Polynomial solutions of the planar Coulomb diamagnetic problem

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It is argued that a set of manifestly normalizable solutions of the two-dimensional Coulomb diamagnetic problem generated by the fine-tuning of the external magnetic field is unphysical. An implication of this for the Hill determinant method is pointed out.

The Coulomb diamagnetic problem remains a fascinating unsolved problem of nonrelativistic quantum mechanics.[1] Its two-dimensional version being separable is naturally more tractable.

This two-dimensional problem has a set of exact solutions which are normalizable. These are obtained for a specific choice of a relevant coupling constant ratio. To keep this report self-contained we first review the origin of these solutions.

Working in cylindrical coordinates (ρ,ϕ) and using dimensionless variables, the primary task is to solve the radial equation

$$R''\frac{1}{\xi}R'+\left[\frac{4E}{\hbar\omega_c}-2m+\frac{\alpha}{\xi}-\frac{m^2}{\xi^2}-\xi^2\right]R=0\ , \qquad (1)$$

the full wave function being

$$\Psi=R(\xi)e^{im\phi}\ . \qquad (2)$$

In Eq. (1) primes denote derivatives and we have used

$$\rho\equiv\nu,\xi,\quad \nu\equiv\sqrt{2\hbar/\mu\omega_c},\quad \omega_c=\frac{|e|B}{\mu c},\quad \alpha^2\equiv\frac{16\mathcal{R}}{\hbar\omega_c}\ . \qquad (3)$$

B is the magnetic field in the Z direction and ω_c the corresponding cyclotron frequency. With $R=\xi^{|m|}e^{-\xi^2/2}v(\xi)$, one gets

$$v''+(p-2\xi^2)v'+(\delta\xi+\alpha)v=0\ , \qquad (4)$$

where

$$p\equiv2|m|+1,\quad \delta\equiv4E/\hbar\omega_c-2m-p-1\ . \qquad (5)$$

Equation (4) admits polynomial solutions which can be obtained by direct substitution, but are seen more compactly by setting

$$v=\sum_n a_n\xi^n,\quad a_0\neq0\ . \qquad (6)$$

This leads to the three-term recursion relation

$$n(n+p-1)a_n+\alpha a_{n-1}+(\delta-2n+4)a_{n-2}=0\ , \qquad (7)$$

with

$$pa_1+\alpha a_0=0\ , \qquad (8)$$

so that $a_1\neq0$ unless $\alpha=0$. The coefficients being successive, polynomial solutions are obtained by demanding an

$$a_k\neq0,\quad a_{k+1}=a_{k+2}=0,\quad k=1,2,\ldots\ . \qquad (9)$$

Hence, a polynomial solution of degree k requires

$$\delta=2k \qquad (10)$$

and $a_{k+1}=0$, i.e.,

$$\begin{vmatrix}\alpha & 1\times p & 0 & 0 & 0 & \cdots & 0\\ \delta & \alpha & 2(p+1) & 0 & 0 & \cdots & 0\\ 0 & \delta-2 & \alpha & 3(p+2) & 0 & \cdots & 0\\ 0 & 0 & & & & & \\ \vdots & \vdots & & & & & k(p+k-1)\\ & & & & (\delta-2k+2) & & \alpha\end{vmatrix}=0\ . \qquad (11)$$

The set of α values that satisfy the tuning condition above leads to polynomial solutions. The structure of Eq. (11) is such that the roots occur in pairs $\pm\alpha$ unless $\alpha=0$ is a root. The $+\alpha$'s correspond to the problem at hand and the $-\alpha$'s determine the polynomial solutions of the problem with the Coulomb potential repulsive. Equation (11) thus determines the polynomial solutions of both problems, albeit with a different number of nodes.

These solutions, which are normalizable, are, all the same, physically unacceptable. A straightforward argument suffices to justify this assertion.

Consider the simplest set of polynomial solutions that is obtained for $k=1$. These are characterized by

$$\delta=2 \Longrightarrow E_{1m}=\frac{\hbar\omega_c}{2}(m+|m|+2) , \quad (12)$$

$$\alpha^2=2p \Longrightarrow 8\mathcal{R}=\hbar\omega_c(2|m|+1) , \quad (13)$$

and

$$\psi_{1m}=\xi^{|m|}e^{-\xi^2/2}\left[1-\frac{\alpha}{p}\xi\right]e^{im\phi} . \quad (14)$$

Such solutions possess one radial node only irrespective of the value taken by m. Furthermore, there is no restriction on the allowed values of m. Thus there is an infinity of such one-node solutions. Let us focus attention on solutions of large positive m. In fact, let $m\to\infty$. One then has

$$E_{1m}\to+4\mathcal{R} , \quad (15)$$

$$\hbar\omega_c\to\frac{4\mathcal{R}}{m}\to 0 \text{ as } m\to\infty . \quad (16)$$

Since $\omega_c\to 0$ and E_{1m} is positive the wave functions must approach those of the scattering states of the two-dimensional Coulomb problem. It is readily seen from Eq. (14) that the set ψ_{1m} does not satisfy this requirement. The Coulombic limit of these solutions is thus erroneous and hence these solutions simply cannot be physical. The same argument applies to other sets of polynomial solutions. There is no choice but to admit that for these special sets of α values there must exist other solutions that are physical.

One must naturally wonder about the origin of such spurious solutions. Although we do not have a definitive answer to this question, a logical possibility immediately suggests itself. Solutions that are normally divergent for arbitrary values of α can accidentally become polynomials and hence normalizable for some special values of α. In this sense such solutions may be termed "zero measure solutions," that is, they have no acceptable continuation as a function of α away from the tuned values. This situation can be obtained in practice. For the sake of illustration we point out two distinct ways in which this can happen.

First, consider a formal class of solutions to Eq. (7) obtained by setting a coefficient $a_{k+1}=0$, where k takes any one of the values $0,1,2,\ldots,$ at a time. This requirement is consistent with Eq. (7). The resulting solutions are clearly divergent, for they lead to undesirable limiting results in the pure Coulomb and Landau limits. However, for the tuned set of α values that make a_{k+2} vanish simultaneously such solutions become polynomials of degree k corresponding to the energies $\delta=2k$.

Second, demand a set of solutions to Eq. (7), such that δ takes the values $2k$, with k selected as before to be an integer. These solutions also diverge, for the energy has no reference to the Coulomb coupling at all and the Landau limit is erroneous. But, again, when α takes one of the values determined by Eq. (11), both a_{k+1} and a_{k+2} vanish and the polynomial solutions reemerge.

Having firmly established that in the given context the polynomials for special α leading to a $\delta=2k$ type of spectrum are illegitimate, we would like to observe that this is not meant to be a general result. In some other context polynomial solutions may legitimately arise upon the tuning of a parameter, due to some compelling circumstance, such as, for example, the sudden emergence of a symmetry of the associated Hamiltonian.[2] This is best illustrated by turning to the very illuminating example of the sextic double-well potential in one dimension. In this case the Hamiltonian can be written as

$$H=p^2-x^2+\beta^2x^6 . \quad (17)$$

The parameter β is a suitable dimensionless ratio of the two coupling constants in the problem. Extracting the leading asymptotic factor $e^{-\beta x^4/4}$ one arrives at a three-term recursion relation that for arbitrary β admits infinite series solutions. But, for $\beta=1/(2k+3)$, $k=0,1,2,\ldots,$ a subset of the solutions turns out to be a set of orthogonal polynomials with a weight factor of $e^{-\beta x^4/4}$. More specifically, for $k=2n$, $n+1$ even-parity polynomial solutions are obtained with node numbers $0,2,4,\ldots,2n$. For $k=2n+1$, $n+1$ odd-parity solutions result. These solutions can be argued to be physical.[3] This pattern emerges, because for these β values the Hamiltonian develops an SL$(2,R)$ symmetry.[4]

Consider now the two-dimensional (2D) Coulomb diamagnetic problem in the same light. A 2D isotropic oscillator problem has an SU(2) symmetry while the 2D Coulomb problem has O(3) symmetry. The combination has merely SO(2) as the residual symmetry. The energy thus depends on $|m|$, so that the levels $\pm m$ are degenerate. Adding to this a suitable Zeeman term we arrive at the Coulomb diamagnetic Hamiltonian. This removes the $\pm m$ degeneracy. Even for the tuned values of α (which incidently form a very irregular pattern unlike the β's of the double-well problem) no special symmetry of the Hamiltonian is noticeable. Hence, from the group-theoretic point of view also, one does not see any reason to expect polynomial solutions.

Finally, we turn to the implications of our result. The problem at hand involves a multiple-step recursion relation. A four-step relation seems to be natural. In this circumstance, the assessment of the normalizability of any proposed solutions becomes a formidable task in practice. Thus indirect additional checks on the feasibility of such solutions are certainly to be welcomed. Our result provides one such check. Regarded as a function of α, the energy is such as not to admit the $\delta=2k$ values corresponding to polynomial solutions for the discrete, but an infinite set of special α values.

This last remark has a nontrivial bearing on the Hill determinant approach to the present problem. Pandey and Varma[5] have recently reported a numerical computation of energies using this approach. Their reported energy trajectories in α space explicitly pass through the energies corresponding to the polynomial solutions for the tuned α values. In view of the arguments presented above such energies are unlikely to converge uniformly to physically acceptable values for all values of α.[6] Thus the theoretical basis of this method, as applied to the class of problems to which the Coulomb diamagnetic problem belongs, deserves to be reviewed. To the best of our knowledge, only for the case of anharmonic-oscillator bound-state problems has the question of the normalizability of the Hill determinant solutions been investigated in depth.[7,8] In the present case this method has the merit of building in the limiting solutions *a priori* that makes such an investigation even more worthwhile.

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[1]D. Kleppner, M. G. Littman, and M. A. Zimmerman, *Rydberg States of Atoms and Molecules,* edited by R. F. Stebbings and F. B. Dunning (Cambridge University Press, Cambridge, England, 1983), pp. 73–116. See also A. R. P. Rau, Nature **325**, 577 (1987), and references therein.

[2]See, e.g., M. Shifman, Int. Mod. Phys. (to be published).

[3]See a forthcoming work by S. C. Chhajlany and V. N. Malnev (unpublished).

[4]The group-theoretic aspects of this problem are discussed by A. M. Peremolov *et al.* (unpublished).

[5]R. K. Pandey and V. S. Varma (unpublished).

[6]One exceptional circumstance, howsoever improbable its realization may be, deserves to be noted. If to each value of α that leads to a polynomial solution there exists a second normalizable solution of the Schrödinger equation with the same node classification and with an energy that lies in an infinitesimal neighborhood of the energy associated with the polynomial solution, then the Hill-determinant-based energies may yet converge uniformly to true energy eigenvalues.

[7]A. Hautot, Phys. Rev. D **33**, 437 (1986).

[8]M. Znojil, Phys. Rev. D **34**, 1224 (1986).