# Quantum-radiation spectra of relativistic particles derived by the correspondence principle

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It is known that ultrarelativistic electrons, when moving nearly parallel to an axis in a single crystal, will emit very intense radiation containing high-energy quanta. The radiation spectra are connected to synchrotron spectra in excessively strong magnetic fields. The latter spectra have previously been calculated by quantum perturbation theory, and are much reduced compared to the corresponding classical spectra. By application of the correspondence principle, it is shown that, for spin-zero particles, the complete quantum spectrum is obtained exactly from the classical spectrum by a simple modification of the frequency variable. The result holds for arbitrary external fields and with respect to angular distribution, polarization, and frequency distribution. Agreement with previous calculations of the synchrotron case is verified. For spin-$\frac{1}{2}$ particles the quantum frequency spectrum can be expressed with satisfactory accuracy by the classical radiation of spin-zero particles. These results may be of assistance in our understanding of radiation by ultrarelativistic particles.

## I. INTRODUCTION

When a relativistic charged particle encounters an external force field, it emits electromagnetic radiation. A basic case is that of circular motion in a constant magnetic field **B**. The corresponding so-called synchrotron spectrum was calculated classically by Schwinger in 1949.[1-3] He subsequently derived[4] the first-order quantum corrections to this classical spectrum, i.e., correction terms linear in $\hbar$. The more complete quantum spectrum was found by Klepikov in 1954,[5] the basis being first-order perturbation for a Dirac electron in strong magnetic fields. These results were further enlarged upon by several authors, in particular Baier and Katkov.[6,7] The change from the classical spectrum can be described in terms of a dimensionless parameter $\chi=(eB/m^2c^3)\hbar\gamma$, where $\gamma=(1-v^2/c^2)^{-1/2}$. When $\chi$ is small, $\chi<0.1$, the classical spectrum obtains. But if $\chi>0.1$ there is a strong quantum reduction. In actual fact, laboratory magnetic fields are not strong enough to realize $\chi>0.1$. Instead, an important case is that of channeling for a relativistic electron, with direction of motion close to an axis or a plane in a single crystal. In channeling there are strong electric forces due to the combined effect of numerous atoms. The forces are essentially transverse to the motion of the particle with an effect simulating that of an exceedingly large magnetic field of slowly varying magnitude. For ultrarelativistic energies, above 100 GeV, $\chi$ becomes large and the electron may readily lose a substantial fraction of its energy in one radiation event.[8,9] The radiation is very intense, being $\sim 10^2$ times the familiar Bethe-Heitler Bremsstrahlung,[10] which latter forms the background belonging to random incidence of the electron beam.

## II. THE USE OF CORRESPONDENCE

The purpose in the following is to show that a relativistic quantum spectrum can be obtained directly from the classical spectrum by a simple change of variable. This result is based on correspondence. In order to explain that, let us consider some general features of radiation theory.

The classical calculation of radiation consists of finding retarded solutions of Maxwell equations for a given motion of a charge $e$, i.e., when $\mathbf{v}(t)/c$ is a given function of time. If the particle has mass $m$, the given classical motion is obtained by means of suitable external forces. Under certain conditions the emitted intensity spectrum and its angular distribution will merge with the quantum result, and the motions will coincide. This happens when $\hbar/m \to 0$; note, for instance, that the above parameter $\chi$, for given $\gamma$ and given rotation frequency $eB/(mc\gamma)$, is proportional to $\hbar/m$. In that limit the momentum and energy of a single photon becomes insignificant, or $\hbar\omega/E_{\rm kin} \to 0$.

As to the direct quantum calculation of the radiation spectra, this consists of treating the radiation interaction as small and using first-order perturbation theory. A typical and simple example is the Bethe-Heitler formula[10] for bremsstrahlung, giving the radiation spectrum emitted by a relativistic electron encountering an atom.

There is another way of calculating the quantum bremsstrahlung spectrum for relativistic electrons, to wit the Weizsäcker-Williams method.[11] In that approach one introduces the rest frame of the electron, where the Lorentz-contracted electric field of an atom is resolved into Fourier components and is considered as a wave packet of free electromagnetic waves. These virtual photons are then scattered individually by the electron according to the relativistic Klein-Nishina differential cross section. Finally, the scattered intensity is transformed back to the laboratory frame. The result is in agreement with the Bethe-Heitler formula for the quantum spectrum. By means of this method, however, one could also obtain the classical spectrum. In fact, instead of the Klein-Nishina formula, use the classical and recoilless

 

Thomson cross section for scattering of light by a free-charged particle, but transform still relativistically between the rest frame and the laboratory frame. In this latter procedure classical concepts have been used throughout.

By combining these two procedures, it appears that we may be able to bypass the Weizsäcker-Williams method, and find the quantum spectrum directly from the classical spectrum in the following manner. The classical spectrum is first calculated from the Maxwell equations. The result is identical to the above description based on the Thomson cross section. If we now introduce the Klein-Nishina cross section instead, we have the quantum spectrum, and the question is whether this can be expressed as a simple substitution in the original classical spectrum. There is some positive evidence for this since Schwinger's first-order quantum corrections are expressed in terms of the classical spectrum only.

The conception of virtual quanta in the above is apparently more general than in a standard Weizsäcker-Williams procedure. In fact, if we take the example of a constant magnetic field **B**, the virtual quanta can of course not be obtained as Fourier components of **B**. But the circular motion could just as well be due to a radial electric field, and then the virtual quanta are recovered. In general it is a question of the behavior of the force as a function of time in, e.g., that Lorentz frame where the particle is at rest when $t=0$; this behavior can be decomposed into harmonic oscillations.

The program sketched here will be carried out in the following sections. It is an attempt to apply the correspondence principle, according to which quantum spectra must contain all features of classical spectra, with due respect to the necessary nonclassical features, cf. also Kramers' formulation of correspondence.[12] In the present case the nonclassical features turn out to be primarily the finite recoil of the charged particle when scattering a photon.

Of old the correspondence principle was employed in order to find properties of atomic states as expressed in terms of finite quantum jumps. In the present case, the knowledge of finite quantum recoil jumps of relativistic electrons is utilized to infer properties of the emitted radiation.

### III. CLASSICAL RADIATION SPECTRUM

We follow now the program outlined in Sec. II, and treat classical radiation. Consider a charged particle, with $\gamma=(1-v^2/c^2)^{-1/2}\gg 1$, moving through an electromagnetic field of force. The retarded solution of the Maxwell equations gives the intensity distribution $dI$ of the emitted radiation. It will be convenient to introduce also the number of quanta emitted, with $dN_{\text{class}}=dI_{\text{class}}/\hbar\omega$, so that

$$dN_{\text{class}}(\omega,\vartheta,\varphi,\mathbf{e}_f)=F(\omega,\vartheta,\varphi,\mathbf{e}_f)\,d\omega\, d\Omega\ , \qquad (1)$$

where $\omega$ is the frequency, $\vartheta$ the angle with direction of particle motion, $\varphi$ the polar angle, $d\Omega$ the differential solid angle, and $\mathbf{e}_f$ the direction of polarization. These quantities are recorded in the laboratory reference frame. We consider then $F$ as a known function.

We calculate next the result (1) by the alternative method of virtual quanta. We describe the initial situation in the momentary rest frame of the particle. The particle meets a certain energy current, with velocity $-\mathbf{v}$, where $v\approx c$. The energy current can be time resolved into Fourier components, with polarizations $\mathbf{e}_i'$ perpendicular to $\mathbf{v}$. Let then $g(k_0',\mathbf{e}_i')dk_0'$ represent the intensity distribution of virtual photons, with momentum between $k_0'$ and $dk_0'$. The dash on these quantities indicates that they belong to the particle rest frame.

We thus have for the incoming number of photons,

$$dN_i(k_0',\mathbf{e}_i')=C\frac{dk_0'}{k_0'}g(k_0',\mathbf{e}_i')\ , \qquad (2)$$

where $C$ is a constant. If the pulse were a $\delta$ function, as in the case of collision with an atom, the function $g$ would be independent of $k_0'$.

The photons are scattered by an angle $\vartheta'$ in a collision with the charged particle. The differential cross section is the classical Thomson cross section $d\sigma_{\text{Th}}$,

$$d\sigma_{\text{Th}}=r_0^2 d\Omega'(\mathbf{e}_f'\cdot\mathbf{e}_i')^2\ , \qquad (3)$$

where $r_0=e^2/mc^2$, $e$ being the charge of the particle, $m$ its mass, and $d\Omega'$ the differential solid angle. The formula describes a particle without magnetic moment and spin. The momentum of the outgoing photon is $\hbar\mathbf{k}'$, and we have $k'=k_0'$, because $\hbar$ is considered vanishingly small in Eq. (3), which implies elastic scattering.

According to Eqs. (2) and (3) the number of scattered photons is

$$dN_f(k',\mathbf{e}_f')=C\frac{dk'}{k'}\sum_{\mathbf{e}_i'}g(k',\mathbf{e}_i')r_0^2 d\Omega'(\mathbf{e}_i'\cdot\mathbf{e}_f')^2\ . \qquad (4)$$

We next transform to the laboratory frame, where the frequency is $\omega$, and the momentum $\hbar\omega/c$, so that

$$\tan\vartheta=\frac{1}{\gamma}\frac{\sin\vartheta'}{v/c-\cos\vartheta'},\quad \varphi=\varphi' \qquad (5)$$

and

$$\omega=k'c\left[\gamma^2\left[\frac{v}{c}-\cos\vartheta'\right]^2+\sin^2\vartheta'\right]^{1/2}\equiv k'\xi\ . \qquad (6)$$

so that $\xi\approx c\gamma(1-\cos\vartheta')$ if $\vartheta\ll 1$ and $\gamma\gg 1$. By Eqs. (4), (5), and (6) we arrive at the classical radiation spectrum

$$dN_{\text{class}}(\omega,\vartheta,\varphi,\mathbf{e}_f)=\frac{C}{\gamma}\frac{d\omega}{\omega}\sum_{\mathbf{e}_i'}g\left[\frac{\omega}{\xi},\mathbf{e}_i'\right]r_0^2 d\Omega'(\mathbf{e}_f'\cdot\mathbf{e}_i')^2\ , \qquad (7)$$

where $\mathbf{e}_f$ is the final polarization, and $d\Omega'$ is connected to $d\Omega$ by Eq. (5).

We can now go back to Eq. (1), relating $F$ to Eq. (7), so that

$$\frac{C}{\gamma}\frac{1}{\omega}\sum_{\mathbf{e}_i'}g\left[\frac{\omega}{\xi},\mathbf{e}_i'\right]r_0^2 d\Omega'(\mathbf{e}_i'\cdot\mathbf{e}_f')^2=F(\omega,\vartheta,\varphi,\mathbf{e}_f)d\Omega\ , \qquad (8)$$

the right-hand side being a known function. If here we sum over final polarizations, writing

$$F(\omega,\vartheta,\varphi)=\sum_{\mathbf{e}_f} F(\omega,\vartheta,\varphi,\mathbf{e}_f) \ ,$$

we find

$$\frac{C}{\gamma}\frac{1}{\omega}\sum_{\mathbf{e}_i'} g\left[\frac{\omega}{\xi},\mathbf{e}_i'\right] r_0^2 d\Omega'[1-(\mathbf{n}'\cdot\mathbf{e}_i')^2]=F(\omega,\vartheta,\varphi)d\Omega \ , \tag{9}$$

where $\mathbf{n}'$ is a unit vector in the direction of $\mathbf{k}'$, and $(\mathbf{n}'\cdot\mathbf{e}_i')=\sin\vartheta'\sin\varphi'$. Integrating next over the solid angle on both sides of (9), we note that the average of $\sin^2\varphi'$ is $\frac{1}{2}$, and thus

$$2\pi\frac{Cr_0^2}{\gamma\omega}\int_{-1}^{+1} d(\cos\vartheta')\sum_{\mathbf{e}_i'} g\left[\frac{\omega}{\xi},\mathbf{e}_i'\right]\tfrac{1}{2}(1+\cos^2\vartheta')=F(\omega) \ , \tag{9'}$$

since $\xi=\xi(\cos\vartheta')$ according to (6). Since the radiation is emitted within an extremely narrow cone, the angular distribution is often not of practical interest, and the function $F(\omega)$ describes the essential property of the spectrum. We shall also utilize (9′) to find the average of $\cos^2\vartheta'$, in cases where $g$ is a simple function.

## IV. QUANTUM SPECTRUM FOR CHARGED PARTICLES

In order to find the quantum spectrum we now have to replace the classical Thomson cross section (3) by the corresponding one in relativistic quantum theory. We introduce therefore the Klein-Nishina cross section, belonging to a Dirac electron in an initially unpolarized state,

$$d\sigma=r_0^2 d\Omega'\left[\left[\frac{k'}{k_0'}\right]^2(\mathbf{e}_f'\cdot\mathbf{e}_i')^2+\frac{k'}{4k_0'}\left[1-\frac{k'}{k_0'}\right]^2\right] , \tag{10}$$

where $k'$ is connected to $k_0'$ by energy and momentum conservation in the collision, or

$$k'=\frac{k_0' mc}{mc+\hbar k_0'(1-\cos\vartheta')}=k_0'\left[1-\frac{\hbar\omega}{E}(1+\psi)\right] , \tag{11}$$

where Eqs. (5) and (6) have been applied, and where

$$1+\psi=\cos\vartheta(1-\cos\vartheta')/(v/c-\cos\vartheta')$$
$$\simeq 1-\vartheta^2/4+\gamma^{-2}/4 \ .$$

Moreover, $E=\gamma mc^2$ is the energy of the particle in the laboratory frame.

In Eq. (10), the first term in the large square brackets is a simple kinematic modification of the Thomson formula. In fact, this term in itself represents the relativistic scattering of photons by a scalar particle. The second term in the brackets is due to the spin and magnetic moment of the electron. It follows from Eq. (11) that the second term is proportional to $\hbar^2$, i.e., to the square of the spin.

The ratio $k'/k_0'$ in the Klein-Nishina formula (10) can now, by (11), be expressed in terms of variables of the laboratory frame. Before doing that, we introduce a simplification by noting that, since $\gamma\gg 1$, all relevant parts of the spectrum are radiated at angles $\vartheta$ very small compared to 1. Thus, a characteristic angle of the spectrum is $\vartheta\sim 1/\gamma$. Therefore, in (11), $1+\psi$ can be replaced by 1. The error, $\sim\gamma^{-2}$, is exceedingly small; already for $\gamma=10$ it would be only $\sim 1\%$, but in channeling radiation for ultrarelativistic electrons one might have $\gamma^{-2}\sim 10^{-10}$.

We can then rewrite (10) as

$$d\sigma=r_0^2 d\Omega'\left[\left[1-\frac{\hbar\omega}{E}\right]^2(\mathbf{e}_f'\cdot\mathbf{e}_i')^2+\frac{1}{4}\left[\frac{\hbar\omega}{E}\right]^2\left[1-\frac{\hbar\omega}{E}\right]\right] , \tag{12}$$

and, for fixed angle $\vartheta'$ or $\vartheta$, cf. (6) and (11),

$$\frac{dk_0'}{k_0'}=\left[1-\frac{\hbar\omega}{E}\right]^{-1}\frac{d\omega}{\omega} \ . \tag{13}$$

Corresponding to the two terms in Eq. (12) there are two contributions to the radiation spectrum, or $dN=dN^{(1)}+dN^{(2)}$, to be handled separately.

Consider the first term in Eq. (12), belonging to spin-zero particles. The procedure is now analogous to that in Sec. III. We start from the formula for the number of incoming photons, Eq. (2), where we can express $k_0'$ in terms of the frequency $\omega$ in the laboratory frame, by substituting (13) for $dk_0'/k_0'$, and introducing

$$k_0'=\frac{k'}{1-\hbar\omega/E}=\frac{\omega}{\xi(1-\hbar\omega/E)}$$

by Eq.(11) with $\psi=0$ and by Eq. (6). Finally we multiply Eq. (2) by the first term in Eq. (12), i.e., the differential cross section, and sum over initial polarizations, obtaining

$$dN^{(1)}(\omega,\vartheta,\varphi,\mathbf{e}_f)=\sum_{\mathbf{e}_i'}\frac{C}{\gamma}\frac{d\omega}{\omega}\left[1-\frac{\hbar\omega}{E}\right]\times g\left[\frac{\omega}{\xi(1-\hbar\omega/E)},\mathbf{e}_i'\right]\times r_0^2 d\Omega'(\mathbf{e}_f'\cdot\mathbf{e}_i')^2 \ . \tag{14}$$

When we define

$$\omega^*=\frac{\omega}{1-\hbar\omega/E} \ , \tag{15}$$

and compare (14) with (7) and (8), we obtain

$$dN^{(1)}(\omega,\vartheta,\varphi,\mathbf{e}_f)=F(\omega^*,\vartheta,\varphi,\mathbf{e}_f)d\omega d\Omega \ , \tag{16}$$

where $F$ is the known function in Eq. (1), calculated directly from the classical Maxwell equations.

Equation (16) represents the central result of the correspondence argument for a charged particle with

$\gamma \gg 1$. It is obtained from the classical spectrum (1) simply by retaining $d\omega$, but changing $\omega$ to $\omega^*$ everywhere else. Likewise, in the radiated intensity, or $dI^{(1)}=\hbar\omega dN^{(1)}$, we must retain $\omega d\omega$ from the classical intensity, and in the remainder change $\omega$ to $\omega^*$.

Our first general conclusion from (16) deals with the first-order quantum correction of Schwinger.[4] To that end we note that the second term in the square brackets in (12), i.e., the spin term leading to $dN^{(2)}$, contains $\hbar^2$ as a factor. Therefore, the first-order quantum correction belongs exclusively to $dN^{(1)}$ or $dI^{(1)}$, and here the stipulated substitution in (16) is exactly the one given by Schwinger; he solved explicitly for synchrotron radiation with constant magnetic field. Therefore, the present derivation is a more general one, belonging to any external field.

The second conclusion from (16) deals with the specific case of synchrotron radiation. The detailed classical spectrum, $F(\omega,\vartheta,\varphi,\mathbf{e}_f)$, was studied by Schwinger,[1] cf. also Jackson.[2] It is confined within an angle $\sim 1/\gamma$ with the direction of motion. For simplicity, let us here consider only the spectrum integrated over angles and polarization. Qualitatively, at low frequencies this intensity spectrum increases quite slowly, proportionally to $\omega^{1/3}$. But above a cut-off frequency $\omega_c$ the spectrum falls off exponentially. This critical frequency is

$$\omega_c=\frac{3}{2}\frac{eB}{mc}\gamma^2=\frac{3}{2}\chi\frac{E}{\hbar} , \tag{17}$$

where $\chi$ is the previously mentioned quantum parameter of field strength, and $E=\gamma mc^2$. Thus, as long as $\chi \ll 1$, the highest energies in the spectrum, $\sim\hbar\omega_c$, remain small compared to the energy of the particle. The intensity spectrum is (cf. Landau and Lifshitz[3])

$$dI_{\text{class}}(\omega)=\frac{e^2\omega d\omega}{\pi\sqrt{3}c\gamma^2}\int_{\omega/\omega_c}^{\infty}K_{5/3}(x)dx \equiv \hbar\omega F(\omega)d\omega . \tag{18}$$

Therefore, Eq. (16) leads to a quantum spectrum for a spinless particle

$$dI^{(1)}(\omega)=\frac{e^2\omega d\omega}{\pi\sqrt{3}c\gamma^2}\int_{\omega^*/\omega_c}^{\infty}K_{5/3}(x)dx , \tag{19}$$

where $\omega^*$ is given by (15). This result is in agreement with the formula quoted by Baier and Katkov[5] for a spinless particle. As mentioned above, Schwinger's first-order quantum correction is also contained in (19). We have thus seen that the correspondence principle can be applied in a precise manner for a radiating charge.

Third, there are several advantages of the present result (16) based on the classical spectrum. Thus, it follows that the essence of the quantum calculations in Refs. 4, 5, 6, and 7 is that one considers jumps of a *free* particle, because the Klein-Nishina formula applies. It is to be noted that the number of quanta meeting the electron in the rest frame does not increase with $\gamma$; only their energy increases proportionally to $\gamma$. When $\chi$ is large, the high-energy quanta even tend not to contribute to scattering. Moreover, the present calculation, as expressed by Eq. (16), is quite flexible and need not be confined to synchrotronlike spectra.

There is also an advantage as compared with the Weizsäcker-Williams method as such, because the virtual quanta are not calculated directly. They would not be readily found, for instance, when there is a constant magnetic field $\mathbf{B}$ all over laboratory space. Instead, they are implicit in the change of motion as seen from the momentary rest frame of the electron.

## V. QUANTUM SPECTRUM DUE TO MAGNETIC MOMENT

The use of the correspondence argument was straightforward for obtaining the radiation spectrum $dN^{(1)}$ in Eq. (16), where only the charge of the electron was involved. But the situation is different when we ask for the spectrum $dN^{(2)}$ due to magnetic moment and spin, and determined by the second term in the Klein-Nishina formula.[12] This difference is not unexpected since the classical spectrum has to be accommodated to one further quantum feature—the quantization of spin—in addition to the quantum recoil effect.

There are two ways of using correspondence. The first one is a simple procedure, but it is only approximate. It starts from the classical and quantum radiation by a charge, e.g., from Eqs. (1), (7), (12), and (14), finally replacing the first term in Eq. (12) by the second term. The second way is an attempt at precise correspondence. It requires knowledge of radiation by a classical magnetic moment moving in a classical orbit, and a comparison between the second term of Eq. (10) and a classical cross section for scattering of light by the magnetic moment of a spinning charged particle. It is then analogous to the previous procedure in Secs. III and IV, only now for a magnetic moment instead of a charge.

It is profitable to follow the first procedure, even though approximations will have to be made in the final stage. The intention is to replace the Thomson cross section by the second term in Eq. (12). The essential difficulty is that the former quantity, but not the latter, contains the factor $\eta=(\mathbf{e}'_f\cdot\mathbf{e}'_i)^2$, depending on angles $\vartheta$, $\varphi$, and polarizations. Now, we need only ask for the frequency distribution of the final spectrum and not for its dependence on angles. Let us therefore integrate over the angular variables. First, we average $\eta$ over the final polarization $\mathbf{e}'_f$ and over the polar angle $\varphi'$. The corresponding average of $\eta$ is equal to $(1+\cos^2\vartheta')/4$, which function is already narrowed down to a value between $\frac{1}{4}$ and $\frac{1}{2}$. Next, we integrate this function over $\vartheta'$ as described in (9′) and obtain the desired average $\bar{\eta}=\bar{\eta}(\omega)$. This function depends implicitly upon the virtual spectrum of photons, since $g(k'_0,\mathbf{e}'_i)=g(\omega^*/\xi,\mathbf{e}'_i)$, and $\xi\simeq c\gamma(1-\cos\vartheta')$ according to Eq. (6). If the function $\bar{\eta}$ is known, we can immediately find the spectrum $dN^{(2)}$ from $dN^{(1)}$. Indeed, the ratio $dN^{(2)}/dN^{(1)}$ is equal to the ratio between the second and first terms in the Klein-Nishina formula (12), with the polarization factor replaced by its average, $\bar{\eta}$. The total spectrum is therefore, using Eq. (16),

$$dN(\omega)=d\omega F(\omega^*)\left[1+\frac{1}{\bar{\eta}}\frac{1}{4}\frac{\hbar^2\omega\omega^*}{E^2}\right], \tag{20}$$

and this is an exact formula for large $\gamma$ in the same sense as Eq. (16). We have achieved this because we renounced on finding the angular dependence of the spectrum. But it remains to find $\bar{\eta}(\omega)$. As we shall see, quite primitive approximations will here give acceptable results.

In order to illustrate the situation, it can be appropriate to study a simple model for the virtual spectrum $g$. For this purpose, suppose that $g$ is proportional to $(k_0')^\nu$ which again varies as $(1-\cos\vartheta')^{-\nu}$. When now we perform the averaging described above, and contained in (9′), we find

$$\bar{\eta}=\frac{1}{2}-\frac{1-\nu}{(2-\nu)(3-\nu)}, \tag{21}$$

where it is known beforehand that $\frac{1}{4}\leq\bar{\eta}\leq\frac{1}{2}$, and thus we must have $\nu\leq 1$, for convergence.

As a further illustration, we can now solve the case of bremsstrahlung. The Lorentz-contracted field of an atom gives a pulse proportional to a $\delta$ function in time. Then, the initial number of quanta in Eq. (2) behaves as $dk_0'/k_0'$, so that $g$ is independent of $k_0'$ and $F(\omega)\propto 1/\omega$. In Eq. (21) we have $\nu=0$ and therefore $\bar{\eta}=\frac{1}{3}$. The spectrum (20) becomes

$$\begin{aligned}dN(\omega)&=\text{const}\cdot\frac{d\omega}{\omega^*}\left[1+\frac{3}{4}\frac{\hbar^2\omega\omega^*}{E^2}\right]\\&=\text{const}\cdot\frac{d\omega}{\omega}\left[1-\frac{\hbar\omega}{E}+\frac{3}{4}\left[\frac{\hbar\omega}{E}\right]^2\right],\end{aligned} \tag{22}$$

where the factor in the second set of large parentheses is the quantum correction to classical bremsstrahlung. It is seen that Eq. (22) is precisely the Bethe-Heitler formula for $\gamma$ large.[10] Moreover, the second term in large parentheses is the first-order quantum correction and belongs to Eq. (16), the radiation from a charge.

We can now look into the synchrotron radiation. From Eqs. (20) and (19) we get

$$dI(\omega)=\frac{e^2\omega d\omega}{\pi\sqrt{3}c\gamma^2}\left[1+\frac{1}{\bar{\eta}}\frac{1}{4}\frac{\hbar^2\omega\omega^*}{E^2}\right]\int_{\omega^*/\omega_c}^{\infty}K_{5/3}(x)dx, \tag{23}$$

where it remains to estimate $\bar{\eta}$. Since the classical synchrotron spectrum is proportional to $\omega^{1/3}$ at lower frequencies, and since this part is mainly responsible for the radiation at large $\gamma$, we introduce $\nu=\frac{1}{3}$ in Eq. (21) and find $\bar{\eta}=\frac{7}{20}$, quite close to its value in bremsstrahlung. Inserting this value in Eq. (23) and comparing with the results of Baier and Katkov for quantum synchrotron radiation of electrons, one finds that if $\chi\leq 5$ our results are correct within a few percent. This primitive estimate of $\bar{\eta}$ is thus satisfactory in a discussion of present measurements of channeling radiation.[9]

The above estimate leading to $\bar{\eta}=\frac{7}{20}$ is not accurate when $\chi$ is very large; for $\chi=10^3$ the error is some 20%. But the following argument shows that $\bar{\eta}$ then tends to the value $\frac{1}{2}$. Notice first that when $\chi$ becomes large, we are concerned with the lowest part of the classical synchrotron distribution. This is because the cut-off frequency $\omega_c$ in Eq. (17) becomes very large compared to $E/\hbar$. Next, from the classical theory it is known (cf. Jackson,[2] p. 676) that the angular width of the lower spectrum is approximately

$$\vartheta\sim\gamma^{-1}(\omega_c/\omega)^{1/3}=\gamma^{-1}(2\chi E/3\hbar\omega)^{1/3},$$

where we have to replace $\omega$ by $\omega^*$. Therefore, although $\vartheta$ is very small, it becomes large compared to $\gamma^{-1}$. We learn from Eq. (5), on the simplified form $\vartheta\approx\gamma^{-1}\cot(\vartheta'/2)$, that eventually $\vartheta'$ must become quite small, and finally from $\eta\rightarrow(1+\cos^2\vartheta')/4$ we find $\bar{\eta}=\frac{1}{2}$. With this value in Eq. (23), the total power for $\chi=10^3$ agrees with Baier and Katkov[6,7] within 1%. It can be surmised that further improvements in this kind of treatment can lead to quite accurate results for all values of $\chi$.

As mentioned previously, there is another, more basic way of introducing correspondence for the spin term in the Klein-Nishina formula. It contains the additional correspondence argument belonging to a transition from classical spin to quantized spin. It also requires quite extensive calculations of radiation and of light scattering by the magnetic moment of a classically spinning particle. The present case of spin $\frac{1}{2}$ turns out to correspond to classical spin parallel to the velocity. For spin-1 particles Baier and Katkov find asymptotically much stronger radiation than for spin-zero and spin-$\frac{1}{2}$ particles. This behavior also reflects properties of classical spin. I shall not here enlarge upon the correspondence argument for magnetic moment and spin.

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