Theory of the anisotropic ferrite wake-field accelerator. I. Azimuthally symmetric case

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The theory of the anisotropic ferrite wake-field accelerator is solved, wherein the ferrite is driven into saturation by a static magnetic field, resulting in a permeability tensor having off-diagonal elements. We show that it is possible to obtain a maximum accelerating gradient of 1.5 MV/[m (nC driver beam charge)] for a driver beam of 0.7-mm rms bunch length. This compares favorably with wake-field accelerators based upon other types of structures.

I. INTRODUCTION

It is clear that in the near future, a new technology will be needed for building high-energy accelerators. If one considers the case of linear electron accelerators, the most powerful of its kind in existence today is located at the Stanford Linear Accelerator Center (SLAC). In the SLAC machine, electrons are accelerated up to 50 GeV over a distance of 2 miles, thus yielding accelerating gradients of about 15 MV/m. If one ever hopes to build TeV electron colliders, certainly it would be advantageous to have in place some technology yielding in excess of 100 MV/m.

One such method which has received considerable attention is called wake-field acceleration. In this scheme, an intense bunch of electrons called the driver beam traverses some medium or structure, giving up part of its energy to the electromagnetic field in its wake. Subsequently, a second dilute bunch of electrons called the witness beam travels through the same medium or structure, and is thus accelerated by the driver's electromagnetic wake field.

The most promising medium to be considered in wake-field acceleration studies was that of a dense plasma which could yield accelerating gradients in excess of 1 GV/m. First proposed by Feinberg,[1,2] it was subsequently studied by Bolotovsky,[3] and in recent times it has gained renewed interest due to the work of Dawson and collaborators.[4] The experimental verification of plasma wake-field acceleration for modest acceleration gradients was provided by Simpson, Rosenzweig, and collaborators.[5] Similar experiments are presently under consideration to be done in the near future by Amatuni and coworkers.[6]

Simpson *et al.*[7] have recently pointed out that a variety of technical problems associated with plasma wake-field acceleration may provide difficult to overcome in trying to build practical accelerators based upon this technique. Nonlinearities in the plasma are poorly understood, although the problem has begun to be explored both theoretically and experimentally.[8-11] But perhaps the most difficult problem has to do with transverse forces in the plasma wake field which make it difficult to control the trajectory of particles within the witness pulse as a result of their alignment errors.

As for structures, Figueroa *et al.* have theoretically and experimentally studied wake-field acceleration in pillbox cavities.[12] Here again transverse forces are a major problem, leading to beam instabilities even within the driver bunch.

More recently, wake-field acceleration using a metal tube lined with a dielectric material has been proposed.[13] In this scheme the transverse forces are not quite so bad as they are in the plasma and pillbox cavities, but they still present a problem.[14-17] The experimental verification of this technique has been provided recently.[18] A good summary of much of the early work relevant to wake-field acceleration in dielectric loaded structures can be found in Ref. 19.

In the present work we explore another structure, a metal tube lined with some ferrite material with a static magnetic field applied along the longitudinal direction which drives the ferrite into saturation. For that reason it is called the anisotropic ferrite wake-field accelerator. Throughout the discussions we use the mks system of units. Previous work on wake fields in magnetized ferrite-lined tubes has been carried out by Nasonov and Shenderovitch, Grishaev *et al.*, and Zakutin *et al.* in the context of the self-acceleration of electron bunches.[20-25] Other early work relevant to this scheme is summarized in Ref. 19. The present theoretical discussion describes the application of magnetized ferrites to the present generation of wake-field experiments. The situation for unmagnetized ferrites in these experiments has been discussed by Callan *et al.*[26] However, in their scheme the relative permeability $\mu_r \sim 50$ and the dielectric constant $\epsilon_r \sim 1.5$, which as they point out are not the parameter values of normal ferrites. For the case of magnetized ferrites, in the regime in which we operate, the magnitudes of the components of the relative permeability tensor are of order unity and the dielectric constant is about ten so that almost any normal ferrite material should suffice.

The next phase of the wake-field acceleration experiments will have driver beam charges of about 100 nC within bunch lengths of less than 10 psec traversing loaded waveguide structures. Thus the physics of such schemes will be pushed into an entirely new regime. No one knows for sure what materials science problems will be posed by such applications. For that reason it is necessary to have a variety of candidates for the loading material to be used, magnetized ferrites being one such additional possibility.

In the next section we derive the electromagnetic fields inside the ferrite. In Sec. III we solve for the electromagnetic fields inside the vacuum hole. In Sec. IV we apply the boundary conditions and solve for the coefficients of the homogeneous electromagnetic field solutions. Finally, in Sec. V we offer some numerical calculations and concluding remarks.

II. FIELD SOLUTIONS INSIDE THE FERRITE

A. The permeability tensor

Using the Gilbert form[27] of the dynamical equation governing the magnetization vector, we can write

$$\frac{d\mathbf{M}}{dt}=\Gamma_e(\mathbf{M}\times\mathbf{H})+\frac{\alpha}{M}\mathbf{M}\times\frac{d\mathbf{M}}{dt}, \tag{1}$$

where $\mathbf{M}$ is the magnetization vector with magnitude M, $\mathbf{H}$ is the magnetic field, α is the damping parameter, and Γ_e is the gyromagnetic ratio given by

$$\Gamma_e=-g(\mu_0 e/2m_e)$$

with e and m_e being the charge and the mass of the electron, μ_0 the permeability of free space, and g the spectroscopic splitting factor. Since $g=2$ for a free electron, in our case $\Gamma_e=-2.21\times10^5$ (rad/sec)/(At/m). We have included nonzero damping in the theory for completeness; numerical analyses show that it has little effect on the results.

Next, we can write

$$\mathbf{H}=H_s\hat{\mathbf{k}}+\mathbf{H}_{\rm rf}(t)-\overleftrightarrow{\mathbf{D}}\cdot\mathbf{M}, \tag{2}$$

$$\mathbf{M}=M_s\hat{\mathbf{k}}+\mathbf{M}_{\rm rf}(t), \tag{3}$$

where the subscripts s and rf refer to the static and oscillatory parts of the fields, $\hat{\mathbf{k}}$ is the unit vector along the longitudinal direction, and the tensor $\overleftrightarrow{\mathbf{D}}$ is the demagnetization factor which can be neglected throughout our discussions since it is automatically included when we impose boundary conditions on the fields.[28] In Cartesian coordinates we have

$$\mathbf{H}_{\rm rf}(t)=(H_x\hat{\mathbf{i}}+H_y\hat{\mathbf{j}}+H_z\hat{\mathbf{k}})e^{i\omega t}, \tag{4}$$

$$\mathbf{M}_{\rm rf}(t)=(M_x\hat{\mathbf{i}}+M_y\hat{\mathbf{j}}+M_z\hat{\mathbf{k}})e^{i\omega t}, \tag{5}$$

where ω is the angular frequency. We make the assumption that

$$|\mathbf{H}_{\rm rf}|\ll H_s, \tag{6}$$

$$|\mathbf{M}_{\rm rf}|\ll M_s. \tag{7}$$

To obtain the magnetic susceptibility of the ferrite, we substitute the expressions for $\mathbf{H}$ and $\mathbf{M}$ in Eqs. (2)–(5) into Eq. (1) and obtain

$$M_x=\frac{\Gamma_e M_s(\Gamma_e H_s-i\omega\alpha)}{(\omega_r+i\omega\alpha)^2-\omega^2}H_x-\frac{i\omega\Gamma_e M_s}{(\omega_r+i\omega\alpha)^2-\omega^2}H_y, \tag{8}$$

$$M_y=\frac{i\omega\Gamma_e M_s}{(\omega_r+i\omega\alpha)^2-\omega^2}H_x+\frac{\Gamma_e M_s(\Gamma_e H_s-i\omega\alpha)}{(\omega_r+i\omega\alpha)^2-\omega^2}H_y, \tag{9}$$

$$M_z=0, \tag{10}$$

where $\omega_r=-\Gamma_e H_s$ is the ferrite resonance frequency. In matrix form, Eqs. (8)–(10) become

$$\begin{bmatrix}M_x\\M_y\\M_z\end{bmatrix}=\begin{bmatrix}\chi & -iK & 0\\ iK & \chi & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}H_x\\H_y\\H_z\end{bmatrix}. \tag{11}$$

If we define

$$\chi=\chi'-i\chi'', \tag{12}$$

$$K=K'-iK'', \tag{13}$$

$$\omega_m=-\Gamma_e M_s, \tag{14}$$

where χ and K have been divided into real and imaginary parts, then we can write

$$\chi'=\frac{\omega_m\omega_r(\omega_r^2-\omega^2)+\omega_m\omega_r\alpha^2\omega^2}{[\omega_r^2-\omega^2(1+\alpha^2)]^2+4\omega^2\omega_r^2\alpha^2}, \tag{15}$$

$$\chi''=\frac{\omega_m\omega\alpha[\omega_r^2+(1+\alpha^2)\omega^2]}{[\omega_r^2-\omega^2(1+\alpha^2)]^2+4\omega^2\omega_r^2\alpha^2}, \tag{16}$$

$$K'=\frac{-\omega_m\omega[\omega_r^2-\omega^2(1+\alpha^2)]}{[\omega_r^2-\omega^2(1+\alpha^2)]^2+4\omega^2\omega_r^2\alpha^2}, \tag{17}$$

$$K''=\frac{-2\omega_m\omega_r\omega^2\alpha}{[\omega_r^2-\omega^2(1+\alpha^2)]^2+4\omega^2\omega_r^2\alpha^2}. \tag{18}$$

In our application of a ferrite-lined metallic tube, it is convenient to use cylindrical coordinates and it is straightforward to show

$$\chi_{rr}=\chi_{\theta\theta}=\chi_{xx}=\chi_{yy}, \tag{19}$$

$$\chi_{r\theta}=\chi_{xy}=-\chi_{\theta r}=-\chi_{yx}. \tag{20}$$

From the susceptibility tensor $\overleftrightarrow{\chi}$ defined by Eq. (11), the permeability tensor $\overleftrightarrow{\mu}$ is

$$\overleftrightarrow{\mu}=\mu_0(\overleftrightarrow{\mathrm{I}}+\overleftrightarrow{\chi})\equiv\mu_0\overleftrightarrow{\mu}_r \tag{21}$$

with

$$\mathbf{B}=\overleftrightarrow{\mu}\cdot\mathbf{H}, \tag{22}$$

where $\overleftrightarrow{\mathrm{I}}$ is the identity matrix and $\overleftrightarrow{\mu}_r$ is the relative permeability and can be written

$$\overleftrightarrow{\mu}_r=\begin{bmatrix}\tilde{\mu}=1+\chi_{rr} & -iK & 0\\ iK & \tilde{\mu} & 0\\ 0 & 0 & \tilde{\mu}_z\end{bmatrix}. \tag{23}$$

Now that we have the permeability tensor inside the ferrite medium, let us next solve Maxwell's equations inside the ferrite.

B. Solutions to Maxwell's equations

Consider a metallic tube lined with a ferrite material as shown in Fig. 1. The ferrite is contained in the region $a \le r \le b$, where r is the radial coordinate. Maxwell's equations inside the ferrite are given by

$$\nabla \times \mathbf{E}^f = -\frac{\partial \mathbf{B}^f}{\partial t} , \tag{24}$$

$$\nabla \times \mathbf{H}^f = \frac{\partial \mathbf{D}^f}{\partial t} , \tag{25}$$

$$\nabla \cdot \mathbf{D}^f = 0 , \tag{26}$$

$$\nabla \cdot \mathbf{B}^f = 0 , \tag{27}$$

with the constitutive equations

$$\mathbf{D}^f = \epsilon_f \mathbf{E}^f , \tag{28}$$

$$\mathbf{B}^f = \mu_0 \tilde{\mu}_r \cdot \mathbf{H}^f . \tag{29}$$

Assuming all oscillatory fields vary as $e^{-\kappa z + i\omega t}$, then the curl equations in cylindrical coordinates for the time-dependent fields become[28]

$$\frac{1}{r}\frac{\partial E_z^f}{\partial \theta} + \kappa E_\theta^f = -i\omega\mu_0(\tilde{\mu} H_r^f - iK H_\theta^f) , \tag{30}$$

$$-\kappa E_r^f - \frac{\partial E_z^f}{\partial r} = -i\omega\mu_0(iK H_r^f + \tilde{\mu} H_\theta^f) , \tag{31}$$

$$\frac{1}{r}\left[\frac{\partial(rE_\theta^f)}{\partial r} - \frac{\partial E_r^f}{\partial \theta}\right] = -i\omega\mu_0\tilde{\mu}_z H_z^f , \tag{32}$$

$$\frac{1}{r}\frac{\partial H_z^f}{\partial \theta} + \kappa H_\theta^f = i\omega\epsilon_f E_r^f , \tag{33}$$

$$-\kappa H_r^f - \frac{\partial H_z^f}{\partial r} = i\omega\epsilon_f E_\theta^f , \tag{34}$$

$$\frac{1}{r}\left[\frac{\partial(rH_\theta^f)}{\partial r} - \frac{\partial H_r^f}{\partial \theta}\right] = i\omega\epsilon_f E_z^f . \tag{35}$$

Since Eqs. (30), (31), (33), and (34) do not contain derivatives of the transverse field components, we can solve for

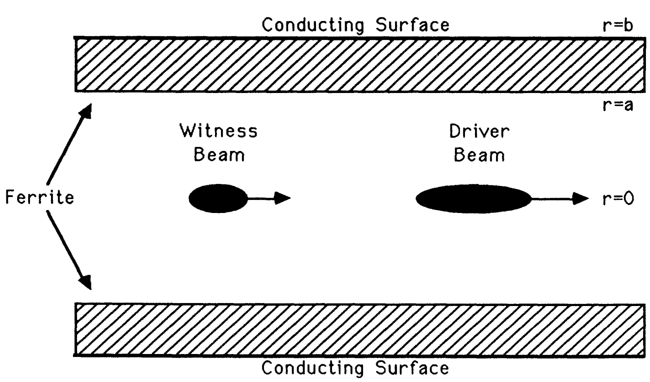

FIG. 1. Ferrite-loaded wake-field structure.

those transverse components in terms of the derivatives of E_z^f and H_z^f. One gets

$$E_r^f = p^f\frac{\partial E_z^f}{\partial r} + q^f\frac{1}{r}\frac{\partial E_z^f}{\partial \theta} + \overline{r}^f\frac{\partial H_z^f}{\partial r} + s^f\frac{1}{r}\frac{\partial H_z^f}{\partial \theta} , \tag{36}$$

$$E_\theta^f = -q^f\frac{\partial E_z^f}{\partial r} + p^f\frac{1}{r}\frac{\partial E_z^f}{\partial \theta} - s^f\frac{\partial H_z^f}{\partial r} + \overline{r}^f\frac{1}{r}\frac{\partial H_z^f}{\partial \theta} , \tag{37}$$

$$H_r^f = t^f\frac{\partial E_z^f}{\partial r} + u^f\frac{1}{r}\frac{\partial E_z^f}{\partial \theta} + p^f\frac{\partial H_z^f}{\partial r} + q^f\frac{1}{r}\frac{\partial H_z^f}{\partial \theta} , \tag{38}$$

$$H_\theta^f = -u^f\frac{\partial E_z^f}{\partial r} + t^f\frac{1}{r}\frac{\partial E_z^f}{\partial \theta} - q^f\frac{\partial H_z^f}{\partial r} + p^f\frac{1}{r}\frac{\partial H_z^f}{\partial \theta} , \tag{39}$$

where

$$p^f = -\kappa(\kappa^2 + \omega^2\epsilon_f\mu_0\tilde{\mu})(\Delta^f)^{-1} , \tag{40}$$

$$q^f = -i\kappa\omega^2\epsilon_f\mu_0 K(\Delta^f)^{-1} , \tag{41}$$

$$\overline{r}^f = \omega\mu_0 K\kappa^2(\Delta^f)^{-1} , \tag{42}$$

$$s^f = -i\omega[\mu_0\tilde{\mu}\kappa^2 + \omega^2\mu_0^2(\tilde{\mu}^2 - K^2)\epsilon_f](\Delta^f)^{-1} , \tag{43}$$

$$t^f = \omega^3\epsilon_f^2\mu_0 K(\Delta^f)^{-1} , \tag{44}$$

$$u^f = i\omega(\epsilon_f\kappa^2 + \omega^2\epsilon_f^2\mu_0\tilde{\mu})(\Delta^f)^{-1} , \tag{45}$$

$$\Delta^f = [\kappa^2 + \omega^2\epsilon_f\mu_0(\tilde{\mu} + K)][\kappa^2 + \omega^2\epsilon_f\mu_0(\tilde{\mu} - K)] . \tag{46}$$

Upon substituting these expressions for the transverse fields back into Eqs. (32) and (35), one gets

$$\nabla_{r,\theta}^2 E_z^f + aE_z^f + bH_z^f = 0 , \tag{47}$$

$$\nabla_{r,\theta}^2 H_z^f + cH_z^f + dE_z^f = 0 , \tag{48}$$

where

$$a = \kappa^2 + \omega^2\epsilon_f\mu_0\frac{\tilde{\mu}^2 - K^2}{\tilde{\mu}} , \tag{49}$$

$$b = \frac{\kappa\omega\mu_0\tilde{\mu}_z K}{\tilde{\mu}} , \tag{50}$$

$$c = \frac{\tilde{\mu}_z}{\tilde{\mu}}(\kappa^2 + \omega^2\epsilon_f\mu_0\tilde{\mu}) , \tag{51}$$

$$d = -\frac{\kappa K\omega\epsilon_f}{\tilde{\mu}} . \tag{52}$$

It is important to note that if either E_z or H_z is equal to zero, then all oscillatory fields vanish. Therefore there are no pure TE or TM modes. This is due to the anisotropy of the dc magnetized ferrite medium. Thus, if one were to turn off H_s, then the off-diagonal components of the permeability tensor would be zero and one would retrieve the usual TE and TM modes propagating through the structure.

In order to solve Eqs. (47) and (48), we introduce functions F_1 and F_2 and parameters g_1 and g_2 such that

$$E_z = F_1 + F_2 , \tag{53}$$

$$H_z = g_1 F_1 + g_2 F_2 \ , \tag{54}$$

$$g_1 \neq g_2 \ . \tag{55}$$

Upon substituting Eqs. (53) and (54) into Eqs. (47) and (48) we arrive at

$$\nabla^2_{r,\theta} F_1 + (a + b g_1) F_1 + \nabla^2_{r,\theta} F_2 + (a + b g_2) F_2 = 0 \ , \tag{56}$$

$$g_1 \nabla^2_{r,\theta} F_1 + (d + c g_1) F_1 + g_2 \nabla^2_{r,\theta} F_2 + (d + c g_2) F_2 = 0 \ . \tag{57}$$

If it is possible to determine g_1 and g_2 such that

$$a + b g_1 = k_1^2, \quad a + b g_2 = k_2^2 \ , \tag{58}$$

$$d + c g_1 = g_1 k_1^2, \quad d + c g_2 = g_2 k_2^2 \ , \tag{59}$$

then

$$\nabla^2_{r,\theta} F_1 + k_1^2 F_1 + \nabla^2_{r,\theta} F_2 + k_2^2 F_2 = 0 \ , \tag{60}$$

$$g_1 (\nabla^2_{r,\theta} F_1 + k_1^2 F_1) + g_2 (\nabla^2_{r,\theta} F_2 + k_2^2 F_2) = 0 \ . \tag{61}$$

But since $g_1 \neq g_2$ then it follows that

$$\nabla^2_{r,\theta} F_1 + k_1^2 F_1 = 0 \ , \tag{62}$$

$$\nabla^2_{r,\theta} F_2 + k_2^2 F_2 = 0 \ , \tag{63}$$

where

$$g_1 = \frac{k_1^2 - a}{b} = \frac{d}{k_1^2 - c} \ , \tag{64}$$

$$g_2 = \frac{k_2^2 - a}{b} = \frac{d}{k_2^2 - c} \ . \tag{65}$$

Also,

$$(k_{1,2}^2 - a)(k_{1,2}^2 - c) = bd \tag{66}$$

so that

$$k_{1,2}^2 = \frac{(a + c) \pm [(a + c)^2 - 4(ac - bd)]^{1/2}}{2} \ . \tag{67}$$

Finally, we have Eq. (53) together with

$$H_z = \frac{k_1^2 - a}{b} F_1 + \frac{k_2^2 - a}{b} F_2 = \frac{1}{b} [-a (F_1 + F_2) + k_1^2 F_1 + k_2^2 F_2] \ , \tag{68}$$

where

$$k_{1,2}^2 = \frac{1}{2} \left[\kappa^2 \left[1 + \frac{\tilde{\mu}_z}{\tilde{\mu}} \right] + \omega^2 \mu_0 \epsilon_f \frac{\tilde{\mu}^2 - K^2}{\tilde{\mu}} + \omega^2 \epsilon_f \mu_0 \tilde{\mu}_z \right] \pm \frac{1}{2} \left\{ \left[\kappa^2 \left[1 - \frac{\tilde{\mu}_z}{\tilde{\mu}} \right] + \omega^2 \mu_0 \epsilon_f \frac{\tilde{\mu}^2 - K^2}{\tilde{\mu}} - \omega^2 \epsilon_f \mu_0 \tilde{\mu}_z \right]^2 - 4 \kappa^2 \omega^2 \epsilon_f \mu_0 \tilde{\mu}_z \left[\frac{K}{\tilde{\mu}} \right]^2 \right\}^{1/2} . \tag{69}$$

Now that we know how to write all electromagnetic field components in terms of the functions F_1 and F_2, we next solve Eqs. (62) and (63) for an azimuthally symmetric geometry. This should give the major contribution to the wake-field acceleration. Assuming no θ dependence, Eqs. (62) and (63) become

$$\frac{\partial^2 F_1}{\partial r^2} + \frac{1}{r} \frac{\partial F_1}{\partial r} + k_1^2 F_1 = 0 \ , \tag{70}$$

and similarly for F_2. The solutions inside the ferrite become

$$F_1(r) = A_1 J_0(k_1 r) + B_1 N_0(k_1 r) \ , \tag{71}$$

$$F_2(r) = A_2 J_0(k_2 r) + B_2 N_0(k_2 r) \ , \tag{72}$$

where J_0 and N_0 are the zeroth-order Bessel functions of the first and second kinds, respectively.

To derive the longitudinal accelerating field behind the driver electron bunch, we will only need expressions for E_z^f, H_z^f, E_θ^f, and H_θ^f. Using Eqs. (37), (39), (53), (54), (71), and (72), we arrive at

$$E_z^f = [A_1 J_0(k_1 r) + A_2 J_0(k_2 r) + B_1 N_0(k_1 r) + B_2 N_0(k_2 r)] e^{-\kappa z + i\omega t} \ , \tag{73}$$

$$H_z^f = \frac{\tilde{\mu}}{\kappa \omega \mu_0 \tilde{\mu}_z K} \left[\left[k_1^2 - \kappa^2 - \omega^2 \epsilon_f \mu_0 \frac{\tilde{\mu}^2 - K^2}{\tilde{\mu}} \right] [A_1 J_0(k_1 r) + B_1 N_0(k_1 r)] + \left[k_2^2 - \kappa^2 - \omega^2 \epsilon_f \mu_0 \frac{\tilde{\mu}^2 - K^2}{\tilde{\mu}} \right] [A_2 J_0(k_2 r) + B_2 N_0(k_2 r)] \right] e^{-\kappa z + i\omega t} \ , \tag{74}$$

$$E_\theta^f = i\kappa\omega^2\epsilon_f\mu_0 K(\Delta^f)^{-1}\{k_1[A_1J_0'(k_1r)+B_1N_0'(k_1r)]+k_2[A_2J_0'(k_2r)+B_2N_0'(k_2r)]\}e^{-\kappa z+i\omega t}$$
$$+\frac{i\tilde\mu}{\kappa\mu_0\tilde\mu_z K}[\mu_0\tilde\mu\kappa^2+\omega^2\mu_0^2(\tilde\mu^2-K^2)\epsilon_f](\Delta^f)^{-1}$$
$$\times\left[\left[k_1^2-\kappa^2-\omega^2\epsilon_f\mu_0\frac{\tilde\mu^2-K^2}{\tilde\mu}\right]\{k_1[A_1J_0'(k_1r)+B_1N_0'(k_1r)]\}\right.$$
$$\left.+\left[k_2^2-\kappa^2-\omega^2\epsilon_f\mu_0\frac{\tilde\mu^2-K^2}{\tilde\mu}\right]\{k_2[A_2J_0'(k_2r)+B_2N_0'(k_2r)]\}\right]e^{-\kappa z+i\omega t}, \quad (75)$$

$$H_\theta^f=-i\omega(\epsilon_f\kappa^2+\omega^2\epsilon_f^2\mu_0\tilde\mu)(\Delta^f)^{-1}\{k_1[A_1J_0'(k_1r)+B_1N_0'(k_1r)]+k_2[A_2J_0'(k_2r)+B_2N_0'(k_2r)]\}e^{-\kappa z+i\omega t}$$
$$+\frac{i\omega\epsilon_f\tilde\mu(\Delta^f)^{-1}}{\tilde\mu_z}\left[\left[k_1^2-\kappa^2-\omega^2\epsilon_f\mu_0\frac{\tilde\mu^2-K^2}{\tilde\mu}\right]\{k_1[A_1J_0'(k_1r)+B_1N_0'(k_1r)]\}\right.$$
$$\left.+\left[k_2^2-\kappa^2-\omega^2\epsilon_f\mu_0\frac{\tilde\mu^2-K^2}{\tilde\mu}\right]\{k_2[A_2J_0'(k_2r)+B_2N_0'(k_2r)]\}\right]e^{-\kappa z+i\omega t}. \quad (76)$$

We now have expressions for the field solutions inside the ferrite medium. The constants A_1, B_1, A_2, and B_2 will be determined later by the boundary conditions. In the next section we proceed to solve for the fields inside the vacuum hole.

III. FIELD SOLUTIONS INSIDE THE VACUUM HOLE

Inside the vacuum hole, Maxwell's equations become

$$\nabla\times\mathbf{E}^h=-\frac{\partial\mathbf{B}^h}{\partial t}, \quad (77)$$

$$\nabla\times\mathbf{H}^h=\mathbf{J}+\frac{\partial\mathbf{D}^h}{\partial t}, \quad (78)$$

$$\nabla\cdot\mathbf{E}^h=\frac{\rho}{\epsilon_0}, \quad (79)$$

$$\nabla\cdot\mathbf{B}^h=0, \quad (80)$$

with the constitutive equations

$$\mathbf{D}^h=\epsilon_0\mathbf{E}^h \quad (81)$$

$$\mathbf{B}^h=\mu_0\mathbf{H}^h. \quad (82)$$

We follow a procedure similar to that in Sec. II, although it is simpler in this case. We first solve for the longitudinal components E_z^h and H_z^h which satisfy

$$\left[\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\begin{bmatrix}E_z^h\\ H_z^h\end{bmatrix}=\begin{bmatrix}\dfrac{1}{\epsilon_0}\dfrac{\partial\rho}{\partial z}+\mu_0\dfrac{\partial J_z}{\partial t}\\ 0\end{bmatrix}. \quad (83)$$

We consider the case where the driver is a line of charge having a Gaussian line distribution. So we have

$$\rho(z-vt,r)=-\frac{Q}{(2\pi)^{3/2}}\frac{\delta(r)}{r\sigma_z}e^{-(z-vt)^2/2\sigma_z^2}, \quad (84)$$

$$\mathbf{J}(z-vt,r)=\rho\mathbf{v}=(0,0,\rho v), \quad (85)$$

where Q is the charge in the driver electron bunch, σ_z is the rms bunch length, and $v=\beta c$ is the speed of the driver, with $c^2=1/\mu_0\epsilon_0$.

Expanding E_z^h and ρ in harmonics

$$E_z^h(z-vt,r)=\int_{-\infty}^{\infty}e^{-(i\omega/v)(z-vt)}E_z^h(\omega,r)d\omega, \quad (86)$$

$$\rho(z-vt,r)=\int_{-\infty}^{\infty}e^{-(i\omega/v)(z-vt)}\rho(\omega,r)d\omega, \quad (87)$$

we get

$$\rho(\omega,r)=-\frac{Q}{4\pi^2}\frac{\delta(r)}{rv}e^{-\omega^2\sigma_z^2/2v^2} \quad (88)$$

and Eq. (83) becomes

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\right]-\frac{\omega^2}{v^2}(1-\beta^2)\right]E_z^h(\omega,r)$$
$$=\frac{iQ\omega}{(2\pi)^{3/2}\epsilon_0 v^2\gamma^2}\frac{\delta(r)}{r\sigma_z}f(\omega) \quad (89)$$

having the solution

$$E_z^h(\omega,r)=\frac{-iQ\omega}{4\pi^2\epsilon_0 v^2\gamma^2}e^{-\omega^2\sigma_z^2/2v^2}K_0\left[\frac{\omega r}{v}(1-\beta^2)^{1/2}\right]$$
$$+CI_0\left[\frac{\omega r}{v}(1-\beta^2)^{1/2}\right], \quad (90)$$

where γ is the Lorentz contraction factor and I_0 and K_0 are the zeroth-order modified Bessel functions. Similarly, for H_z^h we obtain

$$H_z^h(\omega,r)=GI_0\left[\frac{\omega r}{v}(1-\beta^2)^{1/2}\right], \quad (91)$$

where the constants C and G will be determined in the next section by the boundary conditions.

Following the procedure outlined in Sec. III, but specialized to the vacuum, we obtain for the transverse fields

$$E_r^h = \frac{iv\gamma^2}{\omega}\frac{\partial E_z^h}{\partial r} + \frac{i\mu_0 v^2\gamma^2}{\omega r}\frac{\partial H_z^h}{\partial \theta} , \tag{92}$$

$$E_\theta^h = \frac{iv\gamma^2}{\omega r}\frac{\partial E_z^h}{\partial \theta} - \frac{i\mu_0 v^2\gamma^2}{\omega}\frac{\partial H_z^h}{\partial r} , \tag{93}$$

$$H_r^h = \frac{-i\epsilon_0 v^2\gamma^2}{\omega r}\frac{\partial E_z^h}{\partial \theta} + \frac{iv\gamma^2}{\omega}\frac{\partial H_z^h}{\partial r} , \tag{94}$$

$$H_\theta^h = \frac{i\epsilon_0 v^2\gamma^2}{\omega}\frac{\partial E_z^h}{\partial r} + \frac{iv\gamma^2}{\omega r}\frac{\partial H_z^h}{\partial \theta} . \tag{95}$$

Using Eqs. (90)–(95) and assuming no θ dependence for the fields we finally obtain

$$E_z^h(\omega,r) = \frac{-iQ\omega}{4\pi^2\epsilon_0 v^2\gamma^2} e^{-\omega^2\sigma_z^2/2v^2} K_0\left[\frac{\omega r}{v}(1-\beta^2)^{1/2}\right] + CI_0\left[\frac{\omega r}{v}(1-\beta^2)^{1/2}\right] , \tag{96}$$

$$E_r^h(\omega,r) = i\gamma\left[\frac{-iQ\omega}{4\pi^2\epsilon_0 v^2\gamma^2} e^{-\omega^2\sigma_z^2/2v^2} K_0'\left[\frac{\omega r}{v}(1-\beta^2)^{1/2}\right] + CI_0'\left[\frac{\omega r}{v}(1-\beta^2)^{1/2}\right]\right] , \tag{97}$$

$$E_\theta^h(\omega,r) = -i\mu_0 v\gamma G I_0'\left[\frac{\omega r}{v}(1-\beta^2)^{1/2}\right] , \tag{98}$$

$$H_z^h(\omega,r) = G I_0\left[\frac{\omega r}{v}(1-\beta^2)^{1/2}\right] , \tag{99}$$

$$H_r^h(\omega,r) = i\gamma G I_0'\left[\frac{\omega r}{v}(1-\beta^2)^{1/2}\right] , \tag{100}$$

$$H_\theta^h(\omega,r) = i\epsilon_0 v\gamma\left[\frac{-iQ\omega}{4\pi^2\epsilon_0 v^2\gamma^2} e^{-\omega^2\sigma_z^2/2v^2} \times K_0'\left[\frac{\omega r}{v}(1-\beta^2)^{1/2}\right] + CI_0'\left[\frac{\omega r}{v}(1-\beta^2)^{1/2}\right]\right] . \tag{101}$$

In the next section we apply the boundary conditions to solve for the unknown constants in the field expressions.

IV. BOUNDARY CONDITIONS

Since we have to determine six constants: A_1, B_1, A_2, B_2, C, and G, we must impose six boundary conditions. We choose

$$E_z^f(r=b)=0 , \tag{102}$$

$$E_z^f(r=a)=E_z^h(r=a) , \tag{103}$$

$$H_z^f(r=a)=H_z^h(r=a) , \tag{104}$$

$$E_\theta^f(r=a)=E_\theta^h(r=a) , \tag{105}$$

$$H_\theta^f(r=a)=H_\theta^h(r=a) , \tag{106}$$

$$E_\theta^f(r=b)=0 . \tag{107}$$

Also, to match the arguments of the exponentials from the ferrite and vacuum regions we must have $\kappa=i\omega/v$ so that from Eq. (69) we get

$$k_{1,2}^2 = \frac{\omega^2}{2}\left[-\frac{1}{v^2}\left[1+\frac{\tilde{\mu}_z}{\tilde{\mu}}\right] + \mu_0\epsilon_f\frac{\tilde{\mu}^2-K^2}{\tilde{\mu}} + \epsilon_f\mu_0\tilde{\mu}_z\right] \pm \frac{\omega^2}{2}\left\{\left[-\frac{1}{v^2}\left[1-\frac{\tilde{\mu}_z}{\tilde{\mu}}\right] + \mu_0\epsilon_f\frac{\tilde{\mu}^2-K^2}{\tilde{\mu}} - \epsilon_f\mu_0\tilde{\mu}_z\right]^2 + 4\frac{\epsilon_f}{v^2}\mu_0\tilde{\mu}_z\left[\frac{\tilde{K}}{\tilde{\mu}}\right]^2\right\}^{1/2} . \tag{108}$$

Equations (102)–(107) can be written in the form

$$MX=Y , \tag{109}$$

where

$$M = \begin{bmatrix} \alpha_1 & \beta_1 & \delta_1 & \gamma_1 & 0 & 0 \\ \alpha_2 & \beta_2 & \delta_2 & \gamma_2 & \nu_2 & 0 \\ \alpha_3 & \beta_3 & \delta_3 & \gamma_3 & 0 & \lambda_3 \\ \alpha_4 & \beta_4 & \delta_4 & \gamma_4 & 0 & \lambda_4 \\ \alpha_5 & \beta_5 & \delta_5 & \gamma_5 & \nu_5 & 0 \\ \alpha_6 & \beta_6 & \delta_6 & \gamma_6 & 0 & 0 \end{bmatrix} , \tag{110}$$

$$X = \begin{bmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \\ C \\ G \end{bmatrix}, \quad Y = \begin{bmatrix} 0 \\ d_2 \\ 0 \\ 0 \\ d_5 \\ 0 \end{bmatrix} . \tag{111}$$

To find the accelerating field E_z^h inside the vacuum hole, we need the constant C. Using Cramer's rule[29] for solving simultaneous linear equations we find

$$C = \frac{\det M(\nu_i \to d_i)}{\det M} , \tag{112}$$

where in the numerator we mean to replace the coefficients ν_i by the quantities d_i. Thus, for the accelerating wake field, we obtain

$$E_z^h(z-vt,r)=\int_{-\infty}^{\infty} e^{-(i\omega/v)(z-vt)}\left[\frac{\det M(\nu_i\to d_i)}{\det M} I_0\left[\frac{\omega r}{v}(1-\beta^2)^{1/2}\right]-\frac{iQ\omega}{4\pi^2\epsilon_0 v^2\gamma^2} e^{-\omega^2\sigma_z^2/2v^2} K_0\left[\frac{\omega r}{v}(1-\beta^2)^{1/2}\right]\right] d\omega \,. \tag{113}$$

Since the second term can be neglected relative to the first, we have finally

$$E_z^h(z-vt,r)=\int_{-\infty}^{\infty} e^{-(i\omega/v)(z-vt)}\frac{\det M(\nu_i\to d_i)}{\det M} \times I_0\left[\frac{\omega r}{v}(1-\beta^2)^{1/2}\right] d\omega \,. \tag{114}$$

The coefficient C and the expression for $E_z^h(z-vt,r)$ can be evaluated on the computer by summing the residues of the poles in the frequency complex plane. The results for a variety of ferrite parameter values are discussed in the next section.

V. DISCUSSION AND CONCLUSIONS

As an example of an anisotropic ferrite wake-field accelerator, we consider the following per 1 nC of driver charge:

$$\text{rms length of driver bunch } \sigma_z=0.7 \text{ mm} \,, \tag{115}$$

$$\text{energy of driver bunch}=150 \text{ MeV } (\beta=0.999\,994) \,, \tag{116}$$

$$\text{inner ferrite radius } a=3 \text{ mm} \,. \tag{117}$$

These parameter values have been considered for the next series of wake-field experiments at the Argonne Wakefield Accelerator (AWA).[30] Since smaller values of the inner ferrite radius a lead to higher accelerating gradients, we take a as small as possible, provided the driver electron bunch does not scrape the ferrite tube. We also take the damping parameter $\alpha=0$ and $\tilde{\mu}_z=1.0$.

In Tables I–IV we display numerical evaluations of Eq. (114) for $E_z^{h,\max}$ (in [(MV/m)/nC] of driver charge) and the two lowest-frequency pole contributions versus ferrite thickness $\tau=b-a$, applied static magnetic field H_s, saturated static magnetization M_s, and ferrite dielectric constant ϵ_f. We draw the following conclusions.

(a) The ferrite thickness τ is optimized at $\sim$0.5 mm. However, to make machining of the ferrite easier, perhaps one could take $\tau\sim 1.0$ mm without much loss of maximum acceleration gradient.

(b) The ferrite excitation frequencies f_1 and f_2 and $E_z^{h,\max}$ are relatively insensitive to the applied static magnetic field H_s and the saturated static magnetization M_s. However, this does not mean that one can set $H_s=M_s=0$ and obtain the same gradients. On the contrary, all of the theory discussed in this paper is only valid for a saturated ferrite [cf. Eqs. (6) and (7)], so there is no $H_s=M_s=0$ limit of the formulation. Once driven into saturation, the relative independence of $E_z^{h,\max}$, f_1, and f_2 on H_s and M_s is an asset in that any normal ferrite material should suffice, as opposed to the situation for the unmagnetized ferrite wake-field accelerator discussed in Ref. 26. In that reference, the authors point out that normal unsaturated ferrites will not work.

(c) The optimal value of the ferrite relative dielectric constant is between 2 and 3. However, most normal ferrites have dielectric constants between 10 and 20. Thus the optimal realistic value is 10.

To test these ideas in the AWA [cf. Eqs. (115)–(117)], we propose the following optimized anisotropic ferrite wake-field accelerator:

TABLE I. Maximum acceleration gradient $E_z^{h,\max}$ and two lowest-frequency pole contributions f_1 and f_2 vs ferrite thickness τ for $H_s=2\times10^3$ At/m ($H_s=25$ Oe in cgs units), $M_s=3\times10^5$ At/m ($B_s=4\pi M_s=3770$ G in cgs units), $a=3$ mm, $\epsilon_f=10\epsilon_0$.

τ (mm)	f_1 (GHz)	f_2 (GHz)	$E_z^{h,\max}$ [(MV/m)/nC]
0.1	118.6		0.731
0.2	77.67		1.41
0.5	41.64	123.4	1.54
1.0	25.33	67.40	1.31
2.0	15.46	36.91	0.868

TABLE II. Maximum acceleration gradient $E_z^{h,\max}$ and two lowest-frequency pole contributions f_1 and f_2 vs applied static magnetic field H_s for $M_s=10^5$ At/m, $a=3$ mm, $b=4$ mm, $\epsilon_f=10\epsilon_0$.

H_s (At/m)	f_1 (GHz)	f_2 (GHz)	$E_z^{h,\max}$ [(MV/m)/nC]
5×10^2	23.49	66.62	1.47
10^3	23.49	66.62	1.48
2×10^3	23.49	66.62	1.50
3×10^3	23.49	66.62	1.51
4×10^3	23.50	66.62	1.51

TABLE III. Maximum acceleration gradient $E_z^{h,\max}$ and two lowest-frequency pole contributions f_1 and f_2 vs saturated static magnetization M_s for $H_s=2\times10^3$ At/m, $a=3$ mm, $b=4$ mm, $\epsilon_f=10\epsilon_0$.

M_s (At/m)	f_1 (GHz)	f_2 (GHz)	$E_z^{h,\max}$ [(MV/m)/nC]
10^4	23.23	66.52	1.46
10^5	23.49	66.62	1.50
2×10^5	24.24	66.91	1.51
3×10^5	25.33	67.40	1.31
4×10^5	26.48	68.06	1.33
5×10^5	27.34	68.87	1.32

$$b=4 \text{ mm} , \tag{118}$$

$$M_s=10^5 \text{ At/m} , \tag{119}$$

$$H_s=2.0\times10^3 \text{ At/m} . \tag{120}$$

The optimized value of M_s corresponds to a saturated magnetic induction in cgs units of $B_s=4\pi M_s=1257$ G, so that it corresponds to realistic ferrite materials.

A graph of E_z^h versus delay behind the driver is shown in Fig. 2. The maximum accelerating gradient is ~ 1.5 [(MV/m)/nC] of driver charge. The most important pole contributions in the frequency spectrum are 23.5 and 66.6 GHz, while the ferrite resonance frequency is at 70.4 MHz. We have studied damping and found that it has a negligible effect on the above numbers. Finally, the ferrite relative permeability tensors have components [cf. Eq. (23)] $\tilde{\mu}\sim 1$ and $K\sim 0.15$ at f_1.

We can compare the anisotropic ferrite wake-field accelerator with the dielectric wake-field accelerator.[13–17] Considering the same case given in Eqs. (115)–(117), we can optimize in the parameters b and ϵ_d, where the subscript d corresponds to the dielectric material. The optimal values are

$$b=4 \text{ mm} , \tag{121}$$

$$2\epsilon_0 \lesssim \epsilon_d \lesssim 3\epsilon_0 , \tag{122}$$

TABLE IV. Maximum acceleration gradient $E_z^{h,\max}$ and two lowest-frequency pole contributions f_1 and f_2 vs ferrite dielectric constant ϵ_f for $H_s=2\times10^3$ At/m, $M_s=10^5$ At/m, $a=3$ mm, $b=4$ mm.

ϵ_f/ϵ_0	f_1 (GHz)	f_2 (GHz)	$E_z^{h,\max}$ [(MV/m)/nC]
1.2	90.79		1.28
2.0	48.95	167.6	2.06
3.0	39.18	123.3	2.02
4.0	34.46	104.0	1.87
10.0	23.49	66.62	1.50
15.0	19.72	55.42	1.24
20.0	17.38	48.59	1.03

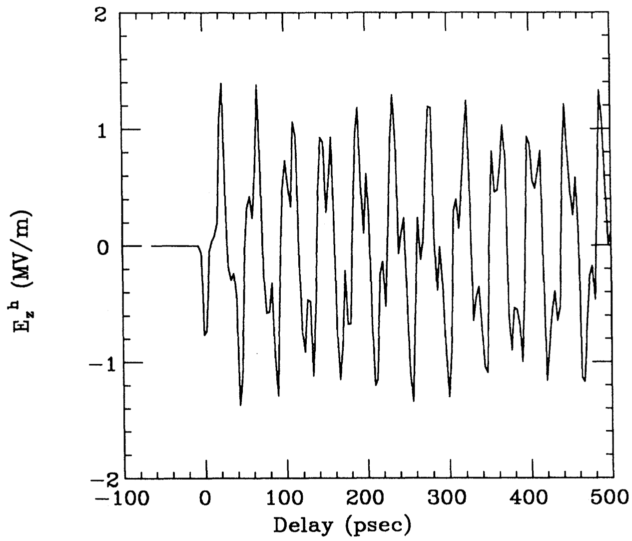

FIG. 2. Accelerating wake field E_z^h vs delay (distance) behind the driver beam for the optimized case: $a=3$ mm, $b=4$ mm, $Q=1$ nC, $\beta=0.999\,994$, $\sigma_z=0.7$ mm, $\alpha=0$, $M_s=10^5$ At/m, $H_s=2\times10^3$ At/m, $\epsilon_f=10\epsilon_0$.

where E_z^h attains a maximum value of ~ 2 [(MV/m)/nC] of driver charge. This accelerating gradient is comparable to (although somewhat higher) than that of the anisotropic ferrite wake-field accelerator, due mainly to the smallness of ϵ_d. But since ferrites and dielectrics give comparable results, the final choice of which material to use will have to be made after further experimental work.

To conclude, we have derived the theory of the anisotropic ferrite wake-field accelerator for the case of azimuthal symmetry and have obtained accelerating gradients of 1.5 [(MV/m)/nC] for realistic ferrite materials and accelerator parameters. By increasing the charge in the driver pulse we should be able to attain accelerating gradients of 100 MV/m. We encourage more work along these lines.

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