PHYSICAL REVIEW A VOLUME 43, NUMBER 10 15 MAY 1991

Extinction of sound by spherical scatterers in a viscous fluid

Mark K. Hinders*
Mechanics Research Laboratory, Boston University College of Engineering, 110 Cummington Street, Boston, Massachusetts 02215
(Received 9 August 1990; revised manuscript received 28 January 1991)

The extinction of sound waves in a viscous fluid due to scattering from viscous fluid and solid elastic spheres is investigated. Exact expressions for scattered and transmitted fields caused by an incident-plane compressional wave of unit amplitude are calculated analytically with both compressional and shear modes supported in the viscous fluid and solid elastic media. Furthermore, it is shown that by boundary coupling, incident compressional acoustic waves generate compressional and shear waves in the scattered and transmitted fields. The general expressions for scattering and extinction cross sections are given, and for the special cases of vanishing viscosity, agreement with well-known solutions is obtained. In the Rayleigh-wave range for small scatterers, simple expressions for the cross sections are obtained, and for lossy scattering, absorption phenomena are shown to dominate scattering phenomena in this range. Finally, the extinction coefficient for a dilute cloud of scatterers is derived and the extinction effects due to viscosity are discussed.

I. INTRODUCTION

The scattering of sound waves by spherical obstacles dates to the works of Rayleigh[1] and Lamb,[2] who considered fixed and movable rigid scatterers, respectively, while the exact solutions for the scattering of plane acoustic waves from inviscid fluid[3] and elastic[4] spheres in inviscid fluids have been found more recently. The inclusion of viscosity in the fluid model was made by Sewell,[5] who considered the small scatterer to be rigid, and Herzfeld,[6] who added the elasticity of the small scatterer. Viscosity greatly complicates the analysis because the fluid medium can then support shear as well as compressional modes, both of which must be accounted for in satisfying the boundary conditions on the scatterer. Other authors have considered models accounting for thermal losses[7-10] and more complicated boundary or scatterer models.[11-14] In the present work, the scattering of plane acoustic waves from a spherical obstacle in a viscous fluid is investigated with the method of Herzfeld (normally referred to as Mie scattering after Mie[15] who is credited with the solution for the corresponding electromagnetic problem), but no restriction is placed on the size of the scatterer, which can be a fluid or a solid sphere. In Sec. II a general solution of the vector differential equation in terms of scalar generating functions is derived. Expansion of incident-plane longitudinal waves is given, and proper boundary conditions for the problem are discussed. Scattering of sound waves by a fluid sphere is analyzed in Sec. III. Important scattering quantities such as intensity, differential, and total cross sections of the scattered field are obtained. Limiting cases for vanishing viscosity η are derived and agreement with known pure acoustic solutions is established. Finally, the extinction coefficient for a cloud of viscous fluid scatterers in a viscous fluid is derived. Section IV then contains similar discussions to those of Sec. III, but for solid scatterers. In Sec. V, we give the approximate solutions for the range of Rayleigh scattering and discuss the results.

II. FLUID MODEL

We assume the fluid medium in which the scatterer is embedded extends to infinity, and both the medium surrounding the scatterer as well as the fluid sphere are described by the same set of equations. We shall assign subscripts 1 and 2 to the parameters of the media outside and inside the scatterer. Following Herzfeld, we write the equation of motion for harmonic time variation $e^{-i\omega t}$ in the viscous fluid as

$$(\nabla^2+K^2)\mathbf{v}-\left[1-\frac{K^2}{k^2}\right]\nabla(\nabla\cdot\mathbf{v})=\mathbf{0}\ . \tag{1}$$

Here $\mathbf{v}$ is the perturbation in the fluid velocity due to the acoustic field and $K=\omega/C_f$ and $k=\omega/c_f$ are the viscous-fluid propagation constants for shear and compressional modes with

$$C_f^2=-i\omega\eta/\rho,\quad c_f^2=(1/\kappa-2i\omega\eta)/\rho \tag{2}$$

defining the shear and compressional wave propagation velocities, respectively. In these expressions $\rho, 1/\kappa, \eta$ are the density, compressibility, and the coefficient viscosity of the fluid.

Compressional and shear waves can be immediately separated by representing $\mathbf{v}=\mathbf{v}_C+\mathbf{v}_S$, of which the former satisfies $\nabla\times\mathbf{v}_C=\mathbf{0}$ and the latter satisfies $\nabla\cdot\mathbf{v}_S=0$. The separation of the two waves in this linear representation indicates that shear and compressional waves propagate independently. However, this is no longer true for higher approximations where it is expected that the two types of modes are coupled.

In order to obtain the acoustic solution, the divergence of both sides of Eq. (1) is taken, so that

$$(\nabla^2+k^2)\nabla\cdot\mathbf{v}_C=0\ . \tag{3}$$

Considering that $\nabla\cdot\mathbf{v}_C=\pi_C$ is determined from (3), a particular solution (compressional wave solution) $\mathbf{v}_C$ of Eq. (1) is

$$\mathbf{v}_C=\frac{-1}{k^2}\nabla\pi_C\ . \tag{4}$$

A more complete solution of Eq. (2) is obtained by adding the complementary solution (shear-wave solution) $\mathbf{v}_S$ of equations

$$(\nabla^2+K^2)\mathbf{v}_S=0,\quad \nabla\cdot\mathbf{v}_S=0\ . \tag{5}$$

The divergenceless nature of the vector $\mathbf{v}_S$ is ensured by writing a solution in the form $\nabla\times\nabla\times\mathbf{a}\pi_S$ where π_S satisfies

$$(\nabla^2+K^2)\pi_S=0\ , \tag{6}$$

$\mathbf{a}$ is a constant vector, and in spherical coordinates the position vector can be substituted for $\mathbf{a}$. Therefore, the shear-wave solution can be represented by

$$\mathbf{v}_S=\frac{1}{K}\nabla\times\nabla\times\mathbf{r}\pi_S\ . \tag{7}$$

We are now able to write the components of the velocity vector in terms of scalar generating functions π_C,π_S:

$$\begin{aligned}
v_r&=v_r^C+v_r^S\\
&=-\frac{1}{(kr)^2}\left[r\frac{\partial}{\partial r}-1\right](r\pi_C)+\frac{1}{Kr^2}l(l+1)(r\pi_S)\ ,\\
v_\theta&=v_\theta^C+v_\theta^S\\
&=-\frac{1}{(kr)^2}\frac{\partial}{\partial\theta}(r\pi_C)+\frac{1}{Kr}\frac{\partial}{\partial r}\frac{\partial}{\partial\theta}(r\pi_S)\ ,\\
v_\phi&=v_\phi^C+v_\phi^S\\
&=-\frac{1}{(kr)^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(r\pi_C)+\frac{1}{Kr}\frac{1}{\sin\theta}\frac{\partial}{\partial r}\frac{\partial}{\partial\phi}(r\pi_S)\ .
\end{aligned} \tag{8}$$

The stress components can be found from

$$\begin{aligned}
\sigma_{rr}&=\frac{i}{\omega\kappa}\nabla\cdot\mathbf{v}+2\eta\frac{\partial}{\partial r}v_r\ ,\\
\sigma_{r\theta}&=\eta\left[\frac{\partial}{\partial r}v_\theta-\frac{v_\theta}{r}+\frac{1}{r}\frac{\partial}{\partial\theta}v_r\right]\ ,
\end{aligned} \tag{9}$$

and

$$\nabla\cdot\mathbf{v}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2v_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}v_\phi\ .$$

The solutions of the scalar wave equation in spherical coordinates are known,[16] one can write

$$r\pi_C=z_l(kr)Y_{lm}^C(\theta,\phi),\quad r\pi_S=z_l(Kr)Y_{lm}^S(\theta,\phi)\ , \tag{10}$$

where $z_l(x)=\sqrt{\pi x/2}\,Z_{l+1/2}(x)$ is the spherical radial function corresponding to the half-order cylindrical Bessel $J_{l+1/2}(x)$ or Hankel $H_{l+1/2}^{(1)}(x)$ functions. $Y_{lm}(\theta,\phi)$ is the tesseral surface harmonic defined as $Y_{lm}(\theta,\phi)=(a_{lm}\cos m\phi+b_{lm}\sin m\phi)P_l^{(m)}(\cos\theta)$. Here $P_l^{(m)}(\cos\theta)$ is the associated Legendre function.

A. Expansion of vector plane wave

In order to study the scattering and attenuation due to scattering with the method of Herzfeld and Mie, an expansion of the incident wave in terms of the spherical wave functions must be given. We take a rectangular system of coordinates with the origin at the center of the sphere with the z direction as the direction of propagation of the incident wave. For an incident-plane compressional wave of unit amplitude the directions of displacement and propagation are both the z direction. The spherical components of the incident wave $\mathbf{v}=\hat{z}e^{ik_1z}$ are therefore

$$\begin{aligned}
v_r^i&=e^{ik_1r\cos\theta}\cos\theta\ ,\\
v_\theta^i&=e^{ik_1r\cos\theta}\sin\theta\ ,\\
v_\phi^i&=0\ .
\end{aligned} \tag{11}$$

We may rewrite these in terms of the potential $\pi_C^i=ike^{ik_1r\cos\theta}$ as

$$\begin{aligned}
v_r^i&=-\frac{1}{k_1^2}\left[\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right](r\pi_C^i)\ ,\\
v_\theta^i&=-\frac{1}{k_1^2}\frac{1}{r^2}\frac{\partial}{\partial\theta}(r\pi_C^i)\ ,\\
v_\phi^i&=-\frac{1}{k_1^2}\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\phi}(r\pi_C^i)\ ,
\end{aligned} \tag{12}$$

which agree with (11).

Using Bauer's formula,

$$e^{ikr\cos\theta}=\sum_{l=0}^{\infty}i^l(2l+1)\frac{z_l(kr)}{(kr)}P_l(\cos\theta)\ , \tag{13}$$

one can write

$$r\pi_C^i=\sum_{l=0}^{\infty}i^{l+1}(2l+1)z_l(k_1r)P_l(\cos\theta)\ , \tag{14}$$

where $z_l(kr)$ is a spherical radial function for a solid sphere, and for the incident wave will be the Ricatti-Bessel function.

B. Boundary conditions

Conditions that hold at a surface separating two fluid media are easily derivable. At such a surface the requirement that the two media remain in contact leads to the conclusion that the velocities must be continuous across the boundary, and the equilibrium of an arbitrary volume which encloses portions of both media leads to the continuity of stress across the spherical surface.

The boundary conditions for stresses then will be written as

$$\sigma_{rr}^{t(2)}=\sigma_{rr}^{i(1)}+\sigma_{rr}^{s(1)},\quad \sigma_{r\theta}^{t(2)}=\sigma_{r\theta}^{i(1)}+\sigma_{r\theta}^{s(1)}\ . \tag{15}$$

The third $\sigma_{r\phi}^{t(2)}=\sigma_{r\phi}^{i(1)}+\sigma_{r\phi}^{s(1)}$ is redundant and will be dropped. The boundary conditions for velocities are

$$v_r^{t(2)}=v_r^{i(1)}+v_r^{s(1)}, \quad v_\theta^{t(2)}=v_\theta^{i(1)}+v_\theta^{s(1)} , \tag{16}$$

where the third $v_\phi^{t(2)}=v_\phi^{i(1)}+v_\phi^{s(1)}$ is also redundant and will be dropped.

III. SCATTERING BY A VISCOUS-FLUID SPHERE

When a purely compressional plane wave of unit amplitude and with wave number k_1 is incident on the surface of the spherical viscous-fluid scatterer, scattered as well as transmitted longitudinal and vertical shear modes will be generated. (Although k_1 is actually complex and will give a damped plane wave, for many real fluids the imaginary part of this compressional wave number will be small enough that we can neglect it in order to consider the propagation of a plane acoustic wave of unit amplitude, neglecting the small damping.[5]) We set the following expressions for scalar potentials of incident, scattered, and transmitted waves:

$$r\pi_C^i=\sum_{l=0}^{\infty} i^{l+1}(2l+1)\psi_l(k_1 r)P_l(\cos\theta) ,$$

$$r\pi_C^s=\sum_{l=0}^{\infty} i^{l+1}(2l+1)\left[\frac{\Delta_1^f}{\Delta_0^f}\right]\zeta_l(k_1 r)P_l(\cos\theta) ,$$

$$r\pi_C^t=\sum_{l=0}^{\infty} i^{l+1}(2l+1)\left[\frac{\Delta_2^f}{\Delta_0^f}\right]\psi_l(k_2 r)P_l(\cos\theta) , \tag{17}$$

$$r\pi_S^s=\sum_{l=0}^{\infty} i^{l+1}(2l+1)\left[\frac{\Delta_3^f}{\Delta_0^f}\right]\zeta_l(K_1 r)P_l(\cos\theta) ,$$

$$r\pi_S^t=\sum_{l=0}^{\infty} i^{l+1}(2l+1)\left[\frac{\Delta_4^f}{\Delta_0^f}\right]\psi_l(K_2 r)P_l(\cos\theta) ,$$

where the Ricatti-Bessel and Ricatti-Hankel functions are defined as

$$\psi_l(x)=\sqrt{x\pi/2}\,J_{l+1/2}(x), \quad \zeta_l(x)=\sqrt{x\pi/2}\,H_{l+1/2}^{(1)}(x)$$

with $J_{l+1/2}(x)$ and $H_{l+1/2}^{(1)}(x)$ the half-order cylindrical Bessel and Hankel functions. The five quantities $\Delta_0^f-\Delta_4^f$ are the unknown model coefficients to be determined.

Using the boundary conditions (15) and (16), and the expressions in Eqs. (8) and (9), four sets of equations will be set in order. By using the identities

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}(\cdot)\right]+\frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2}(\cdot)=-l(l+1)(\cdot) ,$$
$$\frac{\partial^2}{\partial r^2}(\cdot)+\left[\begin{matrix}k^2\\ K^2\end{matrix}\right](\cdot)=\frac{l(l+1)}{r^2}(\cdot) \tag{18}$$

for $(\cdot)$ any of the $r\pi$ in these four equations can be written in a matrix form (see the right-hand column of this page)

$$\left[\begin{matrix}
\frac{-1}{2r}\left[\frac{K_1}{k_1}\right]^2+\frac{l(l+1)+2}{k_1^2r^3}-\frac{2\partial_r}{(k_1r)^2} & \frac{\eta_2}{\eta_1}\left[\frac{1}{2r}\left[\frac{K_2}{k_2}\right]^2-\frac{l(l+1)+2}{k_2^2r^3}+\frac{2\partial_r}{(k_2r)^2}\right] & -\frac{l(l+1)}{K_1r^2}\left[\partial_r-\frac{2}{r}\right] & \frac{\eta_2}{\eta_1}\frac{l(l+1)}{K_2r^2}\left[\partial_r-\frac{2}{r}\right] \\
-\frac{4}{k_1^2r^3}+\frac{2\partial_r}{(k_1r)^2} & \frac{\eta_2}{\eta_1}\left[\frac{4}{k_2^2r^3}-\frac{2\partial_r}{(k_2r)^2}\right] & -\frac{2l(l+1)}{K_1r^3}+\frac{2\partial_r}{K_1r^2}+\frac{K_1}{r} & \frac{\eta_2}{\eta_1}\left[\frac{2l(l+1)}{K_2r^3}-\frac{2\partial_r}{K_2r^2}-\frac{K_2}{r}\right] \\
-\frac{1}{(k_1r)^2}(1-r\partial_r) & \frac{1}{(k_2r)^2}(1-r\partial_r) & -\frac{l(l+1)}{K_1r^2} & \frac{l(l+1)}{K_2r^2} \\
\frac{1}{(k_1r)^2} & -\frac{1}{(k_2r)^2} & -\frac{\partial_r}{K_1r} & \frac{\partial_r}{K_2r}
\end{matrix}\right]
\left[\begin{matrix}(r\pi_C^s)\\ (r\pi_C^t)\\ (r\pi_S^s)\\ (r\pi_S^t)\end{matrix}\right]
=-\left[\begin{matrix}
\frac{1}{2r}\left[\frac{K_1}{k_1}\right]^2-\frac{l(l+1)+2}{k_1^2r^3}+\frac{2\partial_r}{(k_1r)^2} \\
\frac{4}{k_1^2r^3}-\frac{2\partial_r}{(k_1r)^2} \\
\frac{1}{(k_1r)^2}(1-r\partial_r) \\
\frac{1}{(k_1r)^2}
\end{matrix}\right](r\pi_C^i) . \tag{19}$$

In these equations, it is understood that after differentiation r must be substituted by the boundary radius $r=a$. After necessary manipulations we find

$$\begin{aligned}
\Delta_0^f={}&\left[\frac{\eta_2}{\eta_1}-1\right]^2[l(l+1)-2]\left[\frac{k_2aj_l'(k_2a)}{j_l(k_2a)}\frac{K_2a\psi_l'(K_2a)}{\psi_l(K_2a)}-l(l+1)\right]\left[\frac{k_1ah_l'(k_1a)}{h_l(k_1a)}\frac{K_1a\zeta_l'(K_1a)}{\zeta_l(K_1a)}-l(l+1)\right]\\
&+\tfrac{1}{2}(K_1a)^2\left[\frac{\eta_2}{\eta_1}-1\right]\Bigg[\left[\frac{k_2aj_l'(k_2a)}{j_l(k_2a)}\frac{K_2a\psi_l'(K_2a)}{\psi_l(K_2a)}-l(l+1)\right]\left[\frac{K_1a\zeta_l'(K_1a)}{\zeta_l(K_1a)}+2\frac{k_1ah_l'(k_1a)}{h_l(k_1a)}-2l(l+1)\right]\\
&\qquad-\frac{\rho_2}{\rho_1}\left[\frac{k_1ah_l'(k_1a)}{h_l(k_1a)}\frac{K_1a\zeta_l'(K_1a)}{\zeta_l(K_1a)}-l(l+1)\right]\left[\frac{K_2a\psi_l'(K_2a)}{\psi_l(K_2a)}+2\frac{k_2aj_l'(k_2a)}{j_l(k_2a)}-2l(l+1)\right]\Bigg]\\
&+\tfrac{1}{4}(K_1a)^4\left[l(l+1)\left[1-\frac{\rho_2}{\rho_1}\right]^2-\left[\frac{k_2aj_l'(k_2a)}{j_l(k_2a)}-\frac{\rho_2}{\rho_l}\frac{k_1ah_l'(k_1a)}{h_l(k_1a)}\right]\left[\frac{K_2a\psi_l'(K_2a)}{\psi_l(K_2a)}-\frac{\rho_2}{\rho_1}\frac{K_1a\zeta_l'(K_1a)}{\zeta_l(K_1a)}\right]\right], \qquad (20a)
\end{aligned}$$

$$\begin{aligned}
\Delta_l^f={}&\frac{j_l(k_1a)}{h_l(k_1a)}\Bigg\{\left[\frac{\eta_2}{\eta_1}-1\right]^2[l(l+1)-2]\left[\frac{k_2aj_l'(k_2a)}{j_l(k_2a)}\frac{K_2a\psi_l'(K_2a)}{\psi_l(K_2a)}-l(l+1)\right]\left[\frac{k_1aj_l'(k_1a)}{j_l(k_1a)}\frac{K_1a\zeta_l'(K_1a)}{\zeta_l(K_1a)}-l(l+1)\right]\\
&+\tfrac{1}{2}(K_1a)^2\left[\frac{\eta_2}{\eta_1}-1\right]\Bigg[\left[\frac{k_2aj_l'(k_2a)}{j_l(k_2a)}\frac{K_2a\psi_l'(K_2a)}{\psi_l(K_2a)}-l(l+1)\right]\left[\frac{K_1a\zeta_l'(K_1a)}{\zeta_l(K_1a)}+2\frac{k_1aj_l'(k_1a)}{j_l(k_1a)}-2l(l+1)\right]\\
&\qquad-\frac{\rho_2}{\rho_1}\left[\frac{k_1aj_l'(k_1a)}{j_l(k_1a)}\frac{K_1a\zeta_l'(K_1a)}{\zeta_l(K_1a)}-l(l+1)\right]\left[\frac{K_2a\psi_l'(K_2a)}{\psi_l(K_2a)}+2\frac{k_2aj_l'(k_2a)}{j_l(k_2a)}-2l(l+1)\right]\Bigg]\\
&+\tfrac{1}{4}(K_1a)^4\Bigg[l(l+1)\left[1-\frac{\rho_2}{\rho_1}\right]^2\\
&\qquad-\left[\frac{k_2aj_l'(k_2a)}{j_l(k_2a)}-\frac{\rho_2}{\rho_1}\frac{k_1aj_l'(k_1a)}{j_l(k_1a)}\right]\left[\frac{K_2a\psi_l'(K_2a)}{\psi_l(K_2a)}-\frac{\rho_2}{\rho_1}\frac{K_1a\zeta_l'(K_1a)}{\zeta_l(K_1a)}\right]\Bigg]\Bigg\}, \qquad (20b)
\end{aligned}$$

$$\begin{aligned}
\Delta_2^f={}&\frac{j_l(k_1a)}{j_l(k_2a)}\left[\frac{k_1ah_l'(k_1a)}{h_l(k_1a)}-\frac{k_1aj_l'(k_1a)}{j_l(k_1a)}\right]\\
&\times\Bigg\{\frac{1}{4}(K_1a)^4\left[\frac{k_2a\psi_l'(K_2a)}{\psi_l(K_2a)}-\frac{\rho_2}{\rho_1}\frac{K_1a\zeta_l'(K_1a)}{\zeta_l(K_1a)}\right]\\
&\quad+\tfrac{1}{2}(K_1a)^2\left[\frac{\eta_2}{\eta_1}-1\right]\left[\frac{K_1a\zeta_l'(K_1a)}{\zeta_l(K_1a)}\frac{K_2a\psi_l'(K_2a)}{\psi_l(K_2a)}-l(l+1)\left[\frac{K_1a\zeta_l'(K_1a)}{\zeta_l(K_1a)}+\frac{K_2a\psi_l'(K_2a)}{\psi_l(K_2a)}-2\right]\right]\Bigg\}, \qquad (20c)
\end{aligned}$$

$$\begin{aligned}
\Delta_3^f={}&\frac{K_1}{k_1}\frac{j_l(k_1a)}{h_l(K_1a)}\left[\frac{k_1aj_l'(k_1a)}{j_l(k_1a)}-\frac{k_1ah_l'(k_1a)}{h_l(k_1a)}\right]\\
&\times\Bigg\{\frac{1}{4}(K_1a)^4\left[\frac{\eta_2}{\eta_1}-1\right]\frac{\rho_2}{\rho_1}\left[1-\frac{\rho_2}{\rho_2}\right]\\
&\quad+\tfrac{1}{2}(K_1a)^2\left[\frac{\eta_2}{\eta_1}-1\right]\Bigg[\left[\frac{k_2aj_l'(k_2a)}{j_l(k_2a)}\frac{K_2a\psi_l'(K_2a)}{\psi_l(K_2a)}-l(l+1)\right]\\
&\qquad-\frac{\rho_2}{\rho_1}\left[\frac{K_2a\psi_l'(K_2a)}{\psi_l(K_2a)}+2\frac{k_2aj_l'(k_2a)}{j_l(k_2a)}-2l(l+1)\right]\Bigg]\\
&\quad+\left[\frac{\eta_2}{\eta_1}-1\right]^2[l(l+1)-2]\left[\frac{k_2aj_l'(k_2a)}{j_l(k_2a)}\frac{K_2a\psi_l'(K_2a)}{\psi_l(k_2a)}-l(l+1)\right]\Bigg\}, \qquad (20d)
\end{aligned}$$

$$\Delta_4^f = -\frac{K_2}{k_1}\frac{j_l(k_1a)}{j_l(K_2a)}\left[\frac{k_1aj_l'(k_1a)}{j_l(k_1a)} - \frac{k_1ah_l'(k_1a)}{h_l(k_1a)}\right]$$
$$\times\left\{\frac{1}{4}(K_1a)^4\left[1-\frac{\rho_2}{\rho_1}\right]+\tfrac{1}{2}(K_1a)^2\left[\frac{\eta_2}{\eta_1}-1\right]\left[l(l+1)-2+\left[\frac{K_2a\psi_l'(K_2a)}{\psi_l(K_2a)}-1\right]\left[\frac{K_1a\zeta_l'(K_1a)}{\zeta_l(K_1a)}-2\right]\right]\right\}. \tag{20e}$$

Now, we are able to define the scalar scattering potential functions by using the modal coefficients. The scattered and transmitted field components with known potentials $r\pi_C^s, r\pi_S^s$ and $r\pi_C^t, r\pi_S^t$ can easily be obtained from the above equations. Note that the C and S modes are coupled through and at the boundary, even though they propagate independently. The boundary plays the role of a conversion mechanism: the energy of the incident compressional modes is converted to both scattered shear and compressional modes. Upon the reflection of the incident compressional wave on the sphere C and S waves are generated in the scattered field. Therefore, scattered energy will be propagated in terms of C and S waves.

We also note that Δ_0^f is the expression which describes the natural oscillations of a viscous fluid sphere imbedded in an infinitely extended viscous fluid medium of different material properties. In the absence of the incident wave, the right-hand column matrix of (19) disappears, and the left-hand side represents a homogeneous equation for scalar potentials. For all nonzero values of the amplitude parameters, the determinant must vanish. Therefore, a coupled equation for compressional and shear types of oscillations can be obtained. In the case of the oscillation problem, this coupled equation indicates that energy can be converted from compressional waves to the shear waves and vice versa. Oscillations are of purely dilatational type only when $l=0$, which is expected since $l=0$ is the only mode where no transverse wave types exist. For $l\neq 0$ oscillations are coupled, and $\Delta_0^f=0$ is the general expression which describes coupled oscillations of compressional and shear wave modes.

If $\Delta_1^f=0$ or $\Delta_3^f=0$ while Δ_0^f is nonzero the amplitudes of the corresponding scattered C or S waves will be zero, respectively. This indicates that for certain discrete sets of frequencies, either scattered C or scattered S waves vanish. Hence there are also certain discrete sets of frequencies for which the transmitted C or S waves vanish. Moreover, whenever the frequency of the incident wave approaches a characteristic frequency which makes Δ_0^f itself vanish, resonant phenomena will occur. However, the incident frequency is real and the characteristic frequencies are in general complex, so that in reality Δ_0^f can be reduced to a minimum value but never quite to zero, so that the maximum amplitudes at resonance will be finite, not infinite.

A. Scattering cross sections

The total flow of scattered acoustic energy in the radial direction through a closed surface may be represented correctly by the radial component of the energy flux vector which may be decomposed into compressional and shear parts as

$$\mathcal{F}_j = \sigma_{ij}^C v_i^C + \sigma_{ij}^S v_i^S .$$

Significant quantities are intensities, differential cross sections, and total cross sections of the scattered field. Intensities for each mode represent the radiation per unit solid angle, and they are defined as the time-averaged radial component of the far-field energy flux vector:

$$I_{\text{scat}}^C = \frac{\omega^3\rho_1/2}{(k_1r)^3}\sum_{l=0}^{\infty}(2l+1)^2\left|\frac{\Delta_1^f}{\Delta_0^f}\right|[P_l(\cos\theta)]^2$$
$$I_{\text{scat}}^2 = \frac{\omega^3\rho_1/2}{(k_1r)^3}\left|\frac{k_1}{K_1}\right|^3\sum_{l=0}^{\infty}(2l+1)^2\left|\frac{\Delta_3^f}{\Delta_0^f}\right|^2 \times\left[\frac{\partial}{\partial\theta}P_l(\cos\theta)\right]^2, \tag{21}$$
$$I_{\text{inc}}^C = \frac{\omega^3\rho_1/2}{k_1r},$$

where I_{scat}^C, I_{scat}^S, and I_{inc}^S represent the intensities of the scattered C-type waves, S-type waves, and the intensity of the incident C wave which is propagating in the z direction. It is noted that the scattered waves are independent of the polarization angle.

Dividing the intensity of the scattered wave by the intensity of the incident wave, we obtain the differential scattering cross sections of the compressional and shear waves:

$$\frac{d\sigma^C}{d\Omega} = \frac{1}{(k_1r)^2}\sum_{l=0}^{\infty}(2l+1)^2\left|\frac{\Delta_1^f}{\Delta_0^f}\right|^2[P_l(\cos\theta)]^2,$$
$$\frac{d\sigma^S}{d\Omega} = \frac{1}{(k_1r)^2}\left|\frac{k_1}{K_1}\right|^3\sum_{l=1}^{\infty}(2l+1)^2\left|\frac{\Delta_e^f}{\Delta_0^f}\right|^2 \times\left[\frac{\partial}{\partial\theta}P_l(\cos\theta)\right]^2. \tag{22}$$

Integrating these expressions over a spherical surface with radius r, we obtain the total scattering cross sections

$$\sigma^C_{\rm scat}=\frac{4\pi r^2}{(k_1 r)^2}\sum_{l=0}^{\infty}(2l+1)\left|\frac{\Delta_1^f}{\Delta_0^f}\right|^2 ,$$
$$\sigma^S_{\rm scat}=\frac{4\pi r^2}{(k_1 r)^2}\left|\frac{k_1}{K_1}\right|^3\sum_{l=1}^{\infty}(2l+1)l(l+1)\left|\frac{\Delta_3^f}{\Delta_0^f}\right|^2 . \quad (23)$$

We note that there is no $l=0$ mode in the shear wave scattering cross section indicating that $l=1$ is the lowest mode which both shear and compressional waves radiate in the scattered field. Other important quantities are the extinction and absorption cross section. By a known procedure,[17] they can be obtained:

$$\sigma_{\rm ext}=\frac{4\pi r^2}{(k_1 r)^2}\sum_{l=0}^{\infty}(2l+1)\mathcal{R}\left[\frac{\Delta_1^f}{\Delta_0^f}\right] , \quad (24)$$

and by definition $\sigma_{\rm abs}=\sigma_{\rm ext}-(\sigma^C_{\rm scat}+\sigma^S_{\rm scat})$. Because the absorption cross section is zero for lossless scattering the extinction cross section is then equal to the sum of the compressional and shear scattering cross sections. Therefore, by knowing $\Delta_0^f-\Delta_4^f$ we are able to calculate the desired scattering quantities.

B. Scattering of an ordinary acoustic wave from an ordinary acoustic sphere

The consistency of the present formulation can be easily checked by comparing the limit where the viscosity of both media vanishes with the known results of classical acoustic theory. If the shear modulus η is set to zero everywhere, the solutions become those of the scattering of (inviscid) acoustic waves from an inviscid fluid sphere, and the problem is no longer a viscous fluid problem. The characteristic coefficients of the scattered wave now reduce to $\Delta_3^f=0$ and

$$\Delta_0^f=\frac{k_2 a j_l'(k_2 a)}{j_l(k_2 a)}-\frac{\rho_2}{\rho_1}\frac{k_1 a h_l'(k_1 a)}{h_l(k_1 a)} ,$$
$$\Delta_1^f=\frac{j_l(k_1 a)}{h_l(k_1 a)}\left[\frac{k_2 a j_l'(k_2 a)}{j_l(k_2 a)}-\frac{\rho_2}{\rho_1}\frac{k_1 a j_l'(k_1 a)}{j_l(k_1 a)}\right] , \quad (25)$$

which are precisely the coefficients for the scattering problem of acoustic waves from a fluid bubble.[3]

C. Extinction coefficient for cloud of scatterers

The extinction cross section derived for a single sphere represents the fraction of the incident energy which is removed from the plane wave by both scattering and absorption phenomena. This single-sphere result, however, can be applied directly to the consideration of the extinction of an incident plane wave as it passes through a cloud of many spheres. As long as the spheres are identical in composition, randomly distributed, and their volume concentration is small enough that they can be considered to scatter independently, the energy removed from the incident wave by N scatterers is merely N times the energy removed by one.[17]

The plane-wave extinction coefficient α is defined as the energy decay coefficient as the wave passes through the medium. We can then write the energy as[5]

$$E(z)=E_0 e^{-\alpha z} \quad (26)$$

so that $1/\alpha$ gives the distance traveled by the sound before its intensity is diminished in the ratio $1/e$. The extinction coefficient is simply related to the extinction cross section by

$$\alpha=\frac{3}{4}\frac{c}{a}\frac{\sigma_{\rm ext}}{\pi a^2} , \quad (27)$$

where c,a are the volume concentration and common radius of the scatterers. This formula along with (24) and (20) allow the extinction of sound by a cloud of viscous fluid spheres in a viscous fluid to be determined.

IV. SCATTERING BY A SOLID ELASTIC SPHERE

If the spherical scatterer is a solid elastic sphere the formulation of the previous section can still be applied since (1) holds for the elastic medium if we define shear and compressional wave propagation velocities as

$$C_e^2=\mu/\rho, \quad c_e^2=(\lambda+2\mu)/\rho , \quad (28)$$

where ρ is the density of the elastic medium and λ,μ are the Lame parameters. As in (17) we set the following expressions for the scalar potentials of the incident, scattered, and transmitted fields:

$$r\pi_C^i=\sum_{l=0}^{\infty}i^{l+1}(2l+1)\psi_l(k_1 r)P_l(\cos\theta) ,$$
$$r\pi_C^3=\sum_{l=0}^{\infty}i^{l+1}(2l+1)\left[\frac{\Delta_1^e}{\Delta_0^e}\right]\zeta_l(k_1 r)P_l(\cos\theta) ,$$
$$r\pi_C^t=\sum_{l=0}^{\infty}i^{l+1}(2l+1)\left[\frac{\Delta_2^e}{\Delta_0^e}\right]\psi_l(k_2 r)P_l(\cos\theta) , \quad (29)$$
$$r\pi_S^s=\sum_{l=0}^{\infty}i^{l+1}(2l+1)\left[\frac{\Delta_3^e}{\Delta_0^e}\right]\zeta_l(K_1 r)P_l(\cos\theta) ,$$
$$r\pi_S^t=\sum_{l=0}^{\infty}i^{l+1}(2l+1)\left[\frac{\Delta_4^e}{\Delta_0^e}\right]\psi_l(K_2 r)P_l(\cos\theta) ,$$

where $\Delta_0^e-\Delta_4^e$ are five unknown modal coefficients to be determined for the elastic scatterer in a viscous fluid. Boundary conditions remain as in (15) and (16), but with the stress components in the elastic sphere given from

$$\sigma_{rr}^t=\frac{i\lambda_2}{\omega}\nabla\cdot\mathbf{v}+2\frac{i\mu_2}{\omega}\frac{\partial}{\partial r}v_r ,$$
$$\sigma_{r\theta}=\frac{i\mu_2}{\omega}\left[\frac{\partial}{\partial r}v_\theta-\frac{1}{r}v_\theta+\frac{1}{r}\frac{\partial}{\partial\theta}v_r\right] . \quad (30)$$

We can then write the four boundary condition equations in a matrix form similar to (19):

$$
\begin{bmatrix}
\frac{-1}{2r}\left[\frac{K_1}{k_1}\right]^2+\frac{l(l+1+2}{k_1^2r^3}-\frac{2\partial_r}{(k_1r)^2} & \frac{i\mu_2}{\omega\eta_1}\left[\frac{1}{2r}\left[\frac{K_2}{k_2}\right]^2-\frac{l(l+1)+2}{k_2^2r^3}+\frac{2\partial_r}{(k_2r)^2}\right] & -\frac{l(l+1)}{K_1r^2}\left[\partial_r-\frac{2}{r}\right] & \frac{i\mu_2}{\omega\eta_1}\frac{l(l+1)}{K_2r^2}\left[\partial_r-\frac{2}{r}\right] \\
-\frac{4}{k_1^2r^3}+\frac{2\partial_r}{(k_1r)^2} & \frac{i\mu_2}{\omega\eta_1}\left[\frac{4}{k_2^2r^3}-\frac{2\partial_r}{(k_2r)^2}\right] & -\frac{2l(l+1)}{K_1r^3}+\frac{2\partial_r}{K_1r^2}+\frac{K_1}{r} & \frac{i\mu_2}{\omega\eta_1}\left[\frac{2l(l+1)}{K_2r^3}-\frac{2\partial_r}{K_2r^2}-\frac{K_2}{r}\right] \\
-\frac{1}{(k_1r)^2}(1-r\partial_2) & \frac{1}{(k_2r)^2}(1-r\partial_r) & -\frac{l(l(l+1)}{K_1r^2} & \frac{l(l+1)}{K_2r^2} \\
\frac{1}{(k_1r)^2} & -\frac{1}{(k_2r)^2} & -\frac{\partial_r}{K_1r} & \frac{\partial_r}{K_2r}
\end{bmatrix}
$$

$$
\times\begin{bmatrix} (r\pi_C^s) \\ (r\pi_C^t) \\ (r\pi_S^s) \\ (r\pi_S^t) \end{bmatrix}
=-\begin{bmatrix}
\frac{1}{2r}\left[\frac{K_1}{k_1}\right]^2-\frac{l(l+1)+2}{k_1^2r^3}+\frac{2\partial_r}{(k_1r)^2} \\
\frac{4}{k_1^2r^3}-\frac{2\partial_r}{(k_1r)^2} \\
\frac{1}{(k_1r)^2}(1-r\partial_r) \\
\frac{1}{(k_1r)^2}
\end{bmatrix}(r\pi_C^i)\,. \tag{31}
$$

The solution is then given by

$$
\begin{aligned}
\Delta_0^e={}&\left[\frac{i\mu_2}{\omega\eta_1}-1\right]^2[l(l+1)-2]\left[\frac{k_2aj'_1(k_2a)}{j_l(k_2a)}\frac{K_2a\psi'_l(K_2a)}{\psi_l(K_2a)}-l(l+1)\right]\left[\frac{k_1ah'_l(k_1a)}{h_l(k_1a)}\frac{K_1a\zeta'_1(K_1a)}{\zeta_l(K_1a)}-l(l+1)\right] \\
&+\tfrac{1}{2}(K_1a)^2\left[\frac{i\mu_2}{\omega\eta_1}-1\right]\Bigg[\left[\frac{k_2aj'_l(k_2a)}{j_l(k_2a)}\frac{K_2a\psi'_l(K_2a)}{\psi_l(K_2a)}-l(l+1)\right]\left[\frac{K_1a\zeta'_l(K_1a)}{\zeta_l(K_1a)}+2\frac{k_1ah'_l(k_1a)}{h_l(k_1a)}-2l(l+1)\right] \\
&\qquad-\frac{\rho_2}{\rho_1}\left[\frac{k_1ah'_l(k_1a)}{h_l(k_1a)}\frac{K_1a\zeta'_l(K_1a)}{\zeta_l(K_1a)}-l(l+1)\right]\left[\frac{K_2a\psi'_l(K_2a)}{\psi_l(K_2a)}+2\frac{k_2aj'_l(k_2a)}{j_l(k_2a)}-2l(l+1)\right]\Bigg] \\
&+\tfrac{1}{4}(K_1a)^4\left[l(l+1)\left[1-\frac{\rho_2}{\rho_1}\right]^2-\left[\frac{k_2aj'_l(k_2a)}{j_l(k_2a)}-\frac{\rho_2}{\rho_1}\frac{k_1ah'_l(k_1a)}{h_l(k_1a)}\right]\left[\frac{K_2a\psi'_l(K_2a)}{\psi_l(K_2a)}-\frac{\rho_2}{\rho_1}\frac{K_1a\zeta'_l(K_1a)}{\zeta_l(K_1a)}\right]\right],
\end{aligned} \tag{32a}
$$

$$
\begin{aligned}
\Delta_1^e={}&\frac{j_l(k_1a)}{h_l(k_1a)}\Bigg\{\left[\frac{i\mu_2}{\omega\eta_1}-1\right]^2[l(l+1)-2]\left[\frac{k_2aj'_l(k_2a)}{j_l(k_2a)}\frac{K_2a\psi'_l(K_2a)}{\psi_l(K_2a)}-l(l+1)\right]\left[\frac{k_1aj'_l(k_1a)}{j_l(k_1a)}\frac{K_1a\zeta'_l(K_1a)}{\zeta_l(K_1a)}-l(l+1)\right] \\
&+\tfrac{1}{2}(k_1a)^2\left[\frac{i\mu_2}{\omega\eta_1}-1\right]\Bigg[\left[\frac{k_2aj'_l(k_2a)}{j_l(k_2a)}\frac{K_2a\psi'_l(K_2a)}{\psi_l(K_2a)}-l(l+1)\right]\left[\frac{K_1a\zeta'_l(K_1a)}{\zeta_l(K_1a)}+2\frac{k_1aj'_l(k_1a)}{j_l(k_1a)}-2l(l+1)\right] \\
&\qquad-\frac{\rho_2}{\rho_1}\left[\frac{k_1aj'_l(k_1a)}{j_l(k_1a)}\frac{K_1a\zeta'_l(K_1a)}{\zeta_l(K_1a)}-l(l+1)\right]\left[\frac{K_2a\psi'_l(K_2a)}{\psi_l(K_2a)}+2\frac{k_2aj'_l(k_2a)}{j_l(k_2a)} \right. \\
&\qquad\qquad \left. -2l(l+1)\right]\Bigg] \\
&+\tfrac{1}{4}(K_1a)^4\left[l(l+1)\left[1-\frac{\rho_2}{\rho_1}\right]^2-\left[\frac{k_2aj'_l(k_2a)}{j_l(k_2a)}-\frac{\rho_2}{\rho_1}\frac{k_1aj'_l(k_1a)}{j_l(k_1a)}\right]\left[\frac{K_2a\psi'_l(K_2a)}{\psi_l(K_2a)}-\frac{\rho_2}{\rho_1}\frac{K_1a\zeta'_l(K_1a)}{\zeta_l(K_1a)}\right]\right]\Bigg\}
\end{aligned} \tag{32b}
$$

$$
\begin{aligned}
\Delta_2^e={}&\frac{j_l(k_la)}{j_l(k_2a)}\left[\frac{k_1ah'_l(k_1a)}{h_l(k_1a)}-\frac{k_1aj'_l(k_1a)}{j_l(k_1a)}\right] \\
&\times\Bigg\{\frac{1}{4}(K_1a)^4\left[\frac{K_2a\psi'_l(K_2a)}{\psi_l(K_2a)}-\frac{\rho_2}{\rho_1}\frac{K_1a\zeta'_l(K_1a)}{\zeta_l(K_1a)}\right] \\
&+\tfrac{1}{2}(K_1a)^2\left[\frac{i\mu_2}{\omega\eta_1}-1\right]\left[\frac{K_1a\zeta'_l(K_1a)}{\zeta_l(K_1a)}\frac{K_2a\psi'_l)K_2a)}{\psi_l(K_2a)}-l(l+1)\left[\frac{K_1a\zeta'_l(K_1a)}{\zeta_l(K_1a)}+\frac{K_2a\psi'_l(K_2a)}{\psi_l(K_2a)}-2\right]\right]\Bigg\},
\end{aligned} \tag{32c}
$$

$$\Delta_3^e=\frac{K_1}{k_1}\frac{j_l(k_1a)}{h_l(K_1a)}\left[\frac{k_1aj_l'(k_1a)}{j_l(k_1a)}-\frac{k_1ah_l'(k_1a)}{h_l(k_1a)}\right]$$
$$\times\left\{\frac{1}{4}(K_1a)^4\left[\frac{i\mu_2}{\omega\eta_1}-1\right]\frac{\rho_2}{\rho_1}\left[1-\frac{\rho_2}{\rho_1}\right]\right.$$
$$+\tfrac{1}{2}(K_1a)^2\left[\frac{i\mu_2}{\omega\eta_1}-1\right]\left[\left[\frac{k_2aj_l'(k_2a)}{j_l(k_2a)}\frac{K_2a\psi_l'(K_2a)}{\psi_l(K_2a)}-l(l+1)\right]\right.$$
$$\left.-\frac{\rho_2}{\rho_1}\left[\frac{K_2a\psi_l'(K_2a)}{\psi_l(K_2a)}+2\frac{k_2aj_l'(k_2a)}{j_l(k_2a)}-2l(l+1)\right]\right]$$
$$\left.+\left[\frac{i\mu_2}{\omega\eta_1}-1\right]^2[l(l+1)-2]\left[\frac{k_2aj_l'(k_2a)}{j_l(k_2a)}\frac{K_2a\psi_l'(K_2a)}{\psi_l(K_2a)}-l(l+1)\right]\right\}, \tag{32d}$$

$$\Delta_4^e=-\frac{K_2}{K_1}\frac{j_l(k_1a)}{j_l(K_2a)}\left[\frac{k_1aj_l'(k_1a)}{j_l(k_1a)}-\frac{k_1ah_l'(k_1a)}{h_l(k_1a)}\right]$$
$$\times\left\{\frac{1}{4}(K_1a)^4\left[1-\frac{\rho_2}{\rho_1}\right]+\tfrac{1}{2}(K_1a)^2\left[\frac{i\mu_2}{\omega\eta_1}-1\right]\left[l(l+1)-2+\left[\frac{K_2a\psi_l'(K_2a)}{\psi_l(K_2a)}-1\right]\left[\frac{K_1a\zeta_l'(K_1a)}{\zeta_l(K_1a)}-2\right]\right]\right\}. \tag{32e}$$

Expressions obtained in the previous section for intensities and cross sections remain valid, but with the correct expressions for Δ_0^e–Δ_4^e inserted. We write the cross sections for clarity

$$\sigma_{\text{scat}}^C=\frac{4\pi r^2}{(k_1r)^2}\sum_{l=0}^{\infty}(2l+1)\left|\frac{\Delta_1^e}{\Delta_0^e}\right|^2,$$
$$\sigma_{\text{scat}}^S=\frac{4\pi r^2}{(k_1r)^2}\left|\frac{k_1}{K_1}\right|^3\sum_{l=1}^{\infty}(2l+1)l(l+1)\left|\frac{\Delta_3^e}{\Delta_0^e}\right|^2, \tag{33}$$

and

$$\sigma_{\text{ext}}=\frac{4\pi r^2}{(k_1r)^2}\sum_{l=0}^{\infty}(2l+1)\mathcal{R}\left[\frac{\Delta_1^e}{\Delta_0^e}\right], \tag{34}$$

and by definition $\sigma_{\text{abs}}=\sigma_{\text{ext}}-(\sigma_{\text{scat}}^C+\sigma_{\text{scat}}^S)$, and again we see that by knowing Δ_0^e–Δ_4^e we are able to calculate the scattering quantities.

Scattering of ordinary acoustic waves from an elastic sphere

If we set the shear modulus η_1 equal to zero in the medium surrounding the sphere, no shear waves will be present in the scattered field and our results reduce to those of the scattering of ordinary acoustic waves from an elastic sphere. In this case $\Delta_3^e=0$ and

$$\Delta_0^e=\tfrac{1}{2}(K_2a)^2\frac{\rho_1}{\rho_2}\left[\frac{k_2aj_l'(k_2a)}{j_l(k_2a)}\frac{K_2a\psi_l'(K_2a)}{\psi_l(K_2a)}-l(l+1)+\tfrac{1}{2}(K_2a)^2\frac{k_2aj_l'(k_2a)}{j_l(k_2a)}\right]$$
$$+\frac{k_1ah_l'(k_1a)}{h_l(k_1a)}\left[\left[\frac{k_2aj_l'(k_2a)}{j_l(k_2a)}\frac{K_2a\psi_l'(K_2a)}{\psi_l(K_2a)}-l(l+1)\right][2-l(l+1)]\right.$$
$$\left.+\tfrac{1}{2}(K_2a)^2\left[\frac{K_2a\psi_l'(K_2a)}{\psi_l(K_2a)}+2\frac{k_2aj_l'(k_2a)}{j_l(k_2a)}-2l(l+1)+\tfrac{1}{2}(K_2a)^2\right]\right], \tag{35}$$

$$\Delta_1^e=\frac{j_l(k_1a)}{h_l(k_1a)}\left\{-\tfrac{1}{2}(K_2a)^2\frac{\rho_1}{\rho_2}\left[\frac{k_2aj_l'(k_2a)}{j_l(k_2a)}\frac{K_2a\psi_l'(K_2a)}{\psi_l(K_2a)}-l(l+1)+\tfrac{1}{2}(K_2a)^2\frac{k_2aj_l'(k_2a)}{j_l(k_2a)}\right]\right.$$
$$+\frac{k_1aj_l'(k_1a)}{j_l(k_1a)}\left[\left[\frac{k_2aj_l'(k_2a)}{j_l(k_2a)}\frac{K_2a\psi_l'(K_2a)}{\psi_l(K_2a)}-l(l+1)\right][2-l(l+1)]\right.$$
$$\left.\left.+\tfrac{1}{2}(K_2a)^2\left[\frac{K_2a\psi_l'(K_2a)}{\psi_l(K_2a)}+2\frac{k_2aj_l'(k_2a)}{j_l(k_2a)}-2l(l+1)+\tfrac{1}{2}(K_2a)^2\right]\right]\right\}.$$

With these, the expressions given above for longitudinal intensities and cross sections are valid and agree with those of Ref. 4.

V. LIMITING CASES AND DISCUSSION

The Rayleigh-range scattering cross sections are given in a simple form here, since when the scattering is small the radial functions are replaced by the corresponding small-argument approximations and higher powers of the size parameters can be neglected. The general expressions for the extinction cross sections still hold, but now only the first three terms in the summation need to be retained. We write, for either small fluid or small solid scatterers,

$$\sigma^i_{\text{ext}}=\pi a^2\frac{4}{(k_1a)^2}\sum_{l=1}^{3}(2l+1)\mathcal{R}\left[\frac{\Delta_1^i}{\Delta_0^i}\right]\quad(i=f,e)\ , \tag{36}$$

where for the fluid scatterer we have in the Rayleigh limit

$$\begin{aligned}
\left[\frac{\Delta_1^f}{\Delta_0^f}\right]_{l=0}&\approx\tfrac{1}{3}(k_1a)^3\left[1-\frac{\frac{3}{4}(K_1/k_1)^2}{1-\dfrac{\eta_2}{\eta_1}[1-\frac{3}{4}(K_2/k_2)^2]}\right][i+(k_1a)^3]\ ,\\
\left[\frac{\Delta_1^f}{\Delta_0^f}\right]_{l=1}&\approx\tfrac{1}{9}(k_1a)^3\left[1-\frac{\rho_2}{\rho_1}\right]\left[i+\frac{(k_1a)^3}{3}\right]\ ,\\
\left[\frac{\Delta_1^f}{\Delta_0^f}\right]_{l=2}&\approx\tfrac{2}{9}(k_1a)^3\left[\frac{(1-\eta_2/\eta_1)(k_1/K_1)^2}{\frac{5}{2}+(\eta_2/\eta_1-1)[1+\frac{2}{3}(K_1/k_1)^2]}\right]\left[i+\frac{(k_1a)^3}{45}\right]\ ,
\end{aligned}\tag{37}$$

and for small elastic scatterer

$$\begin{aligned}
\left[\frac{\Delta_1^e}{\Delta_0^e}\right]_{l=0}&\approx\tfrac{1}{3}(k_1a)^3\left[1-\frac{\frac{3}{4}(K_1/k_1)^2}{1-\dfrac{i\mu_2}{\omega\eta_1}[1-\frac{3}{4}(K_2/k_2)^2]}\right][i+(k_1a)^3]\ ,\\
\left[\frac{\Delta_1^e}{\Delta_0^e}\right]_{l=1}&\approx\tfrac{1}{9}(k_1a)^3\left[1-\frac{\rho_2}{\rho_1}\right]\left[i+\frac{(k_1a)^3}{3}\right]\ ,\\
\left[\frac{\Delta_1^e}{\Delta_0^e}\right]_{l=2}&\approx\tfrac{2}{9}(k_1a)^3\left[\frac{(1-i\mu_2/\omega\eta_1)(k_1/K_1)^2}{\frac{5}{2}+(i\mu_2/\omega\eta_1-1)[1+\frac{2}{3}(K_1/k_1)^2]}\right]\left[i+\frac{(k_1a)^3}{45}\right]\ .
\end{aligned}\tag{38}$$

It is clear from these expressions that the extinction cross sections are directly proportional to the size parameter of the scatterer when loss is present, and are proportional to the fourth power for loss scattering.

We can also consider the scattering cross sections in the Rayleigh limit. Normalizing the scattering cross sections by the geometric cross section of the scatterer πa^2 we write

$$Q_{\text{scat}}=(\sigma^C_{\text{scat}}+\sigma^S_{\text{scat}})/\pi a^2\ . \tag{39}$$

For a small viscous fluid scatterer in a viscous fluid we find

$$\begin{aligned}
Q^f_{\text{scat}}=\tfrac{4}{9}(k_1a)^4\Bigg[&\left|\frac{(K_1/k_1)^2}{[(K_2/k_2)^2-\frac{4}{3}]\eta_2/\eta_1+\frac{4}{3}}-1\right|^2+\frac{1}{3}\left[1+2\left|\frac{K_1}{k_1}\right|^3\right]\left[1-\frac{\rho_2}{\rho_1}\right]^2\\
&+40\left[2+3\left|\frac{K_1}{k_1}\right|^5\right]\left|\frac{\eta_2/\eta_1-1}{2[3(K_1/k_1)^2+2]\eta_1/\eta_1+[9(K_1/k_1)^2-4]}\right|^2\Bigg]\ .
\end{aligned}$$

Setting the viscosities η_1,η_2 of both media to zero this reduces to

$$Q^f_{\text{scat}}=\tfrac{4}{9}(k_1a)^4\left\{\left[\frac{\rho_1}{\rho_2}\left[\frac{k_2}{k_2}\right]^2-1\right]^2+3\frac{(1-\rho_2/\rho_1)^2}{(1+2\rho_2/\rho_1)^2}\right\}\ , \tag{40}$$

which agrees with the famous result of Rayleigh.[18]

For a small elastic scatterer in a viscous fluid we find

$$Q^e_{\text{scat}}=\tfrac{4}{9}(k_1a)^4\left[\left|\frac{(K_1/k_1)^2}{[(K_1/k_2)^2-\frac{4}{3}](i\mu_2/\omega\eta_1)+\frac{4}{3}}-1\right|^2+\frac{1}{3}\left[1+2\left|\frac{K_1}{k_1}\right|^3\right]\left[1-\frac{\rho_2}{\rho_1}\right]^2\right.$$
$$\left.+40\left[2+3\left|\frac{K_1}{k_1}\right|^5\right]\left|\frac{i\mu_2/\omega\eta_1-1}{2[3(K_1/k_2)^2+2](i\mu_2/\omega\eta_1)+[9(K_1/k_1)^2-4]}\right|^2\right].$$

Setting the viscosity η_1 of the fluid to zero gives

$$Q^e_{\text{scat}}=\tfrac{4}{9}(k_1a)^4\left[\left[\frac{(K_1/k_1)^2}{(K_2/k_2)^2\rho_1/\rho_1-\frac{4}{3}}-1\right]^2+3\left[\frac{1-\rho_1/\rho_2}{2+\rho_1/\rho_2}\right]^2\right], \quad (41)$$

which agrees with Truell's well-known solution.[19]

We see that in each of these the scattering cross section is proportional to the fourth power of the size parameter. Hence for small scatterers the extinction cross section is much larger than the scattering cross section which indicates that absorption dominates over scattering for small scatterers in viscous fluids.

We have formulated scattering of plane monochromatic acoustic waves from viscous fluid and solid elastic spheres in a viscous fluid, and related the results to well-known limiting cases. Some useful conclusions can be drawn from these results.

(i) It is seen that incident waves excite certain modes in the scattered field. For an incident wave that is purely compressional two types of modes—compressional and shear—are excited in the scattered field. For $l=0$ which is the purely dilatational mode, only compressional modes are excited and $l=1$ is the lowest coupled mode which radiates in the scattered field. Excitation of shear modes is due to boundary coupling which results from the continuity condition of the fluid displacement of the boundary. This is a required condition since otherwise the two media would separate.

(ii) Independently of the imposed incident wave the solution of the present problem allows the study of the natural oscillations of the viscous fluid and solid elastic spheres in a viscous fluid. It can be seen that $\Delta_0^f=0$ and $\Delta_0^e=0$ give the natural frequencies of oscillation of the viscous fluid or solid elastic sphere in the viscous fluid, for oscillations of the compressional and shear modes considered here. A second class of shear modes would also be present in a complete study of the natural oscillations of the fluid or elastic sphere in a viscous fluid, but are not discussed in this scattering problem since they are not coupled to the incident sound wave. In our simple model the oscillations are found to be uncoupled and purely dilatational only for zero modes, while for all other modes compressional and shear oscillations are present.

(iii) The general expressions derived here for viscous fluids reduce directly to well-known results when the viscosity coefficients are set to zero. With $\eta_1=\eta_2=0$ in the fluid-fluid case we recover Anderson's solution and with $\eta_1=0$ in the fluid-elastic case we recover Faran's solution.

(iv) In the small sphere approximations the $l=0,1,2$ modes for the scattering cross sections are proportional to the fourth power of the size parameter (product of wave number and sphere radius) which is characteristic of Rayleigh scattering. For lossy scattering, these three modes of the extinction cross section are proportional to the first power of the size parameter. All higher modes are negligible in the small sphere limit and it is seen that absorption phenomena dominate scattering phenomena for scattering from small spheres when loss is present.

ACKNOWLEDGEMENTS

The author wishes to acknowledge the inspiration and guidance of the late Professor Asim Yildiz, and thank B. A. Rhodes, Dr. K. D. Trott, and Professor G. v.H. Sandri for many helpful discussions.

*Present address: Rome Laboratory, Target Characterization Branch, Hanscom AFB, MA 01731.

[1] J. W. S. Rayleigh, Philos. Mag. **41**, 107 (1871); **41**, 187 (1871); **41**, 447 (1871).
[2] H. Lamb, Proc. London Math. Soc. **32**, 11 (1900); **32**, 120 (1900).
[3] V. C. Anderson, J. Acoust. Soc. Am. **22**, 426 (1950).
[4] J. J. Faran, J. Acoust. Soc. Am. **23**, 405 (1951).
[5] C. J. T. Sewell, Philos. Trans. R. Soc. London, Ser. A **210**, 239 (1910).
[6] K. F. Herzfeld, Philos. Mag. **9**, 741 (1930).
[7] P. S. Epstein and R. R. Carhart, J. Acoust. Soc. Am. **25**, 553 (1953).
[8] J. C. F. Chow, J. Acoust. Soc. Am. **36**, 2395 (1964).
[9] J. E. Cole and R. A. Dobbins, J. Atmos. Sci. **27**, 426 (1970).
[10] R. Y. Nishi, Acustica **33**, 65 (1975).
[11] C. A. Miller and L. E. Scriven, J. Fluid Mech. **32**, 417 (1968).
[12] R. B. Jones and R. Schmitz, Physica **122A**, 105 (1983); **122A**, 114 (1983).
[13] B. U. Felderhof and R. B. Jones, Physica **136A**, 77 (1986).
[14] B. U. Felderhof and R. B. Jones, Physicochem. Hydrodyn. **11**, 507 (1989).
[15] G. Mie, Ann. Phys. **25**, 377 (1908).
[16] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953).
[17] M. Born and E. Wolf, *Principles of Optics* (Pergamon, New York, 1964).
[18] A. Yildiz, Nuovo Cimento **30**, 1182 (1963).
[19] C. F. Ying R. Truell, J. Appl. Phys. **27**, 1087 (1956).