Hysteresis of synchronous-asynchronous regimes in a system of two coupled oscillators

M. Poliashenko, S. R. McKay, and C. W. Smith
Department of Physics and Astronomy, University of Maine, Orono, Maine 04469
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Nonisochronism of one oscillator in a system of two coupled nonlinear oscillators leads to an overlap of synchronous and asynchronous regimes. In some cases, two-frequency oscillations may completely destroy synchronization. We derive conditions for the emergence of these competitive two-frequency oscillations and for the destruction of the synchronous regime, which occurs due to the merging of stable and unstable fixed points in phase space. Analysis is done by reducing the averaged coupled oscillator equations to the general nonlinear pendulum equation. Boundaries of each regime derived in this way are in good agreement with those obtained numerically.

I. INTRODUCTION

A system of two coupled oscillators is one of the classical models used for studies of multimodal oscillatory systems. It describes a wide range of processes that take place in biological, chemical, and physical systems.[1–4] In recent years, considerable progress has been made in understanding the dynamics of two-mode systems. In particular, several studies have identified nonoscillatory, one-frequency and two-frequency regimes in two identical weakly coupled, weakly nonlinear oscillators.[1–3] For two oscillators with different characteristics, the width of the synchronization region has been obtained analytically.[4] Experimentally, coupled nonisochronous oscillators show hysteresis between one- and two-frequency oscillations.[5] This competition between different oscillator regimes can substantially change the dynamical behavior of the system and may even destroy the entire synchronous regime of the two oscillators. Analogous behavior is reported for two scalar-coupled, identical oscillators, which exhibit bistability between phase-locked and phase-drift solutions.[1]

In this paper we present a theoretical description of these competition effects for the system of two different weakly nonlinear oscillators with direct coupling, including the dependence of oscillation frequencies upon their amplitudes (nonisochronism). We provide an analytical determination of the conditions and region of existence for competition between synchronous and asynchronous oscillations, by reducing the equations describing such systems to the general nonlinear pendulum equation. This allows certain conclusions about oscillation regimes to be made using a relatively simple mathematical model.

II. DETERMINATION OF THE REGION OF COMPETITION

We consider two linearly coupled oscillators with reactive nonlinearity described in the usual way by the following general equations:

$$\ddot{x}_1-\mu_1F_1(x_1)\dot{x}_1+\omega_1^2G_1(x_1)x_1=Kx_2\ ,$$
$$\ddot{x}_2-\mu_2F_2(x_2)\dot{x}_2+\omega_2^2G_2(x_2)x_2=Kx_1\ . \qquad (1)$$

Here $F_{1,2}$ and $G_{1,2}$ are polynomials of the form

$$F_i(x_i)=1-\nu_ix_i^2,\quad i=1,2$$
$$G_i(x_i)=1-\delta_ix_i^2,\quad i=1,2\ . \qquad (2)$$

This approach is also valid for arbitrary differentiable $F_i(x_i)$ and $G_i(x_i)$.

If the two oscillators are weakly nonlinear and weakly coupled, and the difference in their partial frequencies is small compared with either frequency, solutions of Eqs. (1) may be written in the form

$$x_1(t)=a(t)\cos[\omega t+\phi_1(t)]\ ,$$
$$x_2(t)=b(t)\cos[\omega t+\phi_2(t)]\ , \qquad (3)$$

where $\omega=(\omega_1+\omega_2)/2$ and a, b, ϕ_1, and ϕ_2 are slowly varying functions of time. Using these solutions, we apply the averaging method introduced by Krylov and Bogoliubov[6,7] for weakly nonlinear systems. After first-order averaging, this method yields

$$\frac{da}{d\tau}=(\alpha_a-\gamma_aa^2)a+kb\sin\psi\ , \qquad (4a)$$

$$\frac{db}{d\tau}=(\alpha_b-\gamma_bb^2)b-ka\sin\psi\ , \qquad (4b)$$

$$\frac{d\psi}{d\tau}=-\Delta+\beta a^2-\kappa b^2+k\left[\frac{b}{a}-\frac{a}{b}\right]\cos\psi\ , \qquad (4c)$$

where a,b are oscillation amplitudes for the two oscillators,

$$\alpha_{a,b}=\frac{\mu_{1,2}}{2},\quad \gamma_{a,b}=\frac{\nu_{1,2}\alpha_{a,b}}{4},\quad k=\frac{K}{2\omega}\ .$$

ψ is the phase difference $\phi_2-\phi_1$, k denotes a coefficient of resonant interaction between the two oscillators, $\Delta\equiv\omega_1-\omega_2$ is the detuning of the two partial frequencies, τ is the "slow" time, and coefficients α_a,α_b characterize the linear and γ_a,γ_b nonlinear dissipative features of the oscillators.[8] The coefficients β and κ represent nonisochronic features of the oscillators and are dependent upon the parameters $\delta_{1,2}$ and $\mu_{1,2}$ of Eqs. (1) and (2).

The equilibrium points $a=a_0$, $b=b_0$, $\psi=\psi_0$ of system

(4) are situated along the resonance curves

$$\Delta=\beta a_0^2-\kappa b_0^2\pm\left[\frac{b_0^2}{a_0^2}-1\right]\left[\frac{k^2a_0^2}{b_0^2}-(\alpha_b-\gamma_b b_0^2)^2\right]^{1/2}, \tag{5a}$$

where

$$a_0^2=\frac{1}{2\gamma_a}\{\alpha_a\pm[\alpha_a^2-4\gamma_a b_0^2(\gamma_b b_0^2-\alpha_b)]^{1/2}\}. \tag{5b}$$

Equations (1) and (4) present the general problem of two linearly coupled nonisochronous oscillators. The remainder of the paper will deal with the case when only one of the two oscillators is substantially nonisochronous ($\beta=0$, $\kappa>0$), since this is the case for which synchronous and asynchronous oscillations compete.

A typical resonance curve (with $\beta=0$, $\kappa>0$) is shown in Fig. 1 in the plane of b_0 and Δ (equilibrium amplitude versus detuning). Stable fixed points of the focus type are situated along the solid part of the curve, unstable fixed points of the saddle-node type are indicated by dots, and unstable saddle-focus points are shown as a dashed line. The parabola $\Delta=-\kappa b_0^2$ is sandwiched between the stable segment of the resonance curve and the nearby branch of saddle-node fixed points. Because these branches are so close together along the stable portion, Eq. (5a) implies that, for each equilibrium point,

$$|-\Delta-\kappa b_0^2|\gg\left|\left[\frac{b_0^2}{a_0^2}-1\right]\left[\frac{ka_0^2}{b_0^2}-(\alpha_b-\gamma_b b_0^2)^2\right]^{1/2}\right| =\left|k\left[\frac{b_0}{a_0}-\frac{a_0}{b_0}\right]\cos\psi_0\right|. \tag{6}$$

In general, for any parameter region where condition (6) is valid, the evolution of the phase difference may be determined from the approximate equation

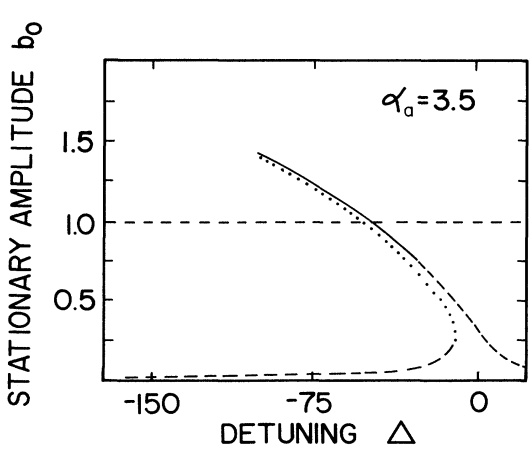

FIG. 1. The resonance curve for system (4) with parameter values $\alpha_a=3.5$; $\alpha_b=1$, $\gamma_a=1$, $\gamma_b=1$, $k=1$, $\beta=0$, and $\kappa=50$. The solid line shows stable node-focus fixed points, the dotted line shows saddle-node fixed points, and the dashed line shows saddle-focus fixed points.

$$\frac{d\psi}{d\tau}=-\Delta-\kappa b^2, \tag{7}$$

Linearizing Eqs. (4) in the neighborhood of the resonant value of the amplitude $b_0=\sqrt{-\Delta/\kappa}$, with amplitude a_0 from Eq. (5b), yields

$$\frac{d\delta_a}{d\tau}=(\alpha_a-\gamma_a a_0^2)a_0+(\alpha_a-3\gamma_a a_0^2)\delta_a+kb_0\sin\psi, \tag{8a}$$

$$\frac{d\delta_b}{d\tau}=(\alpha_b-\gamma_b b_0^2)b_0+(\alpha_b-3\gamma_b b_0^2)\delta_b-ka_0\sin\psi, \tag{8b}$$

$$\frac{d\psi}{d\tau}=-2\kappa b_0\delta_b, \tag{8c}$$

where $\delta_a\equiv a-a_0$, $\delta_b\equiv b-b_0$.

This system of equations (8) has two equilibrium points:

$$\begin{aligned}\bar\delta_a&=-[(\alpha_a-\gamma_a a_0^2)a_0^2\\&\quad+(\alpha_b-\gamma_b b_0^2)b_0^2]/[(\alpha_a-3\gamma_a a_0^2)a_0],\\ \bar\delta_b&=0,\quad \bar\psi_m=m\pi+(-1)^m\arcsin\left[(\alpha_b-\gamma_b b_0^2)\frac{b_0}{a_0k}\right],\\ &\qquad m=0,1\end{aligned} \tag{9}$$

repeated along the ψ axis with period 2π. Stability of these points is determined by the roots λ_i of the following eigenvalue equation:

$$(\alpha_a-3\gamma_a a_0^2-\lambda)[\lambda^2-(\alpha_b-3\gamma_b b_0^2)\lambda-2\kappa k a_0 b_0\cos\bar\psi_m]=0. \tag{10}$$

One of these roots, $\lambda_1=\alpha_a-3\gamma_a a_0^2$, corresponds to that eigenvector in the phase space of Eq. (8) that is directed along the δ_a axis. This eigenvector is common for both fixed points. The two other eigenvectors lie perpendicular to the δ_a axis in the phase plane (ψ,δ_b) and are generally m dependent [See Eq. (9)]. If $\lambda_1<0$, then the first eigenvector provides phase-space compression into the phase plane (ψ,δ_b) in the neighborhood of the equilibrium state $a=a_0$ and $b=b_0$. Stability of the motion in this phase plane is discussed below.

Differentiating Eq. (7) by time τ using Eqs. (8) yields

$$\begin{aligned}\frac{d^2\psi}{d\tau^2}&=-2\kappa b_0\frac{d\delta_b}{d\tau}\\&=-2\kappa b_0^2(\alpha_b-\gamma_b b_0^2)+2\kappa a_0 b_0 k\sin\psi\\&\quad+\frac{d\psi}{d\tau}(\alpha_b-3\gamma_b b_0^2),\end{aligned} \tag{11}$$

a nonlinear physical pendulum equation of the general form

$$\frac{d^2\psi}{d\tau^2}+\nu\frac{d\psi}{d\tau}+\zeta\sin\psi=\gamma, \tag{12}$$

where the dissipation is characterized by

$$\nu = 3\gamma_b b_0^2 - \alpha_b \ ,$$

the rotating moment is

$$\gamma = -2\kappa b_0^2(\alpha_b - \gamma_b b_0^2) \ ,$$

and

$$\zeta = -2\kappa a_0 b_0 k \ .$$

Its phase plane $(\psi, \dot{\psi})$ is equivalent to the phase plane (ψ, δ_b) of system (8). In its phase plane, Eq. (12) has stable focus fixed points F_i and saddle fixed points S_i, which are all located on the ψ axis, evenly spaced with period 2π. If γ is sufficiently small, separatrix C_2, entering S_i, passes above separatrix C_1, which exits the previous saddle point S_{i-1} and enters F_i, the stable fixed point between S_{i-1} and S_i. In this case, from any initial condition, the system arrives at the stable steady state. At a higher value of $\gamma = \gamma_c$, C_1 and C_2 merge to form a periodic curve connecting the saddle points. For $\gamma > \gamma_c$, only trajectories originating below this curve are attracted to the steady state. Initial conditions above the curve flow to the phase rotating solution, which is equivalent to a limit cycle in the phase space of Eq. (8).

An approximate condition for the emergence and existence of this limit cycle can be determined using a Fourier expansion technique, as previously applied to similar equations.[9] This approach is useful for equations such as (12) of the general form

$$\frac{d^2\psi}{d\tau^2} + \nu\frac{d\psi}{d\tau} + R(\psi) = \gamma \ , \tag{13}$$

or, equivalently,

$$\dot{\psi}\frac{d\dot{\psi}}{dx} + \nu\dot{\psi} + R(\psi_0 + x) = \gamma \ , \tag{14}$$

where $R(\psi) = R(\psi + 2\pi)$, ψ_0 is a saddle fixed point, and $x = \psi - \psi_0$. The fixed-point structure of Eq. (12) described above is characteristic of equations of this form, and, since the merged separatrices form a 2π-periodic curve, this function $\dot{\psi}(x)$ can be expanded on the interval $0 < x < 2\pi$ as

$$\dot{\psi}(x) = \sum_{n=1}^{N} b_n \sin\frac{nx}{2} \ . \tag{15}$$

Substituting Eq. (15) into Eq. (14) yields

$$\sum_{n=1}^{2N} A_n \sin\frac{nx}{2} = \gamma - R(\psi_0 + x) \ , \tag{16}$$

where

$$A_n = \nu b_n + \sum_{k=1}^{N} (b_{n+k} + b_{n-k} - b_{k-n})\frac{k b_k}{4} \ , \tag{17}$$

with $b_i = 0$ if $i \le 0$ or $i > N$.

Equation (16) yields

$$A_n = \frac{1}{\pi}\int_0^{2\pi} [\gamma - R(\psi_0 + x)]\sin\frac{nx}{2}dx \ , \tag{18}$$

which, when combined with Eq. (17), gives the coefficients b_i. From Eq. (12), $\sin\psi_0 = \gamma/\zeta$, and, retaining only the first term in the expansion, Eq. (16) yields

$$\frac{b_1^2}{4}\sin x + \nu b_1 \sin\frac{x}{2} = \gamma - \zeta\sin(x + \psi_0) \ . \tag{19}$$

Combining this result with Eqs. (17) and (18) yields

$$\frac{\gamma}{(\zeta^2 - \gamma^2)^{1/4}} \ge \frac{3}{8}\pi\nu \tag{20}$$

for the existence of the limit cycle.[9,10]

We should stress that after the separatrices have formed a loop and the limit cycle appears, the stability of the focus fixed point has not changed and, for some intervals of the parameters in the phase plane, both a stable fixed point and a stable limit cycle coexist. However, when the magnitude of the fraction γ^2/ζ^2 reaches one, stable and unstable fixed points merge and disappear. After such a crisis in the phase space, only rotational motion remains, indicating that the initial system is asynchronous.

Comparison of the results for the stability of the equilibrium states of system (4) [using assumption (6)] shows good agreement with the results of numerical studies of system (4). Indeed, in the parameter interval where equilibrium condition (6) makes sense, system (4) possesses a pair of fixed points (one of them is stable) with very nearby values for their stationary amplitudes (see Fig. 1). Each of the fixed points has one negative eigenvalue. The two other eigenvalues for the stable point are complex conjugates (focus) and, for the unstable point, real and of different sign (saddle). At a certain parameter point, both equilibrium points merge and disappear.

Such a qualitative confirmation of the results of the theoretical analysis leads us to expect good quantitative

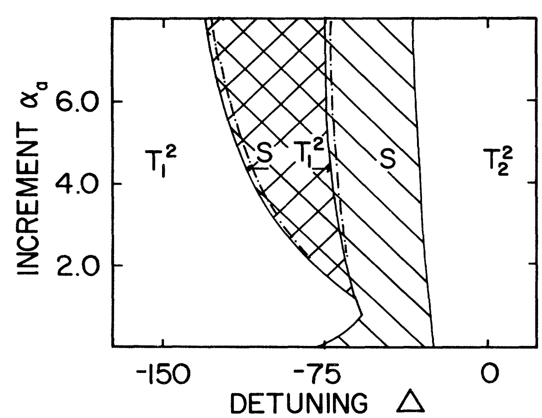

FIG. 2. The bifurcation portrait for system (4) obtained numerically, for parameter values $\alpha_b = 1$, $\gamma_a = 1$, $\gamma_b = 1$, $k = 1$, $\beta = 0$, and $\kappa = 50$. The region of synchronous oscillation (hatched area) and the region where one- and two-frequency regimes compete (double hatched area) are illustrated. The dot-dashed lines show boundaries of the hysteresis region predicted analytically. S denotes synchronous oscillations, a stable fixed point in the phase space of system (4), and $T_{1,2}^2$ denotes asynchronous oscillations, two-dimensional tori in the phase space of system (4). The arrows indicate the boundaries of the hysteresis region.

agreement as well. In particular, condition (20) should define the region in the parameter space where, in the initial system, stable synchronous oscillations compete with pulsations. The collapsing of stable and unstable equilibrium states by exiting the region

$$\zeta^2 \geq \gamma^2 \tag{21}$$

defines the boundary between synchronous and asynchronous motion of the two oscillators.

Figure 2 shows the bifurcation portrait of system (4) in the parameter plane (Δ, α_a) obtained numerically. The region where synchronous oscillations exist is hatched. When moving out of this region by decreasing $|\Delta|$, pulsations emerge gradually as a result of Hopf bifurcation. Increasing $|\Delta|$ leads to a sudden (hard) excitation of beats. When returning back to the synchronization region by decreasing $|\Delta|$, the disappearance of these beats shows hysteresis. This corresponds to the final dimension of the limit cycle losing stability when condition (20) is not satisfied. The hysteresis region of one- and two-frequency oscillations, where synchronous and asynchronous regimes compete, is marked by double hatching. Curves corresponding to the analytically determined boundaries of the hysteresis region, condition (20) and condition (21), are drawn as a dot-dashed line. The latter demonstrates very good agreement with the numerical results. Furthermore, our numerical experiments indicate that the hysteresis of the different oscillation regimes described above is a robust phenomenon, persisting even when factors such as nonisochronism of both oscillators ($\beta \neq 0$) and influence of nonresonant mechanisms of the interactions between the oscillators are included.

To summarize, we have demonstrated that the competition between synchronous and asynchronous motion in a system of two coupled oscillators may be caused by the existence in phase space of two neighboring fixed points, one of which is unstable. Changes of system parameters can lead to closure of separatrices and to the appearance of a limit cycle, which coexists with a stable fixed point.

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