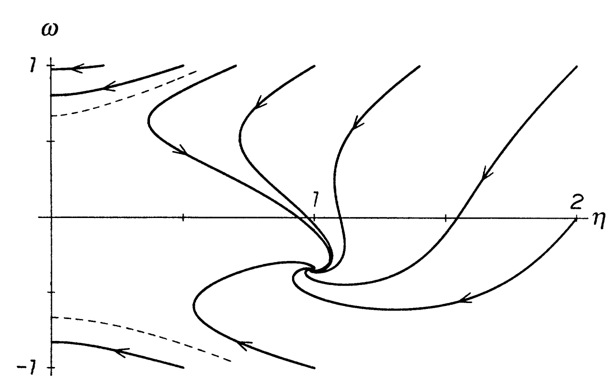

FIG. 2. Phase portrait of (3.3a) and (3.3b) with $\gamma_2=0.056$, $\gamma_0=0.025$, $t_d=0.05$, and $t_c=A=\gamma_n=t_a=B=C=D=E=0$.

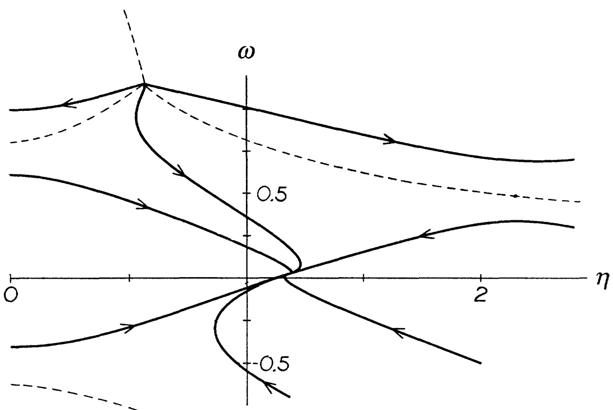

FIG. 3. Phase portrait of (3.3a) and (3.3b) with $\gamma_2=0.056$, $\gamma_0=0.025$, $t_d=0.05$, $t_c=0.034$, $A=0.007$, and $\gamma_n=t_a=B=C=D=E=0$.

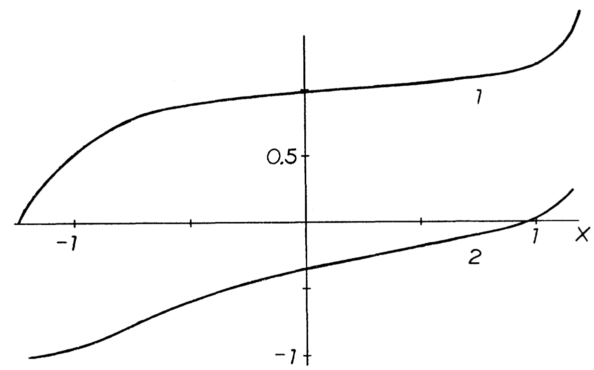

FIG. 4. Variation of $\eta_f(1)$ and $\omega_f(2)$ for the stable equilibrium solution of (3.3a) and (3.3b) with $\gamma_2=0.056$, $\gamma_0=0.025$, and $t_d=0.05$ as function of $X=A/0.007=t_c/0.034$.

The initial field that is local to the first unstable solution evolves toward the second one, toward $\eta=0$, or diverges, depending on its amplitude and frequency. The critical frequencies delimiting the basins of attraction at $\eta=0$ are $\omega_c=-0.62$, 0.81, and 1.47. Similarly, the initial condition local to the third unstable solution evolves toward the second one or diverges for $\omega_i<0.49$ and $\omega_i>0.49$, respectively.

The final amplitude of the stable equilibrium solution can be very sensitive to the variation of A and t_c. For instance, we plotted on Fig. 4 the variation of η_f and ω_f as functions of the parameter $X=A/0.007=t_c/0.034$. This parameter has been chosen to be compatible with the physical model of Sec. II. The case $X=1$ corresponds to the phase plane on Fig. 3. While ω_f behaves smoothly, one observes that η_f vanishes for $X\approx-1.25$ and diverges for $X\approx1.35$. This indicates that a relatively strong nonresonant amplification can be devastating for the transmission of optical solitons.

Finally, let up point out that the absolute stability of the steady-state solution is limited by the growth rate of the tail. This can be estimated from the amplitude of the homogeneous solution (t independent) of (1.2), that is,

$$|u(z)|^2=|u(0)|^2\gamma_0\frac{\exp(2\gamma_0 z)}{\gamma_0+\gamma_n|u(0)|^2[1-\exp(2\gamma_0 z)]} . \tag{4.2}$$

This exponential growth, which saturates to $|u(0)|^2\gamma_0/\gamma_n$ for $\gamma_n\neq0$, tends to predominate over solution (3.4) for sufficiently long propagation length.[8] On its turn, constant solution can become unstable to long-wave modulations. However, for parameter values used here ($\gamma_0=0.025, \gamma_n=0$), the growth is negligible for propagation length $z<50$. Furthermore, the initial pulse (3.1) reached its equilibrium state at $0<z<50$ in all our numerical calculations.

V. CONCLUSIONS

We have studied a nontrivial perturbation of the NLS equation that models the general propagation of a nonresonant optical soliton in a dispersive saturable amplifying medium. The IST perturbation theory was used to obtain the adiabatic evolution of the amplitude, frequency, group velocity, and phase of the fundamental NLS soliton. Amplification (including Raman effect) affects all these parameters while higher-order material dispersion terms change the group velocity and phase only.

An analytic perturbation solution was obtained for the equilibrium solitary wave of the system. Its stability was qualitatively analyzed by numerically solving the evolution equations for the amplitude and frequency in the phase-plane formalism. We then observed the presence of limited basins of attraction for the nonvanishing equilibrium solutions and gave their approximate boundaries.

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