Eddy viscosity of parity-invariant flow

Bérengère Dubrulle
CERFACS, 42 avenue Coriolis, 31057 Toulouse, France
and Observatoire Midi Pyrénées, 14 avenue Belin, 31400 Toulouse, France

Uriel Frisch
Centre National de la Recherche Scientifique,
Observatoire de Nice, Boîte Postale 139, 06003 Nice, France
(Received 30 July 1990; revised manuscript received 25 January 1991)

A general formalism is developed to determine eddy viscosities for incompressible flow of arbitrary dimensionality subject to forcing periodic in space and time. The dynamics of weak large-scale perturbations is obtained by a multiscale analysis. The large-scale behavior is found to be formally diffusive (first order in time, second order in space) whenever the basic flow is parity invariant, that is, possesses a center of symmetry. The eddy viscosity is in general a fourth-order tensor, for which a compact representation is provided. Explicit expressions of the eddy-viscosity tensor are given (i) for basic flow with low Reynolds numbers, and (ii) when the basic flow is layered, i.e., depends only on one space coordinate and time. A special class of layered flow is two-dimensional, time-independent parallel periodic flow, an example of which is the Kolmogorov flow. Such parallel flow acquires a negative-viscosity instability to large-scale perturbations transverse to the basic flow when the molecular viscosity becomes less than the rms value of the stream function of the basic flow. For flows presenting less symmetry than the Kolmogorov flow, the first large-scale instability is usually found not to be transverse, thus breaking the spatial periodicity of the basic flow. Such nontransverse instabilities, observed in a lattice-gas simulation on the Connection Machine, are reported in the companion paper by Hénon and Scholl [following paper, Phys. Rev. A **43**, 5365 (1991)].

I. INTRODUCTION

The idea that molecular transport can be enhanced by turbulent motion is at the very core of turbulence modeling. It has been used since Ludwig Prandtl's time in very diverse fields, including, for example, astrophysics (mixing length theory of energy transport in stellar interiors) and engineering (k-ϵ modeling of complex flows).

Transport of *scalar* quantities on scales much larger than the (energy) scale of the turbulent flow, indeed, typically, leads to enhanced diffusion. However, transport of *vector* quantities, such as momentum or magnetic fields (i) need not be diffusive [instances of nondiffusive behavior are the α effect governing large-scale instabilities in magnetohydrodynamics[1] (MHD) and its hydrodynamical counterpart[2,3]], (ii) may result in *depleted* rather than enhanced diffusion, possibly leading to negative-viscosity[4] instabilities, as found for the Kolmogorov flow.[5,6]

Eddy viscosities, which control the large-scale transport of momentum, are mostly introduced in a phenomenological way or derived within closure approximations.[7,8] A systematic theory is needed to define the frame of validity of the concept and to specify the steps involved in evaluating eddy viscosities.

Various instances of systematic theories exist already for special time-independent flows. This includes parallel periodic flow[5,6,9] and two-dimensional periodic flow with vanishing nonlinearity.[10,11] Generalization of such approaches involves, as we shall see, at least one novel conceptual difficulty, connected with parity invariance.

To bring out this difficulty we define now the framework of our investigation. We assume that there is a basic incompressible flow $(p,\mathbf{u})$, where the pressure p and the velocity $\mathbf{u}$ are periodic in the space variable $\mathbf{x}$ and in the time variable t.[12] This flow is a solution of the D-dimensional incompressible Navier-Stokes equations with a force $\mathbf{f}$,

$$\begin{aligned} &\boldsymbol{\partial}\cdot\mathbf{u}=0, \\ &\partial_t\mathbf{u}+(\mathbf{u}\cdot\boldsymbol{\partial})\mathbf{u}=-\boldsymbol{\partial}p+\nu\partial^2\mathbf{u}+\mathbf{f}\ . \end{aligned} \tag{1}$$

The force depends only on space and time and is periodic in all the variables. Its space-time average over the periodicities, $\langle \mathbf{f}\rangle$, vanishes, and so does $\langle \mathbf{u}\rangle$.[13] Note that spatial gradients are here denoted by the symbol $\boldsymbol{\partial}$.

Our interest is to find the way the flow is modified when subject to a weak large-scale perturbation $(P,\mathbf{W})$. Before engaging in systematic studies of asymptotic expansions we provide some heuristic insight, using a mean-field argument in the sense of Moffatt.[14] The (periodicity-averaged) Reynolds stresses $R_{ij}=\langle u_i u_j\rangle$ of the basic flow are modified through the perturbation by an amount δR_{ij} which contributes to the large-scale momentum flux. In the linear approximation, it can be expanded in a gradient Taylor series:

$$\delta R_{ij}=-\alpha_{ijl}W_l-\nu_{ijlm}\partial_l W_m+O(\partial^2 W).$$

When only the second term on the right-hand side is present, the change in the Reynolds stresses is linearly proportional to the large-scale velocity gradient. Thus there is an (eddy-diffusive) modification of molecular transport of momentum. However, the presence of the first term, directly proportional to the large-scale velocity, cannot in general be ruled out. Indeed, the presence of the force driving the small-scale flow breaks the Galilean invariance, so that a uniform large-scale flow with vanishing gradient may induce nontrivial changes in the small-scale dynamics. This effect, known as the anisotropic kinetic alpha (AKA) effect, is the hydrodynamical counterpart of the α effect of MHD.[1-3,14] In order for the large-scale dynamics to be dominated by eddy-viscosity effects, the AKA effect must vanish. There are a number of special cases where this happens (see Sec. II and Ref. 3). Previous systematic studies of eddy viscosities (see, e.g., Refs. 5, 6, 9, and 10) used flows for which the AKA effect vanishes. One exception is Kraichnan's integral equation technique for eddy-transport coefficients.[15] He noticed the importance of parity invariance in avoiding AKA-type terms. We shall comment on this work at the end of Sec. II D.

For general multidimensional time-dependent flow, the simplest assumption which guarantees the absence of an AKA effect is *parity invariance,* i.e., the presence of a center of spatial symmetry. Indeed, it is clear that the entries of the tensor α_{ijl} are pseudoscalars which change sign under space reversal. Thus parity invariance rules out first-order space derivatives in the large-scale dynamics.

The paper is organized as follows. Section II contains the general theory of the eddy viscosity. It is divided into several subsections. Section II A presents useful notation and properties of the linearized Navier-Stokes operator. The asymptotic multiscale formalism is presented in Sec. II B. It is an adaptation of a formalism developed for transport of scalars and magnetic fields.[16] The assumption of parity invariance is used for the first time in Sec. II C. The general expression of the eddy viscosity (tensor) is given in Sec. II D together with some comments on the derivation. Section III treats the special case of low-Reynolds-number flow for which the eddy viscosity can be obtained perturbatively. Section IV is devoted to layered flow for which explicit results are available without the low-Reynolds-number restriction. Detailed results for parallel time-independent flow (including variants of the Kolmogorov flow) are given in Sec. IV A. Concluding remarks are made in Sec. V.

II. MULTISCALE FORMALISM FOR THE EDDY VISCOSITIES

A. The linearized Navier-Stokes operator

Let $(p,\mathbf{u})$ denote any solution, called the *basic flow,* of (1) with the same periodicity as the force $\mathbf{f}$ (say, 2π) and with vanishing space-time average.

We introduce a perturbation by the following substitutions:

$$p \to p+\eta P, \quad \mathbf{u}\to\mathbf{u}+\eta\mathbf{W}, \quad \eta\to 0. \tag{2}$$

The strength of the perturbation is controlled by the small parameter η. Note that we reserve the notation ϵ to denote the separation of scales (Sec. II B). Substituting (2) in (1), we obtain to order η the following linearized Navier-Stokes equations:[17]

$$\begin{aligned}&\partial\cdot\mathbf{W}=0,\\ &\partial_t W_i+\partial_j(u_iW_j+u_jW_i)=-\partial_iP+\nu\partial^2W_i\ .\end{aligned} \tag{3}$$

Except for some remarks in the conclusion, we shall limit our investigations to the linearized Navier-Stokes equations. Technically, this means that the limit $\eta\to 0$ is taken before the limit of large-scale separation ($\epsilon\to 0$).

An operator formalism will permit compact notation in subsequent developments. Equation (3) is rewritten as

$$\mathcal{A}\begin{bmatrix}P\\ \mathbf{W}\end{bmatrix}\equiv\begin{bmatrix}\mathcal{A}_{PP} & \mathcal{A}_{PW}\\ \mathcal{A}_{WP} & \mathcal{A}_{WW}\end{bmatrix}\begin{bmatrix}P\\ \mathbf{W}\end{bmatrix}=0. \tag{4}$$

The D components of the velocity field perturbation $\mathbf{W}$ and the pressure perturbation P have been lumped into a $(D+1)$-dimensional vector $(P,\mathbf{W})$. [Capital Greek letters such as $\mathbf{\Psi}$ will also be used to denote $(D+1)$-vectors.] The linearized Navier-Stokes operator $\mathcal{A}$ in (4) is made of operator-valued matrix blocks defined by

$$\begin{aligned}&\mathcal{A}_{PP}=0,\quad \mathcal{A}_{PW_i}=\partial_i,\quad \mathcal{A}_{W_iP}=\partial_i,\\ &\mathcal{A}_{W_iW_j}=(\partial_t-\nu\partial^2)\delta_{ij}+\partial_k(\delta_{ij}u_k\bullet+\delta_{kj}u_i\bullet).\end{aligned} \tag{5}$$

Here, the bullet symbol $\bullet$ indicates that u_k and u_i act multiplicatively.

When the linearized Navier-Stokes operator $\mathcal{A}$ is restricted to functions which have the same space-time periodicity as the force, it will be denoted by A (and similarly for its matrix blocks such as A_{PW}).

We now discuss some useful properties of the operator A.

We define the *parity* operator $\mathcal{P}$ as simultaneous reversal of position and velocity with no change in the pressure:

$$\mathcal{P}\mathbf{x}=-\mathbf{x},\quad \mathcal{P}\mathbf{u}=-\mathbf{u},\quad \mathcal{P}p=p\ . \tag{6}$$

When $\mathcal{P}(p,\mathbf{u})=(p,\mathbf{u})$ the basic flow is said to be *parity invariant.* (This may require a change of the origin of coordinates.) For parity-invariant basic flow, the linearized Navier-Stokes operator has the following transformation rules [an immediate consequence of the definitions (5)]:

$$\mathcal{P}A_{PW}=-A_{PW},\quad \mathcal{P}A_{WP}=-A_{WP},\quad \mathcal{P}A_{WW}=A_{WW}. \tag{7}$$

In the sequel we shall have to solve various problems of the form

$$A\mathbf{\Psi}=\mathbf{\Phi}, \tag{8}$$

where $\mathbf{\Phi}$ is a prescribed $(D+1)$-dimensional vector field which is space-time periodic. Since all entries of A given in (5) begin with space or time derivatives on the left, it follows that a necessary condition for solvability of (8) is

$$\langle\mathbf{\Phi}\rangle=0. \tag{9}$$

Note that this is equivalent to stating that $\mathbf{\Phi}$ is orthogonal (in the sense of the L^2 inner product on periodic functions) to constants. The latter are obviously in the null space of the adjoint $A^\dagger$ of A. To make (9) a necessary and sufficient condition of solvability of (8), we shall assume that *the restriction* $\tilde{A}$ *of the operator* A *to periodic functions of zero mean value is invertible.* Its inverse will be denoted $\tilde{A}^{-1}$. We observe that for large viscosities, $\tilde{A}^{-1}$ may be constructed perturbatively, starting from the $(\partial_t - \nu\partial^2)^{-1}$, the inverse of the heat operator, which is uniquely defined on periodic functions of zero mean value. Using $\tilde{A}^{-1}$ we can give an explicit representation of the general solution to (8). We assume that the solvability condition (9) holds; we write $\boldsymbol{\Psi} = \langle \boldsymbol{\Psi} \rangle + \tilde{\boldsymbol{\Psi}}$, and substitute into (8). Operating with $\tilde{A}^{-1}$ on the left, we obtain the general solution of (8):

$$\boldsymbol{\Psi} = \tilde{A}^{-1}\boldsymbol{\Phi} + (I - \tilde{A}^{-1}A)\langle \boldsymbol{\Psi} \rangle, \tag{10}$$

where the mean value $\langle \boldsymbol{\Psi} \rangle$ can be taken arbitrary.

In particular, taking $\boldsymbol{\Phi} = \mathbf{0}$, we obtain a general representation of the nullspace of A in the form

$$\Xi = (I - \tilde{A}^{-1}A)\langle \Xi \rangle, \tag{11}$$

with arbitrary Ξ.

B. Asymptotic analysis

We now consider solutions of (3) which do not have the same periodicity as the basic flow. The perturbation $(P, \mathbf{W})$ is assumed to have nontrivial spatial variations on a scale $O(\epsilon^{-1})$. If diffusive behavior is present on large scales, it will take place on a time scale $O(\epsilon^{-2})$. It is thus appropriate to use (as in Ref. 16) a multiscale formalism with the "fast" variables t and $\mathbf{x}$ and the "slow" variables

$$T = \epsilon^2 t, \quad \mathbf{X} = \epsilon \mathbf{x}. \tag{12}$$

In the multiscale technique, the perturbed flow $(P, \mathbf{W})$ is considered to be a function both of fast and slow variables which are independent (see, for example, Ref. 18). The basic flow $(p, \mathbf{u})$ depends only on fast variables. Space-time periodicity is assumed for all the dependences on fast variables. The appropriate dynamical equations are obtained by applying the decomposition rule for time and space derivatives:

$$\partial_t \to \partial_t + \epsilon^2 \partial_T, \quad \boldsymbol{\partial} \to \boldsymbol{\partial} + \epsilon \boldsymbol{\nabla}, \tag{13}$$

where $\boldsymbol{\nabla}$ denotes partial derivatives with respect to $\mathbf{X}$. Use of (13) in (3) leads to

$$(A + \epsilon B + \epsilon^2 C)\begin{bmatrix} P \\ \mathbf{W} \end{bmatrix} = 0, \tag{14}$$

where A which depends only on fast variables, has already been defined (5) and B and C are given by

$$B = \begin{bmatrix} 0 & B_{PW} \\ B_{WP} & B_{WW} \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 0 & C_{WW} \end{bmatrix}. \tag{15}$$

The various blocks in (15) are given by

$$B_{PW_i} = \nabla_i, \quad B_{W_iP} = \nabla_i,$$
$$B_{W_iW_j} = -2\nu\delta_{ij}\partial_k\nabla_k + \nabla_k(\delta_{ij}u_k \bullet + \delta_{jk}u_i \bullet), \tag{16}$$

and

$$C_{W_iW_j} = (\partial_T - \nu\nabla^2)\delta_{ij}. \tag{17}$$

Since B is linear in slow space derivatives, it may be further decomposed as

$$B = B^l \nabla_l,$$
$$B^l_{PW_i} = B^l_{W_iP} = \delta_{il}, \tag{18}$$
$$B^l_{W_iW_j} = -2\nu\delta_{ij}\partial_l + \delta_{ij}u_l \bullet + \delta_{jl}u_i \bullet.$$

We look for a solution of (14) which can be expanded in powers of ϵ,

$$\begin{bmatrix} P \\ \mathbf{W} \end{bmatrix} = \begin{bmatrix} P^{(0)} \\ \mathbf{W}^{(0)} \end{bmatrix} + \epsilon \begin{bmatrix} P^{(1)} \\ \mathbf{W}^{(1)} \end{bmatrix} + \epsilon^2 \begin{bmatrix} P^{(2)} \\ \mathbf{W}^{(2)} \end{bmatrix} + O(\epsilon^3). \tag{19}$$

We substitute (19) into (14) and require that the result vanishes to all orders in ϵ. This leads to a hierarchy of equations of which we shall need only the first three, namely,

$$A\begin{bmatrix} P^{(0)} \\ \mathbf{W}^{(0)} \end{bmatrix} = 0, \tag{20}$$

$$A\begin{bmatrix} P^{(1)} \\ \mathbf{W}^{(1)} \end{bmatrix} + B\begin{bmatrix} P^{(0)} \\ \mathbf{W}^{(0)} \end{bmatrix} = 0, \tag{21}$$

$$A\begin{bmatrix} P^{(2)} \\ \mathbf{W}^{(2)} \end{bmatrix} + B\begin{bmatrix} P^{(1)} \\ \mathbf{W}^{(1)} \end{bmatrix} + C\begin{bmatrix} P^{(0)} \\ \mathbf{W}^{(0)} \end{bmatrix} = 0. \tag{22}$$

We now proceed to solve these equations. The key is that the operator A depends only on fast variables, so that any of the above equations has the form (8) discussed in Sec. II A, provided we consider the dependence on slow variables as being parametric. Averaging is from now on understood to be only over fast variables.

From (20) it follows that $(P^{(0)}, \mathbf{W}^{(0)})$ is in the null space of A; therefore by (11),

$$\begin{bmatrix} P^{(0)} \\ \mathbf{W}^{(0)} \end{bmatrix} = (I - \tilde{A}^{-1}A)\begin{bmatrix} \langle P^{(0)} \rangle \\ \langle \mathbf{W}^{(0)} \rangle \end{bmatrix}. \tag{23}$$

Using this in (21), we obtain

$$A\begin{bmatrix} P^{(1)} \\ \mathbf{W}^{(1)} \end{bmatrix} + B(I - \tilde{A}^{-1}A)\begin{bmatrix} \langle P^{(0)} \rangle \\ \langle \mathbf{W}^{(0)} \rangle \end{bmatrix} = 0. \tag{24}$$

This equation for $(P^{(1)}, \mathbf{W}^{(1)})$ will be solvable provided the second term has vanishing average:

$$\left\langle B(I - \tilde{A}^{-1}A)\begin{bmatrix} \langle P^{(0)} \rangle \\ \langle \mathbf{W}^{(0)} \rangle \end{bmatrix} \right\rangle = 0. \tag{25}$$

C. Parity invariance

The condition (25), which expresses solvability to order ϵ, is a major source of difficulty in the eddy-viscosity analysis. To understand this, we perform a block decomposition of (25). We observe that (i) only the block B_{WW} in the operator B involves fast variables [see Eq. (16)], and (ii) that the average of any quantity having an operator $\tilde{A}^{-1}$ on the left vanishes since $\tilde{A}$ maps zero mean-value functions onto zero mean-value functions. It follows that (25) may be rewritten

$$\nabla\cdot\langle \mathbf{W}^{(0)}\rangle=0,$$
$$\nabla\langle P^{(0)}\rangle+\langle B_{WW}(I-\tilde{A}^{-1}A)_{WW}\langle \mathbf{W}^{(0)}\rangle\rangle=\mathbf{0}. \tag{26}$$

Here, the notation $(O)_{WW}$ is used to denote the WW block of the operator O. We now decompose further, using the fact that B is linear in slow derivatives [cf. (18)] and that the slow gradient ∇ commutes with operators depending only on fast variables. We thus obtain

$$\nabla_j\langle W_j^{(0)}\rangle=0,$$
$$\nabla_i\langle P^{(0)}\rangle-\alpha_{ijl}\nabla_j\langle W_l^{(0)}\rangle=0, \tag{27}$$

where

$$\alpha_{ijl}=-\langle B^j_{W_iW_n}(I-\tilde{A}^{-1}A)_{W_nW_l}\rangle. \tag{28}$$

In (28) we have introduced a new notation: Given any operator M, its average $\langle M\rangle$ is defined as an operator acting only on functions which are independent of the fast variables by

$$\langle M\rangle\langle\Psi\rangle\equiv\langle M\langle\Psi\rangle\rangle\ . \tag{29}$$

It may be checked that α_{ijl} is the same third-order tensor as considered in the Introduction. When this tensor is nonvanishing, the conditions (27) prevent us from choosing the leading term $\langle \mathbf{W}^{(0)}\rangle$ of the large-scale perturbation in an arbitrary way at $T=0$ and the analysis breaks down. The physical reason for this breakdown has been discussed in the Introduction: when an AKA effect is present the large-scale dynamics is of first order in space and in time, so that the appropriate scaling of the slow time is not as in (12), but instead $T=\epsilon t$ (see Refs. 2 and 3).

From now on, we assume that the basic flow $(p,\mathbf{u})$ is *parity invariant.* This ensures the vanishing of the α_{ijl} tensor. Indeed, by the definition (6) of parity, it follows that the operator matrix block $B^j_{W_iW_n}$ given by (18) changes sign under parity. By (7) $(I-\tilde{A}^{-1}A)_{W_nW_l}$ remains invariant under parity. Hence the average in the right-hand side of (28) vanishes.

Thus, for parity-invariant basic flow, the solvability condition associated with (21) reduces to

$$\nabla_j\langle W_j^{(0)}\rangle=0 \tag{30}$$

and

$$\nabla_i\langle P^{(0)}\rangle=0. \tag{31}$$

It follows from (31) that $\langle P^{(0)}\rangle$ is constant in the space variable. Since the pressure always appears with a gradient, this constant will be taken zero without loss of generality.

D. General expression of the eddy viscosity and comments

To obtain the eddy viscosity, we must proceed to order $O(\epsilon^2)$. For this, we solve (21), using (10):

$$\begin{bmatrix}P^{(1)}\\ \mathbf{W}^{(1)}\end{bmatrix}=-\tilde{A}^{-1}B(I-\tilde{A}^{-1}A)\begin{bmatrix}\langle P^{(0)}\rangle\\ \langle \mathbf{W}^{(0)}\rangle\end{bmatrix}+(I-\tilde{A}^{-1}A)\begin{bmatrix}\langle P^{(1)}\rangle\\ \langle \mathbf{W}^{(1)}\rangle\end{bmatrix}, \tag{32}$$

where $\langle P^{(1)}\rangle$ and $\langle \mathbf{W}^{(1)}\rangle$ are so far arbitrary functions of the slow variables.

We now express the condition of solvability of (22), considered as an equation for $(P^{(2)},\mathbf{W}^{(2)})$: the sum of the averages of the second and third terms must vanish. Using (23) and (32), we obtain

$$\langle B\tilde{A}^{-1}B(I-\tilde{A}^{-1}A)\rangle\begin{bmatrix}\langle P^{(0)}\rangle\\ \langle \mathbf{W}^{(0)}\rangle\end{bmatrix}-\langle B(I-\tilde{A}^{-1}A)\rangle\begin{bmatrix}\langle P^{(1)}\rangle\\ \langle \mathbf{W}^{(1)}\rangle\end{bmatrix}-\langle C(I-\tilde{A}^{-1}A)\rangle\begin{bmatrix}\langle P^{(0)}\rangle\\ \langle \mathbf{W}^{(0)}\rangle\end{bmatrix}=0. \tag{33}$$

We finally separate (33) into its P and $\mathbf{W}$ components. The simplification of the second term is done just as before for the second term in (25). We also use the fact that averages of the form $\langle B_{PW}\tilde{A}^{-1}\cdots\rangle$ and $\langle C\tilde{A}^{-1}\cdots\rangle$ vanish because B_{PW} and C have no dependence on fast variables. We thus obtain

$$\nabla_j\langle W_j^{(1)}\rangle=0 \tag{34}$$

and

$$\nabla_j\langle W_j^{(0)}\rangle=0, \tag{35}$$

$$\partial_T\langle W_i^{(0)}\rangle=\nu_{ijlm}\nabla_j\nabla_l\langle W_m^{(0)}\rangle-\nabla_i\langle P^{(1)}\rangle, \tag{36}$$

where

$$\nu_{ijlm}=\nu\delta_{jl}\delta_{im}+\langle[B^j\tilde{A}^{-1}B^l(I-\tilde{A}^{-1}A)]_{W_iW_m}\rangle. \tag{37}$$

The system of Eqs. (35) and (36) will be referred to as the *mean-field equations.* It is formally a (pressure-modified) diffusion equation with an *eddy-viscosity* tensor ν_{ijlm} given by (37). Note that (35) has been repeated from (27): Indeed, in the mean-field equations, the divergence-free condition (30) stems from solvability to order $O(\epsilon)$, while the eddy-diffusion equation emerges to order $O(\epsilon^2)$.

A number of comments are now in order. Equation (37) gives the eddy viscosity in *compact* form, but not in *explicit* form, since it involves the inversion of the linearized Navier-Stokes operator $\tilde{A}$, an inversion which can be performed explicitly only in special cases (see Secs. III and IV).

In the general anisotropic case, the eddy viscosity is a fourth-order tensor, as is to be expected since it linearly connects the large-scale velocity gradient and the resulting momentum flux, which are both second-order tensors.[19] In general the eddy-viscous term in (36) does not have a vanishing divergence (in spite of the vanishing of the divergence of $\langle \mathbf{W}^{(0)} \rangle$); hence there is a nontrivial large-scale pressure field $\langle P^{(1)} \rangle$.

In the isotropic case, we have $\nu_{ijlm} = \nu_{\text{eddy}} \delta_{jl} \delta_{im}$ and (35) and (36) reduce to

$$\partial_T \langle \mathbf{W}^{(0)} \rangle = \nu_{\text{eddy}} \nabla^2 \langle \mathbf{W}^{(0)} \rangle, \quad \nabla \cdot \langle \mathbf{W}^{(0)} \rangle = 0, \tag{38}$$

where $\nu_{\text{eddy}} = D^{-2} \nu_{ijji}$ is the usual eddy viscosity. Isotropy is obtained when the basic flow is random, isotropic, homogeneous, and stationary, rather than deterministic and space-time periodic. Our formalism remains essentially valid in the random case, provided averages are reinterpreted as ensemble averages. Even in the deterministic case, isotropy is possible if the basic flow has a discrete invariance group ensuring the isotropy of fourth-order tensor. Similar questions have been investigated in connection with lattice gases.[20] Here, we mention only that in two dimensions, a basic flow invariant under rotations of 60° has an isotropic eddy viscosity. This is the case for example of the system of triangular eddies considered in Ref. 10.

Eddy diffusivities, which characterize the diffusion of passive scalars, are easily shown to be positive.[16] Nothing similar holds for eddy viscosities, even in the isotropic case, although in the latter case no specific example is known with a negative eddy viscosity (we shall come back to this in the conclusion). A simple instance of an anisotropic flow which can have a *negative eddy viscosity* is the Kolmogorov flow (see Sec. IV A). Physically, the difference between eddy diffusivities and eddy viscosities comes from the observation that, contrary to a passive scalar, large-scale momentum is not just advected by the basic flow, but can be enhanced by shear.

In order to satisfy the solvability condition (25) which arises to order $O(\epsilon)$, we have assumed parity invariance. Other conditions may suffice for special cases, such as layered flow (Sec. IV). A well-known class of three-dimensional helical flows, which are clearly not parity invariant, are the Arnold-Beltrami-Childress (ABC) flows.[21,22] Still, it may be shown that the solvability condition (25) is satisfied as a consequence of the flows having the Beltrami property (vorticity and velocity are parallel).

Finally, we observe that Kraichnan has proposed an alternative strategy to derive exact formulas for eddy-transport coefficients.[15] Instead of performing a multiscale expansion on the basic equations (say, the linearized Navier-Stokes equations), he derives an exact integral equation which is then solved perturbatively, assuming separation of scales. In the scalar case his technique leads very quickly to an expression for the eddy-diffusivity tensor which is identical to one derived by multiscale techniques in Ref. 16. In the vector case, to obtain the correct eddy-viscosity tensor, it may be necessary to iterate Kraichnan's integral equation to the next to leading order to capture all the relevant terms.

III. LOW-REYNOLDS-NUMBER FLOW

When the Reynolds number of the basic flow is small or, equivalently, when the viscosity is large, the linearized Navier-Stokes operator A is very close to the heat operator $\partial_t - \nu\partial^2$. Thus the inverse of its restriction $\tilde{A}$ to functions of zero mean value can be calculated perturbatively in powers of $1/\nu$. For large viscosities, the basic flow may have a rapid variation with the fast time variable, on a time scale $O(1/\nu)$. This will be the case if the force $\mathbf{f}$ in the nonlinear Navier-Stokes equation (1) itself involves the time scale $O(1/\nu)$ or smaller. We shall thus introduce a new fast time variable

$$\tau = \nu t . \tag{39}$$

From now on, in this section, it is understood that $\mathbf{u}$ is a function of τ and $\mathbf{x}$. As before, $\mathbf{u}$ is assumed parity invariant.

Our first aim is to perturbatively calculate the inverse of $\tilde{A}$. To avoid proliferation of indices, we shall here use vector notation. We still use the block decomposition introduced in Sec. II A. $\tilde{A}$, given by (5), is here rewritten as

$$\tilde{A} = \begin{bmatrix} 0 & \boldsymbol{\partial}\cdot \\ \boldsymbol{\partial} & \nu\mathcal{H} + (\boldsymbol{\partial}\mathbf{u})^{\dagger} + \mathbf{u}\boldsymbol{\partial}\cdot + \mathbf{u}\cdot\boldsymbol{\partial} \end{bmatrix}, \tag{40}$$

where

$$\mathcal{H} = \partial_\tau - \partial^2 \tag{41}$$

is the heat (equation) operator and $(\boldsymbol{\partial}\mathbf{u})^{\dagger}$ is defined by

$$((\boldsymbol{\partial}\mathbf{u})^{\dagger}\mathbf{W})_i = W_j \partial_j u_i . \tag{42}$$

Here it is understood that partial-derivative operators (in fast or slow variables) such as $\boldsymbol{\partial}$, $\boldsymbol{\nabla}$, and $\mathcal{H}$ act on anything to the right unless immediately preceded by an open parenthesis, in which case they act only within the corresponding parenthetical group. The centered dot $\cdot$ is used to denote a scalar product of the vector (or operator-vector) entity immediately to the left with the first vector entity encountered on the right; the rightmost scalar products are to be performed first.

The inversion of $\tilde{A}$ is done by solving the equation $\tilde{A}\boldsymbol{\Psi} = \boldsymbol{\Phi}$, after block decomposition, perturbatively in powers of ν^{-1}. After simple algebra, we obtain

$$\tilde{A}^{-1} = \begin{bmatrix} -\nu\partial^{-2}\mathcal{H} + O(\nu^0) & \partial^{-2}\boldsymbol{\partial}\cdot + O(\nu^{-1})\cdot \\ \partial^{-2}\boldsymbol{\partial} & \nu^{-1}\mathcal{H}^{-1}(I - \partial^{-2}\boldsymbol{\partial}\boldsymbol{\partial}\cdot) + O(\nu^{-2}) \end{bmatrix} . \tag{43}$$

We can now calculate the eddy-viscosity tensor, given by (37). As an intermediate step, we evaluate the vector

$$\begin{bmatrix} 0 \\ \mathbf{S} \end{bmatrix} \equiv \langle B\tilde{A}^{-1} B (I - \tilde{A}^{-1} A) \rangle \begin{bmatrix} 0 \\ \mathbf{Q} \end{bmatrix} \tag{44}$$

where $\mathbf{Q}=\langle \mathbf{W}^{(0)}\rangle$. Note that, in view of (37), the first term on the right-hand side of Eq. (36) for the mean large-scale field is (the ith component of) $\nu\nabla^2\mathbf{Q}+\mathbf{S}$. In our compact notation, the operator B, given by (15) reads

$$B=\begin{bmatrix} 0 & \nabla\cdot \\ \nabla & -2\nu\partial\cdot\nabla+\mathbf{u}\cdot\nabla+\mathbf{u}\nabla\cdot \end{bmatrix} . \tag{45}$$

It is now straightforward but somewhat tedious to perform the various block matrix multiplications involved in (44). In this process, simplifications result from using the conditions $\partial\cdot\mathbf{u}=0$ and $\nabla\cdot\langle \mathbf{W}^{(0)}\rangle=0$. We finally obtain

$$\mathbf{S}=\nu^{-1}(-\langle \mathbf{u}\cdot\nabla\partial^{-2}\partial\nabla\cdot\mathcal{H}^{-1}\mathbf{Q}\cdot\partial\mathbf{u}\rangle-\langle \mathbf{u}\nabla\cdot\partial^{-2}\partial\nabla\cdot\mathcal{H}^{-1}\mathbf{Q}\cdot\partial\mathbf{u}\rangle+2\langle \mathbf{u}\cdot\nabla\mathcal{H}^{-2}\partial\cdot\nabla\mathbf{Q}\cdot\partial\mathbf{u}\rangle+2\langle \mathbf{u}\nabla\cdot\mathcal{H}^{-2}\partial\cdot\nabla\mathbf{Q}\cdot\partial\mathbf{u}\rangle$$
$$+\langle \mathbf{u}\cdot\nabla\mathcal{H}^{-1}\mathfrak{P}\mathbf{u}\cdot\nabla\mathbf{Q}\rangle+\langle \mathbf{u}\nabla\cdot\mathcal{H}^{-1}\mathfrak{P}\mathbf{u}\cdot\nabla\mathbf{Q}\rangle)+O(\nu^{-2}), \tag{46}$$

where

$$\mathfrak{P}\equiv\mathbf{I}-\partial^{-2}\partial\partial\cdot \tag{47}$$

is the operator of projection on divergence-free functions.

Identifying the coefficients of $\nabla_j\nabla_l\langle W_m^{(0)}\rangle$, we obtain the following low-Reynolds-number expansion for the eddy-viscosity tensor:

$$\nu_{ijlm}=\nu\delta_{im}\delta_{jl}+\nu^{-1}(-2\langle u_j\mathcal{H}^{-1}\partial^{-2}\partial_i\partial_m u_l\rangle-2\langle u_i\mathcal{H}^{-1}\partial^{-2}\partial_j\partial_m u_l\rangle+\langle u_j\mathcal{H}^{-1}u_l\rangle\delta_{im}+\langle u_i\mathcal{H}^{-1}u_l\rangle\delta_{jm}$$
$$+2\langle u_j\mathcal{H}^{-2}\partial_l\partial_m u_i\rangle+2\langle u_i\mathcal{H}^{-2}\partial_l\partial_m u_j\rangle)+O(\nu^{-2}). \tag{48}$$

Obviously, this eddy viscosity is always positive (in the sense that it cannot lead to large-scale instabilities), since it is dominated by the molecular contribution. If we assume *isotropy,* as explained in the discussion of Sec. II D, we can simplify the final expression, using identities implied by isotropy and incompressibility, namely,

$$\begin{aligned}
&\langle u_i\mathcal{H}^{-1}u_k\rangle=\frac{1}{D}\delta_{ik}\langle u_l\mathcal{H}^{-1}u_l\rangle,\\
&\langle u_i\mathcal{H}^{-1}\partial^{-2}\partial_j\partial_l u_k\rangle=N[\delta_{ij}\delta_{kl}+\delta_{il}\delta_{jk}-(D+1)\delta_{ik}\delta_{jl}]\langle u_m\mathcal{H}^{-1}u_m\rangle,\\
&\langle u_i\mathcal{H}^{-2}\partial_j\partial_l u_k\rangle=N[\delta_{ij}\delta_{kl}+\delta_{il}\delta_{jk}-(D+1)\delta_{ik}\delta_{jl}]\langle u_m\mathcal{H}^{-2}\partial^2 u_m\rangle,\\
&N=\frac{1}{D(2-D-D^2)} .
\end{aligned} \tag{49}$$

Using (49) and $\nabla\cdot\langle \mathbf{W}^{(0)}\rangle=0$ in (48), we obtain

$$\nu_{\text{eddy}}=\nu\left[1+\frac{N}{\nu^2}[(2+D-D^2)\langle u_m\mathcal{H}^{-1}u_m\rangle+4\langle u_m\mathcal{H}^{-2}\partial^2 u_m\rangle]+O(\nu^{-3})\right] . \tag{50}$$

Let us now specialize to the case where the basic flow $\mathbf{u}$ is time independent or, equivalently, where it varies on a time scale slower than $O(\nu^{-1})$. The heat operator reduces then to $\mathcal{H}=-\partial^2$; hence

$$\nu_{\text{eddy}}=\begin{cases}\nu\left[1-\dfrac{1}{2\nu^2}\langle \mathbf{u}\cdot\partial^{-2}\mathbf{u}\rangle+O(\nu^{-3})\right] & \text{for } D=2,\\[2ex] \nu\left[1-\dfrac{4}{15\nu^2}\langle \mathbf{u}\cdot\partial^{-2}\mathbf{u}\rangle+O(\nu^{-3})\right] & \text{for } D=3.\end{cases} \tag{51}$$

Since the Laplacian and its inverse ∂^{-2} are negative operators (as may be seen by noting that their Fourier representations are $-k^2$ and $-1/k^2$, respectively), it follows that in the low-Reynolds-number isotropic case, the (small) correction *enhances* the molecular viscosity.

IV. LAYERED FLOW

By layered flow, we understand the case where the basic flow $\mathbf{u}$ depends only on one space coordinate, say x_1, and on time. The general assumptions used throughout this paper (incompressibility, periodicity, vanishing mean value) are kept. As for parity invariance, it may not be needed, as we shall see. Since $\langle \mathbf{u}\rangle=\mathbf{0}$, the first component of the flow vanishes and we can write

$$\mathbf{u}=(0,\mathbf{u}_\perp(x_1,t)), \tag{52}$$

where $\mathbf{u}_\perp$ is a $(D-1)$-dimensional vector. Note that we do not restrict the (slow) space dependence of the large-scale perturbation: it may depend both on X_1 and on transverse slow coordinates $\mathbf{X}_\perp$. For the layered case, it is convenient to use a decomposition in which we single out not only the pressure component but also the first vector component, so that the operators are represented as three by three matrices. For example, the linearized Navier-Stokes operator (5) becomes

$$\tilde{A}=\begin{bmatrix} 0 & \partial_1 & 0\\ \partial_1 & \mathfrak{H}_1 & 0\\ 0 & \partial_1\mathbf{u}_\perp & \mathfrak{H}_1\end{bmatrix} \tag{53}$$

where

$$\partial_1 = \frac{\partial}{\partial x_1}, \quad \mathfrak{H}_1 = \partial_t - \nu \partial_1^2. \tag{54}$$

The meaning of the notation is as explained after (42) in Sec. III.

An equation of the form $\tilde{A}\Psi = \Phi$ can now be solved explicitly and non-perturbatively, although it involves partial differential operators with non-constant coefficients (through $\mathbf{u}_\perp$). This is easily seen to be a consequence of incompressibility. The inverse of $\tilde{A}$ is given by

$$\tilde{A}^{-1} = \begin{bmatrix} -\partial_1^{-2}\mathfrak{H}_1 & \partial_1^{-1} & 0 \\ \partial_1^{-1} & 0 & 0 \\ -\mathfrak{H}_1^{-1}\partial_1 \mathbf{u}_\perp \partial_1^{-1} & 0 & \mathfrak{H}_1^{-1} \end{bmatrix}. \tag{55}$$

Here, ∂_1^{-1} is defined as the (unique) inverse of ∂_1 for functions of zero mean value, namely,

$$(\partial_1^{-1} f)(x) \equiv \int_0^\infty f(x')dx' + \frac{1}{2\pi}\int_0^{2\pi} x' f(x')dx', \tag{56}$$

and ∂_1^{-2} is the square of ∂_1^{-1}. In our block notation, the operator B given by (15) reads

$$B = \begin{bmatrix} 0 & \nabla_1 & \boldsymbol{\nabla}_\perp \\ \nabla_1 & -2\nu\partial_1\nabla_1 + \mathbf{u}_\perp\cdot\boldsymbol{\nabla}_\perp & 0 \\ \boldsymbol{\nabla}_\perp & \mathbf{u}_\perp\nabla_1 & -2\nu\partial_1\nabla_1 + \mathbf{u}_\perp\cdot\boldsymbol{\nabla}_\perp + \mathbf{u}_\perp\boldsymbol{\nabla}_\perp\cdot \end{bmatrix}, \tag{57}$$

where ∇_1 and $\boldsymbol{\nabla}_\perp$ are slow space derivatives with respect to X_1 and $\mathbf{X}_\perp$, respectively.

The general theory of the eddy viscosity, developed in Secs. II B–II D can now be worked out in fully explicit form. This requires somewhat lengthy matrix multiplications; the intermediate steps are not very enlightening and so will not be given.

The solvability condition, discussed in Sec. II C can be put in the following form for layered flow:

$$\langle \mathbf{u}_\perp\cdot\boldsymbol{\nabla}_\perp q[(\mathfrak{H}_1^{-1})^\dagger - \mathfrak{H}_1^{-1}]\partial_1\mathbf{u}_\perp\rangle = 0, \quad \forall\ \boldsymbol{\nabla}_\perp q, \tag{58}$$

where $\mathfrak{H}_1$ is the heat operator defined in (54) and $(\mathfrak{H}_1^{-1})^\dagger$ is the adjoint of its inverse. This condition is of course satisfied for parity-invariant flow. It is noteworthy that when the basic flow is layered and time independent, the condition is also satisfied. Indeed, the heat operator reduces then to $-\nu\partial_1^2$, which is self-adjoint, so that the left-hand side of (58) vanishes.

Rather than writing the explicit form for the eddy viscosity (37), we shall now give the explicit form for the mean-field equations (35) and (36). With the notation

$$\langle \mathbf{W}^{(0)}\rangle = (q, \mathbf{Q}_\perp), \quad \langle P^{(1)}\rangle = P', \tag{59}$$

we obtain

$$\begin{aligned} &\nabla_{1q} + \boldsymbol{\nabla}_\perp\cdot\mathbf{Q}_\perp = 0, \\ &\partial_T q = -\nabla_1 P' + \nu\nabla^2 q - \langle \mathbf{u}_\perp\cdot\boldsymbol{\nabla}_\perp\mathfrak{H}_1^{-1}\mathbf{u}_\perp\cdot\boldsymbol{\nabla}_\perp q\rangle, \\ &\partial_T\mathbf{Q}_\perp = -\boldsymbol{\nabla}_\perp P' + \nu\nabla^2\mathbf{Q}_\perp + \mathbf{S}_\perp, \end{aligned} \tag{60}$$

where

$$\begin{aligned} \mathbf{S}_\perp = &-\langle \mathbf{u}_\perp\nabla_1\mathfrak{H}_1^{-1}\mathbf{u}_\perp\cdot\boldsymbol{\nabla}_\perp q\rangle + \langle \mathbf{u}_\perp\cdot\boldsymbol{\nabla}_\perp\mathfrak{H}_1^{-1}\partial_1\mathbf{u}_\perp\mathfrak{H}_1^{-1}\mathbf{u}_\perp\cdot\boldsymbol{\nabla}_\perp q\rangle + \langle \mathbf{u}_\perp\cdot\boldsymbol{\nabla}_\perp\mathfrak{H}_1^{-1}\mathbf{u}_\perp\cdot\boldsymbol{\nabla}^\perp\mathbf{Q}_\perp\rangle + 2\nu\langle \mathbf{u}_\perp\cdot\boldsymbol{\nabla}_\perp\mathfrak{H}_1^{-2}\partial_1^2\mathbf{u}_\perp\nabla_1 q\rangle \\ &-\langle \mathbf{u}_\perp\cdot\boldsymbol{\nabla}_\perp\mathfrak{H}_1^{-1}\mathbf{u}_\perp\cdot\boldsymbol{\nabla}_\perp q\,\mathfrak{H}_1^{-1}\partial_1\mathbf{u}_\perp\rangle - \langle \mathbf{u}_\perp\cdot\boldsymbol{\nabla}_\perp\mathfrak{H}_1^{-1}\mathbf{u}_\perp\mathfrak{H}_1^{-1}\partial_1\mathbf{u}_\perp\cdot\boldsymbol{\nabla}_\perp q\rangle + \langle \mathbf{u}_\perp\nabla_\perp\cdot\mathfrak{H}_1^{-1}\partial_1\mathbf{u}_\perp\mathfrak{H}_1^{-1}\mathbf{u}_\perp\cdot\boldsymbol{\nabla}_\perp q\rangle \\ &+\langle \mathbf{u}_\perp\boldsymbol{\nabla}_\perp\cdot\mathfrak{H}_1^{-1}\mathbf{u}_\perp\cdot\boldsymbol{\nabla}_\perp\mathbf{Q}_\perp\rangle + 2\nu\langle \mathbf{u}_\perp\boldsymbol{\nabla}_\perp\cdot\mathfrak{H}_1^{-2}\partial_1^2\mathbf{u}_\perp\nabla_1 q\rangle - \langle \mathbf{u}_\perp\boldsymbol{\nabla}_\perp\cdot\mathfrak{H}_1^{-1}\mathbf{u}_\perp\cdot\boldsymbol{\nabla}_\perp q\,\mathfrak{H}_1^{-1}\partial_1\mathbf{u}_\perp\rangle \\ &-\langle \mathbf{u}_\perp\boldsymbol{\nabla}_\perp\cdot\mathfrak{H}_1^{-1}\mathbf{u}_\perp\mathfrak{H}_1^{-1}\partial_1\mathbf{u}_\perp\cdot\boldsymbol{\nabla}_\perp q\rangle. \end{aligned} \tag{61}$$

Note that all the q and $\mathbf{Q}_\perp$ factors could be taken out of the averages, since they do not depend on fast variables. This would, however, make the mean-field equations even more cumbersome, because it requires the use of indices.

Many special cases can be worked out in detail from the above general mean-field equations (60). In the next section, we shall be able to carry our investigation further and obtain explicit stability results by restricting the class of layered flows.

A. Parallel time-independent flow

Parallel flow is the subset of layered flow which has $D=2$, so that the stream lines of the basic flow are all parallel. In the time-independent case, the basic flow has the form

$$u_1 = 0, \quad u_\perp = u_2 = -\partial_1\psi, \tag{62}$$

where the stream function ψ is an arbitrary 2π-periodic function with zero mean value. A special case is the *Kolmogorov flow*,[23] for which $\psi = \cos x_1$.

Parallel time-independent flow is a solution of the Navier-Stokes equations (1) with a force $\mathbf{f} = (0, f_2)$, where $f_2 = -\nu\partial_1^2 u_2$. Note that, for this case, only the viscous and force terms are nonvanishing in (1).

The eddy viscosity of parallel flow can in principle be obtained using a stream function formalism simpler than our general D-dimensional formalism.[24] Here, it suffices, however, to specialize the analysis of Sec. IV. For parallel time-independent flow, the mean-field equations (60) take the following form:

$$\begin{aligned} &\nabla_{1q} + \nabla_2 Q = 0, \\ &\partial_T q = -\nabla_1 P' + \nu\nabla^2 q - \frac{\lambda}{\nu}\nabla_2^2 q, \\ &\partial_T Q = -\nabla_2 P' + \nu\nabla^2 Q + \frac{7\lambda}{\nu}\nabla_2^2 Q - \frac{6\mu}{\nu^2}\nabla_2^2 q. \end{aligned} \tag{63}$$

Here, q and Q denote the X_1 and X_2 components of the mean field $\langle \mathbf{W}^{(0)} \rangle$ and

$$\lambda = \langle \psi^2 \rangle, \quad \mu = \tfrac{1}{2} \langle \psi^3 \rangle. \tag{64}$$

The eddy viscosities can be read off (63). A particularly simple result is obtained when the large-scale perturbation is *transverse* to the basic flow: $Q=0$ and q and P' depend only on X_2 and T. Thus (63) reduces to

$$\partial_T q = \left[\nu - \frac{1}{\nu} \langle \psi^2 \rangle \right] \nabla_2^2 q. \tag{65}$$

Hence the transverse eddy viscosity is given by

$$\nu_{\text{eddy}}^{\perp} = \nu - \frac{1}{\nu} \langle \psi^2 \rangle, \tag{66}$$

a result obtained, for example, in Refs. 25 and 9. Note that the eddy viscosity changes sign at

$$\nu = \nu_c^{\perp} = \langle \psi^2 \rangle^{1/2}. \tag{67}$$

For the Kolmogorov flow $\nu_c^{\perp} = \sqrt{1/2}$.[23]

Before discussing the case of general nontransverse perturbations, we observe that the eddy diffusivity governing the large-scale behavior of scalars is easily obtained for parallel time-independent flow using the technique of Ref. 16. The result is

$$\kappa_{\text{eddy}}^{\perp} = \kappa + \frac{1}{\kappa} \langle \psi^2 \rangle, \tag{68}$$

where κ is the molecular diffusivity. Note that (66) has a *minus* sign where (68) has a plus sign: for scalars, eddy motion can only enhance transport. In contrast, the transport of vector quantities, such as momentum, can be depleted by eddy motion. In the special case under investigation, the discrepancy between (66) and (68) can be traced back to a pressure effect.

We now turn to nontransverse perturbations. We look for solutions of (63) of the form

$$\begin{aligned} &q = \hat{q} \Lambda, \quad Q = \hat{Q} \Lambda, \quad P' = \hat{P}' = \Lambda, \\ &\Lambda = \exp(sT + i\alpha X_1 + i\beta X_2). \end{aligned} \tag{69}$$

This leads easily to the following dispersion relation:

$$\begin{aligned} s &= \frac{F(\alpha,\beta)}{\alpha^2+\beta^2}, \\ F(\alpha,\beta) &= -\nu\alpha^4 - \left[2\nu + \frac{7\lambda}{\nu} \right] \alpha^2\beta^2 \\ &\quad - \frac{6\mu}{\nu^2}\alpha\beta^3 + \left[\frac{\lambda}{\nu} - \nu \right] \beta^4. \end{aligned} \tag{70}$$

The stability of the basic flow to weak large-scale perturbation is controlled by the sign of $F(\alpha,\beta)$. It is seen that, for large enough ν, the function F is negative for arbitrary α and β, which implies stability. When $\nu < \nu_c^{\perp} = \sqrt{\lambda}$, the function F is positive for $\alpha=0$. Hence there is a critical value ν_c of the viscosity where stability is lost for some direction of the perturbation wave vector (α,β). This value may be obtained by demanding that the quartic equation $F(\alpha,1)=0$ have a double root. When $\mu=0$ the double root is at $\alpha=0$. Hence, for small μ, the critical viscosity may be obtained perturbatively by ignoring higher than quadratic terms. The result, expressed directly in terms of the moments of the stream function ψ of the basic flow, is

$$\nu_c = \langle \psi^2 \rangle^{1/2} \left[1 + \frac{1}{8} \frac{\langle \psi^3 \rangle^2}{\langle \psi^2 \rangle^3} + O(\langle \psi^3 \rangle^4) \right]. \tag{71}$$

The direction of the wave vector (α_c,β_c) for $\nu=\nu_c$ is given by

$$\frac{\alpha_c}{\beta_c} = -\frac{1}{6} \frac{\langle \psi^3 \rangle}{\langle \psi^2 \rangle^{3/2}} + O(\langle \psi^3 \rangle^3). \tag{72}$$

It follows from (72) that when the Kolmogorov flow is modified, for example, by adding to $\psi = \cos x_1$ a small perturbation $\sigma \cos 2x_1$, then the first large-scale instability obtained when the viscosity is decreased is no more transverse but *tilted* (by an angle proportional to σ for small σ). Indeed, the perturbation breaks one of the fundamental symmetries of the Kolmogorov flow, the symmetry with respect to the line $x_1 = \pi/2$.

More generally, it is shown in the companion paper[26] that when perturbations of arbitrary functional form (respecting the periodicity) and of arbitrary strength are considered, the tilt angle can take any value between $-30°$ and $+30°$.

We observe that all the stability results in this section can be extended to layered time-independent flow with $D>2$ components: if $\mathbf{e}_1$ and $\mathbf{k}$ are the unit vector of the x_1 axis and the wave vector of the perturbation (assumed to be a plane wave in the coordinates other than x_1), then the projection of the perturbation $\mathbf{w}$ on the two-dimensional space spanned by $\mathbf{e}_1$ and $\mathbf{k}$ decouples from the $D-2$ perpendicular components, the latter behaving as passive scalars.[27]

An example of a nontransverse large-scale instability of parallel time-independent flow, obtained by direct numerical simulation, is shown in the companion paper by Hénon and Scholl.[26]

V. CONCLUDING REMARKS

We begin by summarizing the main results. The central assumption needed for a systematic theory of the eddy viscosity is a wide separation between the scale of the basic flow and that of the perturbation. Parity invariance has been identified as a key constraint on flow with space-time periodic forcing. It allows a (leading-order) diffusive response to weak large-scale perturbations. The eddy viscosity is a fourth-order tensor with a compact expression (37). This expression involves (i) a fully explicit operator B, given by (15) and (16), and (ii) the operator $\tilde{A}^{-1}$, the inverse of the linearized Navier-Stokes operator (4) restricted to periodic functions of zero mean value. The latter does not in general possess a closed-form expression and may have to be evaluated numerically or perturbatively (Sec. III). An important exception is the class of *layered flow* of arbitrary dimensionality (Sec. IV). For these a fully explicit (but not so compact) expression of the eddy viscosity is provided by Eqs. (60) and (61). For layered flow, the constraint of parity invariance may be replaced by time independence. Detailed results are given for the special case of *parallel time-independent flow* (Sec. IV A). Such flow generally undergoes a *negative-viscosity*[4] large-scale instability for a critical value of the viscosity.

The only flows known so far possessing a negative eddy viscosity are highly anisotropic. It is an open question whether *isotropic* flow (in the sense of the discussion at the end of Sec. II D) can have a negative eddy viscosity. In the isotropic case, results are available only in the low-Reynolds-number case (Sec. III); the eddy viscosity is then dominated by its molecular value which is positive and the small correction stemming from the eddy motion is itself positive. As the Reynolds number is increased this trend need not persist. A numerical search for negative eddy viscosities could be done by an optimization strategy within the class of time-independent two-dimensional flow with sixfold rotational symmetry, a condition which implies isotropy of fourth-order tensors.

Flow regimes presenting negative-viscosity large-scale instabilities cannot be studied within the restricted framework assumed in this paper, because a negative diffusion equation constitutes an ill-posed problem. A correct theory should include dissipative and nonlinear terms as well. Actually, such a theory has already been worked out for the Kolmogorov flow.[5,6,25,28,29] We summarize some salient qualitative features. When the Reynolds number is just slightly above the critical value $\sqrt{2}$, the appropriate asymptotic equation has been obtained by Nepomnyachtchyi[5] and is a special case of the Cahn-Hilliard equation.[30,31] This equation includes second and fourth-order space derivatives and a cubic nonlinear term. The second-order derivative term, which has a negative-viscosity coefficient, is destabilizing, while the fourth-order derivative term is stabilizing. Their balance determines the wave number of maximum linear growth rate. Nonlinear mechanisms then produce a succession of long-lasting quasiequilibrium states; eventually, in a finite system, a stable steady equilibrium emerges with the dominant excitation at the minimum available wave number.[29] The Kolmogorov flow has very special symmetries. For example, it is parity invariant and symmetrical with respect to $x_1=\pi/2$. For other parallel time-independent flow, the nonlinear large-scale dynamics is quite different: the leading order is governed by a Korteweg–de Vries (KdV) equation, while amplitudes are selected by the next order.[27]

We now make some comments on possible applications of the results obtained in this paper. Experimentally, two-dimensional flow subject to spatially periodic forcing can be realized by magnetic action on a thin layer of an electrolyte.[32] As discussed in Refs. 6 and 32, the standard analysis for the Kolmogorov flow must then be modified to include a (usually strong) linear friction term coming from the interaction with the bottom of the containing vessel. This does not affect the result that the first large-scale instability to appear, when increasing the Reynolds number, is transverse to the basic flow.[33] Modified flows with less symmetry should also be investigated experimentally and may display interesting nonlinear regimes.

We turn to issues of theoretical interest. Very few rigorous results are available for the three-dimensional Navier-Stokes equations (see, e.g. Ref. 34). Global (for all times) existence, regularity, and uniqueness results are available only for small Reynolds numbers. There is no such result for flow in an unbounded domain and with infinite energy. Consider the special case of unbounded flow driven by a space-time periodic force, and such that the Reynolds (or Grasshof[35]) number, based on the spatial periodicity, is small. Proving existence, regularity, and uniqueness within the class of solutions with the same spatial periodicity is easy. Without that restriction, it may not even be true. Indeed, if the force is chiral (not parity invariant), a large-scale instability of the AKA type is possible.[2,3] If, however, parity invariance holds, then our analysis indicates that large-scale instabilities are ruled out and that existence, regularity, and uniqueness will hold irrespective of the size of the domain. Indeed, large-scale perturbations should be diffusively damped, since at low Reynolds numbers, the eddy viscosity is dominated by its molecular contribution. A rigorous proof of this result could be based on homogenization techniques.[18]

The concept of eddy viscosity plays a central role in the renormalization-group (RG) theory of turbulence.[36,37] The usual approach borrows heavily from the theory of critical phenomena developed by Wilson.[38] Our theory of the eddy viscosity can be used to construct an alternative approach which is more in the tradition of applied mathematics. This requires an iterated multiscale expansion which takes advantage of the observation that, in RG problems, the dominant interactions are between widely separated scales.[39] Details will be discussed elsewhere.

The main field of application of eddy-viscosity ideas, as mentioned in the Introduction, is in turbulence modeling. In the so-called large-eddy simulations, the effect of subgrid-scale motion is often modeled by an eddy-viscosity term. It is natural to ask if our systematic theory of the eddy viscosity provides some justification for such procedures. Consider, for example, the somewhat idealized case of turbulence which is homogeneous and parity invariant (in a statistical sense). Let us denote by $\mathbf{u}^>$ the (unknown) subgrid-scale velocity field, i.e., the contribution to the velocity field of the Fourier components having a wave number greater than the cutoff $K_{\max}$ imposed by the coarseness of the numerical grid. The effect of $\mathbf{u}^>$ on *weak* perturbations at scales *much larger* than $K_{\max}^{-1}$ may be represented by an eddy viscosity (tensor). In reality there is usually neither the required separation of scales nor of intensity. Furthermore, the eddy viscosity need not be positive, especially so when the small-scale flow has a quasilayered structure. If the eddy viscosity is negative, higher-order derivative terms will have to be included as well. If, however, the subgrid-scale eddy viscosity is positive, then its strongest effect will be on contiguous supergrid scales having wave numbers just less that $K_{\max}$, so that the use of the eddy viscosity becomes questionable. Still, the concept of eddy viscosity has been probably the most fruitful idea in turbulence, as far as practical calculation of turbulent flow is concerned. A reasonably complete and systematic theory of the eddy viscosity is now available. It is a bit paradoxical that such a theory becomes inapplicable just where it may be most useful.

ACKNOWLEDGMENTS

We have benefited from discussions with E. Aurell, R. H. Kraichnan, A. A. Nepomnyachtchyi, Z. S. She, and with students of the DEA Turbulence et Systèmes Dynamiques of Nice University. This work was supported by the European Community under Grant No. SC1-0212-C.

[1]M. Steenbeck, F. Krause, and K.-H. Rädler, Z. Naturforsch. **21**, 369 (1966).
[2]U. Frisch, Z. S. She, and P. L. Sulem, Physica **28D**, 382 (1987).
[3]P. L. Sulem, Z. S. She, H. Scholl, and U. Frisch, J. Fluid Mech. **205**, 341 (1989).
[4]V. P. Starr, *Physics of Negative Viscosity Phenomena* (McGraw-Hill, New York, 1968).
[5]A. A. Nepomnyachtchyi, Prikl. Mat. Mekh. **40**, 886 (1976).
[6]G. I. Sivashinsky, Physica **17D**, 243 (1985).
[7]Kraichnan, R. H., J. Atmos. Sci. **33**, 1521 (1976).
[8]M. Lesieur, *Turbulence in Fluids* (Nijhoff, Dordrecht, 1987).
[9]K. Gotoh, M. Yamada, and J. Mizushima, J. Fluid Mech. **127**, 45 (1983).
[10]G. I. Sivashinsky and V. Yakhot, Phys. Fluids **28**, 1040 (1985).
[11]M. A. Brutyan and P. L. Krapivsky (unpublished).
[12]Note that the constant background density ρ_0 has been absorbed into the definition of the pressure.
[13]Most of what follows can be extended to the case where the basic flow and the force are random homogeneous and stationary.
[14]H. K. Moffat, J. Fluid Mech. **65**, 1 (1974).
[15]R. H. Kraichnan, Complex Syst. **1**, 805 (1987).
[16]U. Frisch, in *Lecture Notes on Turbulence,* edited by J. R. Herring and J. C. McWilliams (World Scientific, Singapore, 1989), p. 234.
[17]Summation on repeated indices is henceforth assumed.
[18]A. Bensoussan, J.-L. Lions, and G. Papanicolaou, *Asymptotic Analysis for Periodic Structures* (North-Holland, Amsterdam, 1978).
[19]G. Rüdiger, *Differential Rotation and Stellar Convection* (Akademie-Verlag, Berlin 1989).
[20]U. Frisch, D. d'Humières, B. Hasslacher, P. Lallemand, Y. Pomeau, and J.-P. Rivet, Complex Syst. **1**, 648 (1987); also reprinted in Ref. 16.
[21]V. I. Arnold, C. R. Acad. Sci. Paris **261**, 17 (1965).
[22]T. Dombre, U. Frisch, J. M. Greene, M. Hénon, A. Mehr, and A. M. Soward, J. Fluid Mech. **167**, 353 (1986).
[23]L. D. Meshalkin and Ia. G. Sinai, Prikl. Math. Mekh. **25**, 1140 (1961); J. Appl. Math. Mech. **25**, 1700 (1961).
[24]This was done in Ref. 6 for the case of the Kolmogorov flow.
[25]A. A. Nepomnyachtchyi, Gidrodin. Teploobmen **9**, 77 (1976).
[26]M. Hénon, and H. Scholl, following paper, Phys. Rev. A **43**, 5365 (1991).
[27]A. A. Nepomnyachtchyi (private communication).
[28]J. S. Green, J. Fluid Mech. **62**, 273 (1974).
[29]Z. S. She, Phys. Lett. A **124**, 161 (1987).
[30]J. W. Cahn and J. E. Hilliard, J. Chem Phys. **28**, 258 (1958).
[31]A. Novick-Cohen and L. A. Segel, Physica **10D**, 277 (1984).
[32]N. F. Bondarenko, M. Z. Gak, and F. V. Dolzhansky, Atmos. Ocean. Phys. **15**, 711 (1979).
[33]Actually, nontransverse instabilities are sometimes observed experimentally, but seem to arise from the boundary conditions [F. V. Dolzhansky and D. Yu. Manin (private communication)].
[34]J. C. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires* (Dunod-Gauthier-Villard, Paris 1969).
[35]The Grasshof number is a Reynolds number reexpressed in terms of the strength of the force.
[36]D. Forster, M. Nelson, and M. Stephen, Phys. Rev. A **16**, 732 (1977).
[37]V. Yakhot and S. A. Orszag, J. Sci. Comput. **1**, 3 (1986); Phys. Rev. Lett. **57**, 1722 (1986).
[38]K. G. Wilson, Phys. Rev. Lett. **28**, 548 (1972).
[39]J. D. Fournier and U. Frisch, Phys. Rev. A **17**, 747 (1978).