Relaxation in the subcritical pitchfork bifurcation: From critical to Gaussian scaling

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We study the crossover in the relaxation dynamics from the situation at the bifurcation point of a subcritical pitchfork bifurcation to a situation well above the bifurcation point. An approximation for the stochastic paths is given that allows the calculation of the statistical properties of the escape times. These properties can be described by a generating function that exhibits dynamical scaling of critical nature at the bifurcation point and Gaussian dynamical scaling well above the bifurcation point.

I. INTRODUCTION

One of the phenomena in which statistical fluctuations play a crucial role in nonequilibrium physics is the transient dynamics associated with the relaxation from states that have lost their global stability due to changes of appropriate control parameters. An important quantity in the characterization of the relaxation dynamics is the lifetime of such states: The time that the system takes to leave the vicinity of the initial state in some appropriate configuration space is a random quantity. The statistics of this random quantity is mathematically described by a first-passage-time (FPT) distribution. The mean first-passage time (MFPT) is identified with the lifetime of the initial state. There are standard techniques[1] to calculate PT statistics for Markovian processes. A useful alternative route to these techniques focuses on the individual stochastic paths of the process and extracts the FPT statistics from some approximation of these paths. This alternative approach allows for a complete description of the relaxation process connecting the initial stages of the dynamics which determine the PT with the late stages of approach to a final state. It also permits one to address questions of dynamical scaling. From a practical point of view this approach is useful in the calculation of PT statistics in situations in which standard techniques do not hold, such as, for example, problems with time-dependent parameters[2] or processes driven by nonwhite noise.[3] It is also useful in problems described by several variables in which standard techniques do not lead easily to explicit results.[4]

A description in terms of individual stochastic paths is well known for the relaxation from the state that becomes unstable in a supercritical pitchfork bifurcation.[5,6] In a first approximation the paths are described by a deterministic trajectory with a random initial condition. The relaxation dynamics from this type of unstable state follows, initially, Gaussian statistics. More complicated is the description of the relaxation from the state that loses its stability in a saddle-node bifurcation[7] or in a subcritical pitchfork bifurcation.[8] Those are states of marginal stability for which Gaussian statistics or, equivalently, linear theory does not hold at any time of the decay process. In a previous work,[8] a description was given of the stochastic paths of the relaxation process occurring precisely at the bifurcation point of the subcritical pitchfork bifurcation. In this paper we extend that description to arbitrary values of the control parameter above the bifurcation point. This is relevant from the experimental point of view due to the fact that, in practice, it is difficult to sit precisely at the bifurcation point.[9] Here we are particularly interested in the crossover from the relaxation dynamics at the bifurcation point[8] to the relaxation dynamics well above this point. The decay process for values of the control parameter well above the bifurcation point is essentially the same in the subcritical and supercritical pitchfork bifurcations.

A central question addressed in our study is that of dynamical scaling. In a first general sense we say that there is dynamical scaling when the statistical properties of a stochastic process in some time regime can be obtained by a time-dependent mapping of another simpler and known stochastic process. This sort of scaling is well known for the relaxation dynamics in the supercritical pitchfork bifurcation: The time-dependent mapping is the deterministic dynamics and the mapped process is a Gaussian one describing the initial linear regime. This sort of scaling does not hold for the relaxation process in the subcritical pitchfork bifurcation.[8] However, scaling properties for auxiliary processes and relevant physical quantities based on non-Gaussian statistics were found to hold for relaxation at the bifurcation point.[8] Here we discuss that such properties no longer hold for arbitrary values of the control parameter. In particular we analyze the scaling properties of an escape time that we identify with the PT. A scaling property holds when this time can be expressed as a function of a random variable with known statistics. When this property is satisfied

the PT statistics can be calculated easily. Such scaling is found at the bifurcation point with a non-Gaussian random variable (exceptional critical scaling) and well above the bifurcation point (Gaussian scaling). The situation well above the bifurcation point coincides with the description of the relaxation in the supercritical pitchfork bifurcation. For intermediate values of the control parameter scaling does not hold, but we introduce a scalinglike ansatz that is useful in the computation of the passage-time statistics.

The outline of the paper is as follows. In Sec. II we summarize some general properties of the relaxation dynamics. The MFPT is calculated using standard techniques and the phenomenon of transient trimodality is discussed. In Sec. III we develop our approximation for the stochastic paths of the relaxation dynamics. Results for generating functions associated with the calculation of PT statistics are given. Section IV includes the discussion of the scaling properties for the escape times. Details of the calculations are given in the Appendixes.

II. GENERAL PROPERTIES OF THE RELAXATION DYNAMICS

We associate[8] the normal form of the subcritical pitchfork bifurcation with the following stochastic equation for an order paramenter $x(t)$:

$$\frac{dx(t)}{dt} = ax(t) + bx^3(t) - cx^5(t) + \sqrt{\varepsilon}\xi(t) \qquad (2.1)$$

with $b, c > 0$. The parameter ε measures the strength of the fluctuations and $\xi(t)$ is a Gaussian white noise of zero mean and correlation $\langle \xi(t)\xi(t') \rangle = 2\delta(t - t')$. Equation (2.1) can be written in terms of a potential as

$$\frac{dx(t)}{dt} = -\frac{dV(x)}{dx} + \sqrt{\varepsilon}\xi(t) \,, \qquad (2.2)$$

$$V(x) = -\frac{ax^2}{2} - \frac{bx^4}{4} + \frac{cx^6}{6} \,. \qquad (2.3)$$

The bifurcation occurs at $a = 0$. The steady-state solution $x = 0$, locally stable for $a < 0$, becomes unstable for $a > 0$. The two equivalent stable steady states for $a > 0$ are at

$$x = \pm x_0 = \pm \left(\frac{b}{2c} + \frac{(b^2 + 4ac)^{1/2}}{2c} \right)^{1/2} \,. \qquad (2.4)$$

The form of the potential $V(x)$ and the associated bifurcation diagram is shown in Fig. 1.

In a previous paper[8] we considered the relaxation from $x = 0$ to $\pm x_0$ at the bifurcation point $a = 0$. Here we are interested in the same relaxation process, but for an arbitrary value of $a > 0$. Physically we imagine the system being for times $t < 0$ in a steady state $x = 0$ associated with a large negative value a_0 of the control parameter and fluctuations around $x = 0$ are very small. At time $t = 0$ the control parameter a is switched to a value above the bifurcation point, $a \geq 0$. The system finds itself in a nonstable steady state and relaxes to $x = \pm x_0$ due to fluctuations. For $a = 0$, the system relaxes from a state of marginal stability at the top of a locally flat potential.[8] For very large values of a the relaxational dynamics has the same characteristics as in the well-known case[5,6] of the decay from an unstable state in a supercritical pitchfork bifurcation. We are interested in the crossover between these two situations of $a = 0$ and a very large.

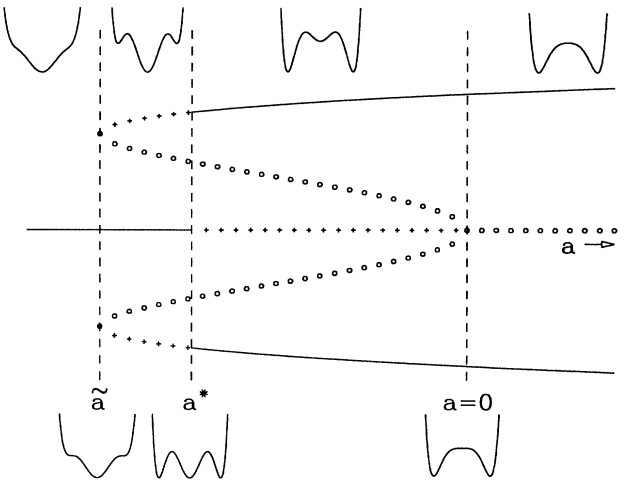

FIG. 1. Bifurcation diagram corresponding to the sixth-order potential $V(x)$ defined in (2.3). Solid lines denote stable states. Dots and crosses stand for unstable and metastable states, respectively, where $\tilde{a} = -b^2/(4c)$ and $a^* = -(3/4)b^2/(4c)$.

In the relaxation from $x = 0$ the saturation term $cx^5(t)$ is only relevant in the late stages of evolution, when the system approaches $\pm x_0$. The early and intermediate regimes of relaxation are then described by (2.1) with $c = 0$:

$$\frac{dx(t)}{dt} = ax(t) + bx^3(t) + \sqrt{\varepsilon}\xi(t) \,. \qquad (2.5)$$

The relevant dimensionless parameter in (2.5) is

$$k = \frac{a}{\sqrt{b\varepsilon}} \,. \qquad (2.6)$$

This parameter measures the relative importance of the linear term in comparison with the nonlinear term and the noise intensity. If $k \ll 1$ the linear term plays no important role and the system behaves as in the marginal case $a = 0$. On the other hand, the normal form of the supercritical pitchfork bifurcation is given by the following dynamical equation:

$$\frac{dx(t)}{dt} = ax(t) - bx^3(t) + \sqrt{\varepsilon}\xi(t) \,. \qquad (2.7)$$

Starting from $x(0) = 0$ this equation describes the decay from an unstable state. We note the different roles of the nonlinear term bx^3 in (2.1) and (2.7). In the first case this

term contributes to the departure from $x = 0$, whereas in (2.7) it accounts for saturation. If k is large enough, the linear term in (2.5) dominates during the early and intermediate regimes of the relaxation from $x = 0$. In such a case, and during these regimes, the relaxational dynamics described by (2.1) and (2.7) coincide.

The role of the parameter k and the differences in the relaxational dynamics for different values of k can be made evident in two complementary ways. The first one is the calculation of the lifetime of the state $x = 0$. The second one is the evolution of the time-dependent probability distribution $P(x,t)$. The lifetime can be identified with the mean first-passage time for x^2 to reach a value $x_{\rm th}^2$ starting at $x = 0$. From the standard theory of stochastic process, we have[1,10] that such MFPT and the variance of the FPT are given by

$$\langle T(0 \to x_{\rm th}^2)\rangle = \frac{1}{\varepsilon}\int_0^{x_{\rm th}} dx_1 \int_0^{x_1} dx_2 e^{[V(x_1)-V(x_2)]/\varepsilon}, \tag{2.8}$$

$$\begin{aligned}[\Delta T(0 \to x_{\rm th}^2)]^2 &\equiv \langle T^2\rangle - \langle T\rangle^2 \\ &= \frac{4}{\varepsilon^2}\int_0^{x_{\rm th}} dx_1 \int_0^{x_1} dx_2 \int_0^{x_2} dx_3 \int_0^{x_3} dx_4 \exp\{[V(x_1)+V(x_2)-V(x_3)-V(x_4)]/\varepsilon\},\end{aligned} \tag{2.9}$$

where $V(x)$ is given by (2.3). The asymptotic evaluation of (2.8) for small ε gives[11-13]

$$T(0 \to x_{\rm th}^2) = \frac{1}{\sqrt{b\varepsilon}}\Phi(k) + \frac{1}{2a}\ln\left|1+\frac{a}{4bx_{\rm th}^2}\right| + O\left(\frac{\varepsilon}{b^2 x_{\rm th}^6}\right). \tag{2.10}$$

This indicates that T scales with $(b\varepsilon)^{-1/2}$. The function $\Phi(k)$is given by

$$\Phi(k) = \tfrac{1}{2}\int_0^\infty e^{-ku}K_0(u^2)du = \sum_{n=0}^{\infty}(-1)^n B_n \frac{k^n}{n!}, \tag{2.11}$$

where $K_0(u)$ is the modified Bessel function of zeroth order[14] and

$$B_n = \frac{\sqrt{2}}{16}2^{n/2}\left[\Gamma\left(\frac{n+1}{4}\right)\right]^2. \tag{2.12}$$

The second contribution in (2.10) does not depend on the noise intensity, but it depends on the details of the potential and the barrier position. Equation (2.10) identifies that the dominant contribution to $\langle T\rangle\,(b\varepsilon)^{1/2}$ is a universal function $\Phi(k)$ of the parameter k. This implies a scaling result for different values of a, b, and ε similar to the one discussed in Ref. 7 for the saddle-node bifurcation. Scaling of this form has been evidenced for relaxation dynamics in a laser with saturable absorber.[13] The function $\Phi(k)$ is analytic for all the values of k and it has interesting asymptotic behaviors. For $k = 0$ we recover the results for the MFPT for the relaxation from a marginal state[10] in the limit $\varepsilon \ll 1$. The result (2.10) interpolates between the MFPT for the relaxation from a metastable state and the MFPT for relaxation from an unstable state: For $a < 0$ all the terms in the series are positive and a large value of T is obtained. In fact, for $a < 0$ relaxation occurs via activation through an energy barrier of height $\Delta V = a^2/(4b)$. When $|k| \gg 1$, very far from marginality, and $a < 0$, we are in the high-barrier limit ($\Delta V/\varepsilon \gg 1$) in which the series (2.11) converges to

$$\Phi(k) \approx \frac{\pi}{\sqrt{2}k}e^{k^2/4}. \tag{2.13}$$

This result reproduces the Kramers escape time.[15] On the other hand, for $a > 0$, (2.11) is an alternating series which in the limit $|k| \gg 1$ converges to

$$\Phi(k) \approx \frac{1}{2k}\ln k^2. \tag{2.14}$$

This value of $\Phi(k)$ replaced in (2.10) reproduces the MFPT time for the decay from an unstable state:

$$\langle T\rangle = \frac{1}{2a}\ln\frac{ax_{\rm th}^2}{\varepsilon}. \tag{2.15}$$

An asymptotic evaluation for small ε of the variance from (2.9) gives

$$(\Delta T)^2 = \frac{1}{b\varepsilon}\tilde{\Phi}(k) + O\left(\frac{\varepsilon}{b^2 x_{\rm th}^6}\right) \tag{2.16}$$

with

$$\tilde{\Phi}(k) = \sum_{n=0}^{\infty}(-1)^n \tilde{B}_n \frac{k^n}{n!},$$

$$\tilde{B}_n = \tfrac{1}{2}\Gamma\left(\frac{n+2}{2}\right)\int_0^1 du_1 \int_0^{u_1} du_2 \int_0^{u_2} du_3 \frac{1}{\sqrt{u_1u_2u_3(1+u_1^2)(1+u_2^2)}(1-u_3^2)^{1+n/2}} \times \left[\left(\frac{u_2^2+1}{u_1^2+1}\right)^{1/2}(1+u_1) - u_2 - u_3\right]^n. \tag{2.17}$$

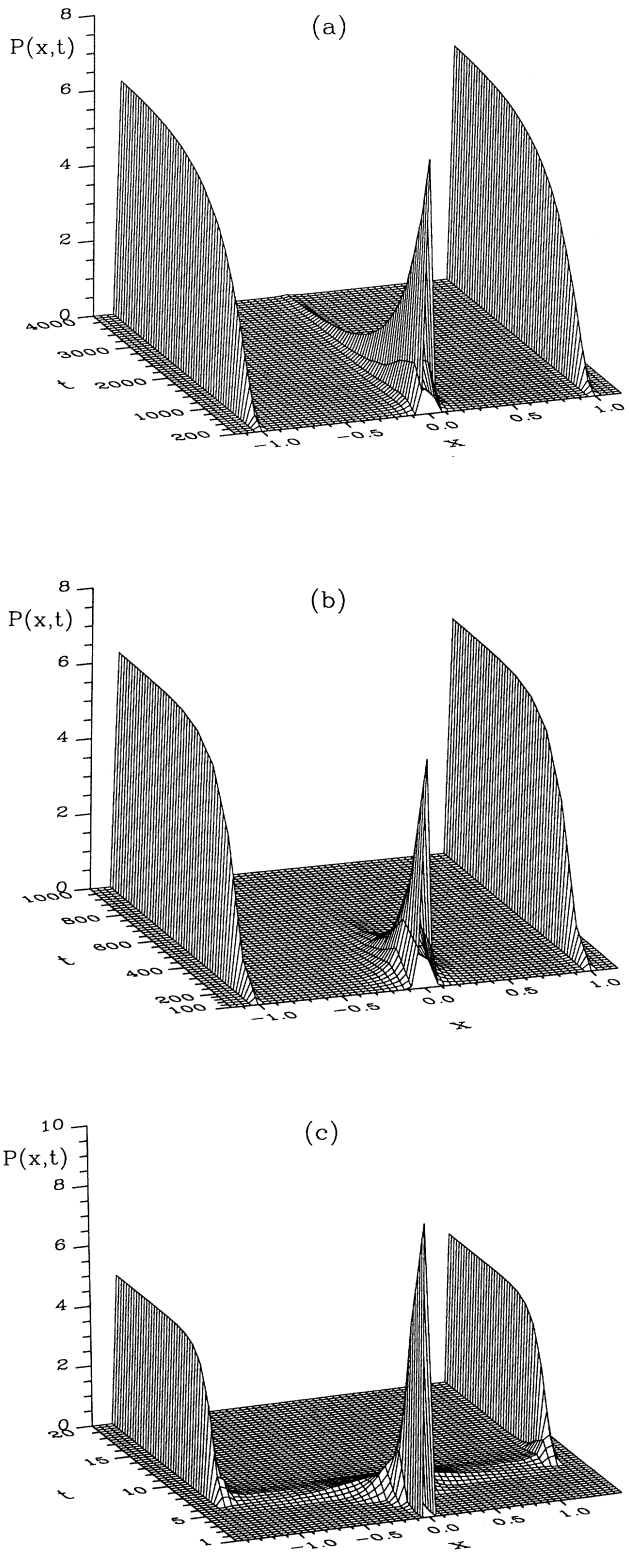

FIG. 2. Time-dependent probability distribution $P(x,t)$ from a numerical simulation of the process (2.1). (a) $k = 0.01$ ($a = 10^{-5}$), (b) $k = 10$ ($a = 10^{-3}$), (c) $k = 1000$ ($a = 1$).

A numerical evaluation of (2.19) for the first four coeficients gives $\tilde{B}_0 = 0.5520$, $\tilde{B}_1 = 0.5075$, $\tilde{B}_2 = 0.6139$, and $\tilde{B}_3 = 0.8971$.

Close to the bifurcation point (2.9) and (2.16) are approximated by

$$\langle T\rangle = \frac{1}{\sqrt{b\varepsilon}}(B_0 - kB_1) + \frac{1}{2a}\ln\left|1 + \frac{a}{4bx_{\rm th}^2}\right|, \tag{2.18}$$

$$\Delta T = \sqrt{\tilde{B}_0/b\varepsilon}\left(1 - k\frac{\tilde{B}_1}{2\tilde{B}_0}\right). \tag{2.19}$$

Eliminating k between these two equations we obtain a linear relation between ΔT and $\langle T\rangle$,

$$\Delta T = m + n\langle T\rangle, \tag{2.20}$$

where the slope n is a constant independent of any parameter value

$$n = \frac{\tilde{B}_1}{2B_1\sqrt{\tilde{B}_0}} = 0.8697. \tag{2.21}$$

Similar results have been obtained for the relaxation close to a saddle-node bifurcation but with different slope (see Ref. 7).

As a second way of describing the differences in relaxational dynamics for different values of k we consider the time-dependent probability distribution $P(x,t)$ obtained from a numerical simulation[16] of (2.1) with initial condition $P(x,t=0) = \delta(x)$. It is seen in Fig. 2 that for $k \ll 1$ and starting with a distribution peaked at the origin, two peaks at $\pm x_0$ appear immediately. As time goes on, the initial peak becomes smaller and the peaks at $\pm x_0$ larger. In the steady state only the side peaks remain, but during the transient $P(x,t)$ has three peaks, that is, there is a transient trimodality. This is the analog of the transient bimodality which appears in relaxation close to a saddle-node bifurcation.[17,18] For larger values of k the transient is shorter, but trimodality remains. For asymptotically large k trimodality disappears (Fig. 2) and the behavior is similar to the one in the relaxation from an unstable state in the supercritical pitchfork bifurcation: The initial peak broadens, becomes flat, and it disappears as the side peaks emerge. Trimodality is associated with the form of the potential (2.3) in the following sense: When the system leaves the initial state close to $x \approx 0$ the deterministic terms dominate the evolution. The term $bx^3(t)$ gives an evolution to the final steady state, which is fast compared with the time that the system needs to leave the vicinity of $x = 0$ and to the standard deviation of the FPT. This implies that at a given time, the probability of finding the system in an intermediate position is very small, while we have large probability peaks at $x \approx 0$ and around $\pm x_0$. In the decay from an unstable state the leaving time and its standard deviation are smaller in comparison with the deterministic evolution. Then the probability of finding the system

in an intermediate position at a given time is quite large and there is no transient trimodality.

In the following sections we will construct a description of the relaxation process for intermediate values of k that includes the decay from a state of marginal stability ($k = 0$) and the decay from an unstable state ($k \gg 1$) as extreme cases. In particular we are interested in the connection between two scaling descriptions: Gaussian scaling ($k \gg 1$) and critical exceptional scaling at $k = 0$.

III. ESCAPE PROCESS

We aim a useful approximation to the individual paths describing (2.5). In order to clarify ideas we consider a typical trajectory obtained from a numerical simulation of (2.1) with initial condition $x(0) = 0$. Simulations for different values of a are shown in Fig. 3. We have used the same sequence of random numbers in all the cases. We distinguish three stages of evolution. In the first stage the system remains close to its initial state. In the second stage the system leaves the vicinity of $x = 0$, and in the third stage of evolution the process fluctuates around $\pm x_0$. Let us start our analysis by looking at the trajectory for a small value of k ($k = 0.01$). In the first stage the random term $\sqrt{\varepsilon}\xi(t)$ has an important role in the evolution. We can identify two time regions. In the initial one the system is so close to $x = 0$ that the deterministic contributions in (2.1) can be neglected in front of $\sqrt{\varepsilon}\xi(t)$. The process evolves like a purely diffusive one (Wiener process),

$$x(t) = \sqrt{\varepsilon}W(t) \equiv \sqrt{\varepsilon}\int_0^t ds\,\xi(s)\ . \tag{3.1}$$

In the second time region the system is still close to $x \approx 0$, but the deterministic terms also contribute to the dynamics and the trajectory clearly differs from the purely diffusive one. Finally the system departs from $x \approx 0$, starting the second stage of the evolution. This departure occurs at time t^*, which is rather sharply defined for each individual trajectory. We refer to the escape process as the evolution up to the time t^*. The duration of the two time regions of this first stage depends on the value of k. As k increases the deterministic terms become important sooner and the escape time is smaller. For k large enough the deterministic term $ax(t)$ contributes from the very beginning and the distinction between two time regions disappears. This is what happens in the first stage of the decay from an unstable state in the supercritical pitchfork bifurcation.

In the second stage of evolution the deterministic contributions in (2.1) play the main role, whereas the noise term can be neglected. The system leaves the vicinity of $x \approx 0$ and evolves to $\pm x_0$. For small values of k the time spent in this second stage is much smaller than the escape time t^*. Following Ref. 18 we call transition time t_d the time that the system spends in this region. This time is determined by deterministic considerations. An important point to be noted is that the existence of transient trimodality depends on the relation between two time scales: It exist whenever t_d is smaller than the standard deviation of the FPT, ΔT.[18] This relation can be characterized by the parameter

$$\mu = \frac{t_d}{\Delta T}, \tag{3.2}$$

which represents the probability of finding the system in the second stage of evolution.[18] In the decay from a marginal state $a = 0$, $\Delta T \approx 1/\sqrt{b\varepsilon}$, so that $\mu \to 0$ for $\varepsilon \to 0$ and transient trimodality exists. In the decay from an unstable state[5,6]

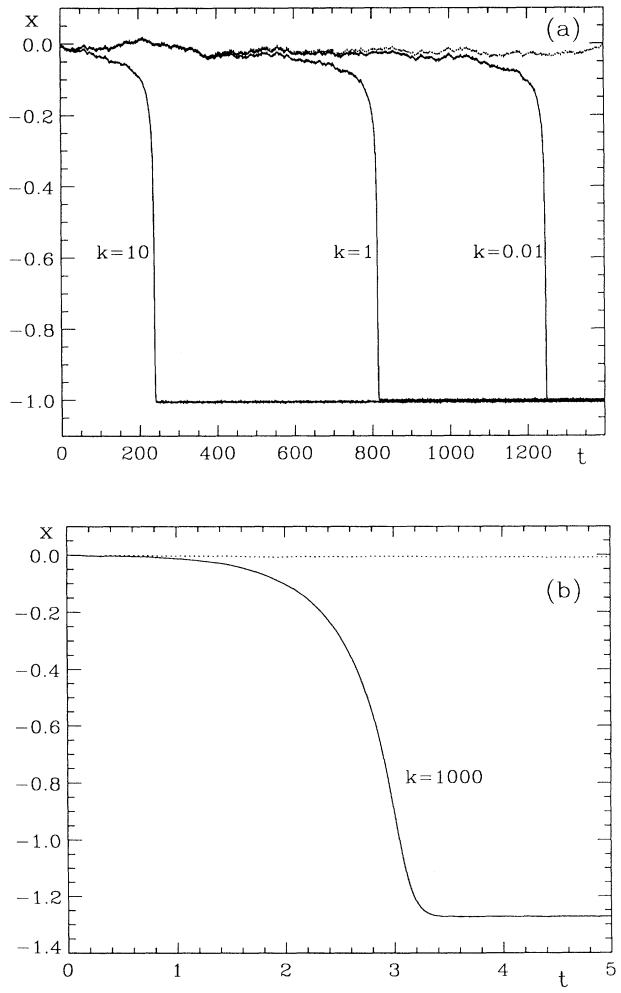

FIG. 3. Single realization of (2.1) for different values of a. (a) Solid lines correspond, from left to right, to $a = 10^{-2}$ ($k = 10$), $a = 10^{-3}$ ($k = 1$), $a = 10^{-5}$ ($k = 0.01$). (b) The solid line corresponds to $a = 1$ ($k = 1000$). The dotted line corresponds to the Wiener process (3.1).

$$\Delta T = \frac{1}{2a}\sqrt{\psi'(1)}, \tag{3.3}$$

where ψ is the digamma function.[14] This variance does not depend on the noise intensity and for large enough a, μ can have a finite value, so that transient trimodality disappears.

We are here concerned with the escape process connecting the first and second stages for different values of k. Rescaling x and t in Eq. (2.5) as

$$x = \alpha x_R\ , \quad t = \beta\tau, \tag{3.4}$$

with $\alpha = (\varepsilon/b)^{1/4}$ and $\beta = (b\varepsilon)^{-1/2}$ we obtain

$$\frac{dx_R(\tau)}{d\tau} = kx_R(\tau) + x_R^3(\tau) + \eta(\tau)\ , \tag{3.5}$$

where $\eta(\tau)$ is a Gaussian white noise of zero mean and correlation $\langle\eta(\tau)\eta(\tau')\rangle = 2\delta(\tau - \tau')$. To find an approximation for this process we proceed in a similar way as for the case $a = 0$.[8] We write the rescaled process $x_R(\tau)$ as the ratio of two stochastic processes

$$x_R(\tau) \equiv \frac{z_R(\tau)}{\sqrt{y_R(\tau)}}\ . \tag{3.6}$$

Then (2.5) is equivalent to the set of equations

$$\frac{dz_R(\tau)}{d\tau} = kz_R(\tau) + \sqrt{y_R(\tau)}\eta(\tau), \tag{3.7}$$

$$\frac{dy_R(\tau)}{d\tau} = -2z_R^2(\tau), \tag{3.8}$$

with $z_R(0) = x_R(0)$, $y_R(0) = 1$. Basically the idea is to decompose the process $x(\tau)$ into two parts, a process $z_R(\tau)$ associated with the linear evolution and a process $y_R(\tau)$, which takes into account the nonlinear terms of the dynamical equation. For $k \ll 1$, initially $y_R(\tau) \approx 1$, then $z_R(\tau)$, and also $x_R(\tau)$, are essentially Wiener processes. As time goes on $y_R(\tau)$ decreases, consequently $z_R(\tau)$ has negligible fluctuations, and the evolution becomes deterministic. For $k \gg 1$, even from the very beginning, the evolution for $z_R(\tau)$ is not purely diffusive since the contribution of the term $kz_R(\tau)$ is important. This term makes $z_R(\tau)$ increase faster, so the system escapes earlier. This term is also responsible for the disappearance of the separation in two time regions in the first stage of the temporal evolution.

It is possible to obtain an approximate solution for the coupled equations (3.7) and (3.8) in an iterative way. We start from $y_R(\tau) = y_R(0) = 1$, and solving (3.7) we obtain

$$z_R(\tau) = e^{k\tau}\int_0^\tau e^{-k\tau'}\eta(\tau')d\tau'\ . \tag{3.9}$$

Substituting in (3.8) we have

$$y_R(\tau) = 1 - 2\int_0^\tau e^{2k\tau'}\left(\int_0^{\tau'} e^{-k\tau''}\eta(\tau'')d\tau''\right)^2 d\tau'\ . \tag{3.10}$$

In this way the approximation for the escape process is

$$x_R(\tau) = \frac{e^{k\tau}\int_0^\tau e^{-k\tau'}\eta(\tau')d\tau'}{\left[1 - 2\int_0^\tau e^{2k\tau'}\left(\int_0^{\tau'} e^{-k\tau''}\eta(\tau'')d\tau''\right)^2 d\tau'\right]^{1/2}}. \tag{3.11}$$

In Fig. 4 we show a numerical simulation of a single trajectory of the process given by (2.1) and the approximation (3.11) for the same sequence of random numbers and for different values of k. The agreement is remarkably good for $k \to 0$ and it becomes better for large values of k. In fact, for $k = 0$ we recover the results of Ref. 8 [see Eq. (2.14)]. It is possible to obtain an improvement upon (3.11) by going to the next order in the iterative process, but the first-order approximation (3.11) contains the main features of the process. We remark that (3.11) is not a scaling approximation for the process since $x_R(t)$ is given as a function of two nonindependent stochastic processes, namely $\int_0^\tau e^{-k\tau'}\xi(\tau')d\tau'$ and $\int_0^\tau e^{2k\tau'}\left(\int_0^{\tau'} e^{-k\tau''}\xi(\tau'')d\tau''\right)^2 d\tau'$. We have checked that the approximated process (3.11) gives a very good approximation to the time-dependent probability distributions $P(x,t)$ shown in Fig. 2.

Let us analyze, in terms of (3.11), the mechanism responsible for the escape from the vicinity of the initial condition $x = 0$. For $k \ll 1$ the numerator of (3.11) is

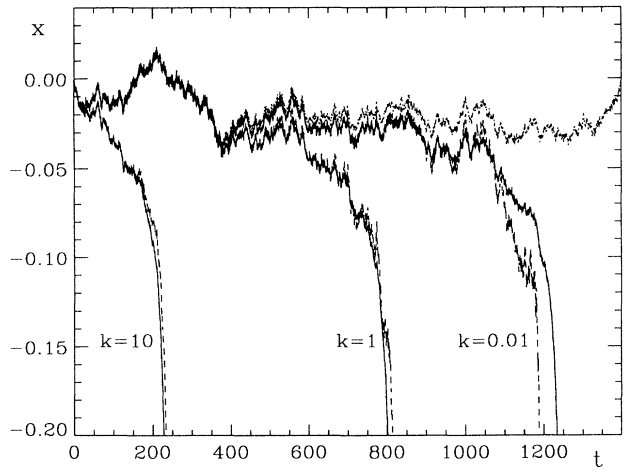

FIG. 4. Different stochastic paths for a given sequence of random numbers and different values of k. Solid lines correspond to a simulation of the exact process (2.1) for the same values of k as in Fig. 3. Dashed lines correspond to the approximation (3.11) for the same values of k. The dotted line corresponds to the Wiener process (3.1).

a stochastic process very close to the Wiener process (in fact, for $k = 0$, it is the Wiener process). This makes the system fluctuate around $x = 0$, but does not provide an escape mechanism. It is the denominator $y_R^{1/2}(\tau)$ that is responsible for that. Equation (3.10) can be rewritten as

$$y_R(\tau) = 1 - 2\Omega(\tau), \tag{3.12}$$

where

$$\Omega(\tau) = \int_0^\tau e^{2k\tau'} \left(\int_0^{\tau'} e^{-k\tau''} \eta(\tau'') d\tau'' \right)^2 d\tau'. \tag{3.13}$$

$\Omega(\tau)$ is a positive definite stochastic process which increases monotonically in time, reaching a time τ^* such that $y_R(\tau) \to 0$ and $x_R(\tau) \to \infty$ [the divergence in $x(\tau)$ is due to the fact that we are neglecting the saturation effects]. We will identify the time τ^* at which $y_R(\tau) \to 0$ with the escape time and we call $y_R(\tau)$ the escape process. This identification was discussed for $k = 0$ in Ref. 8. Its meaning for $k \gg 1$ can be understood as follows: For $k \gg 1$ and small times

$$x_R(t) \simeq z_R(t) \simeq e^{k\tau} h_R(\tau), \tag{3.14}$$

where

$$h_R(\tau) = \int_0^\tau e^{-k\tau'} \eta(\tau') d\tau' . \tag{3.15}$$

Equation (3.14) coincides in rescaled variables with the linear approximation for the decay of an unstable state obtained from (2.7) neglecting the saturation term.[5,6] For large k, $x_R(\tau)$ given by (3.14) grows exponentially in time and is the dominant mechanism of escape. Nevertheless, as (3.14) grows, $\Omega(t)$ also grows since in this case

$$\Omega(\tau) = \int_0^\tau \left[e^{k\tau'} h_R(\tau') \right]^2 d\tau'. \tag{3.16}$$

Thus, for times at which (3.14) becomes large, $y_R(\tau) \to 0$ also.

We note that the escape process $y_R(\tau)$ depends only on one stochastic process, so it is easier to analyze than the full process $x_R(\tau)$. In addition, it turns out that the statistical properties of $y_R(\tau)$ are exactly known since it is possible to calculate the generating function for $\Omega(\tau)$ (see Appendix A)

$$G_\Omega(\lambda, \tau) \equiv \left\langle e^{-\lambda\Omega(\tau)} \right\rangle = e^{-k\tau/2} \left| \cosh\left(\tau\sqrt{k^2 + 4\lambda}\right) - \frac{k}{\sqrt{k^2 + 4\lambda}} \sinh\left(\tau\sqrt{k^2 + 4\lambda}\right) \right|^{-1/2} . \tag{3.17}$$

The escape process $y_R(\tau)$ is simply related to a process[8] with direct physical meaning $\phi(\tau)$ defined as

$$\phi(\tau) \equiv \int_0^\tau x_R^2(\tau') d\tau' = -\tfrac{1}{2} \ln y_R(\tau) . \tag{3.18}$$

The relation (3.18) is verified independently of the approximation used for $y_R(\tau)$. The generating function for the process $\phi(t)$ is derived in Appendix B,

$$\begin{aligned} G_\phi(\lambda, \tau) &\equiv \left\langle e^{-\lambda\phi(\tau)} \right\rangle \\ &= \left\langle y_R^{-\lambda/2}(\tau) \right\rangle \\ &= \frac{1}{\Gamma(\lambda/2)} \int_0^\infty u^{\lambda/2-1} e^{-u} G_\Omega(-2u, \tau) du. \end{aligned} \tag{3.19}$$

In particular

$$\langle \phi(\tau) \rangle = \tfrac{1}{2} \textstyle\int_0^\infty u^{-1} e^{-u} \left[G_\Omega(-2u, \tau) - 1 \right] du . \tag{3.20}$$

These results for the process ϕ imply a scaling law: The statistics of $\phi(\tau)$ are completely determined by a transformation of the statistics of the process $\Omega(\tau)$. We recall that no such scaling law exists in general for the process $x(t)$. In fact, (3.20) and (3.18) give the only form of scaling that is preserved for all values of k.

IV. STATISTICS OF ESCAPE TIMES

We have computed the escape-time probability distribution $P(t^*)$ from a numerical simulation of the process $y(t)$. We recall that t^* has been defined by $y(t^*) = 0$. The distribution $P(t^*)$ is compared in Fig. 5 with the first-passage-time distribution $P(T)$ obtained from numerical simulation of the exact equation (2.1). The distribution

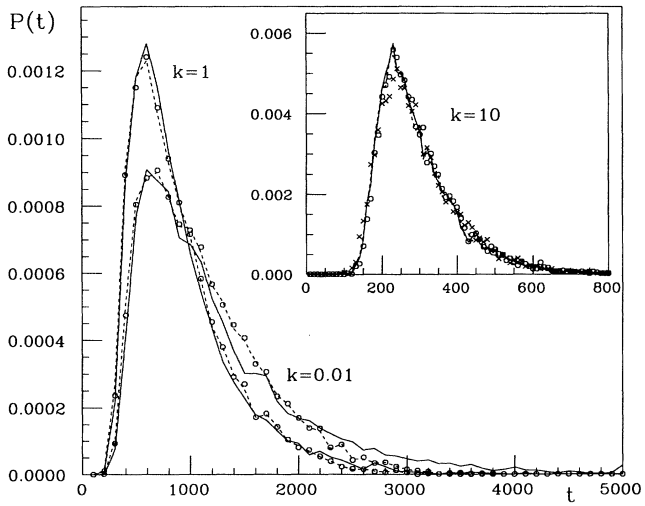

FIG. 5. First-passage-time distribution. Solid line corresponds to a simulation of the exact process (2.1) and dashed line corresponds to the aproximation (3.11), both having reached $x^* = 0.3$ for first time. Circles correspond to the distribution of times at which the denominator of (3.11) vanishes for first time. From top to bottom, $k = 1, 0.01$. Inset: $k = 10$, crosses correspond to the passage-time distribution associated with (2.7) given by (4.1) with $x_{\rm th}^2 = a/b$.

$P(T)$ is defined as the probability that $x(t)$ reaches a given threshold value $x_{\rm th}$ at time T for the first time. $x_{\rm th}$ is choosen in such a way that the process has clearly escaped from the vicinity of $x = 0$. It is seen that $P(t^*)$ gives a good representation of $P(T)$ for different values of k. For $k \ll 1$ there exists a small discrepancy in the tail of the distribution. This discrepancy corresponds to rare stochastic paths which stay a long time around $x = 0$ and are not well approximated by (3.11). As k increases these long-lived events become more rare and the discrepancy disappears. It is also seen that for $k \gg 1$, $P(t^*)$ approaches the first-passage-time distribution associated with the process (2.7). We note that the PT distribution for the process (2.1) is essentially independent of the precise value of $x_{\rm th}$. This is due to the fast deterministic evolution once the system leaves the surroundings of $x = 0$. For the relaxation from an unstable state the PT distribution is shifted in time when changing $x_{\rm th}$. Indeed, such PT distribution is obtained[2,3,6] from the linear approximation to the process given by (2.7) ($b = 0$) and considering times of interest $k\tau \gg 1$:

$$t^* = \frac{1}{2a}\ln\frac{x_{\rm th}^2}{\varepsilon h^2(\infty)}, \tag{4.1}$$

where $h(\infty)$ is a Gaussian variable defined by

$$h(\infty) \equiv \int_0^\infty e^{-at'}\xi(t')dt' . \tag{4.2}$$

We see below that the PT distribution approaches (4.1) for $k \gg 1$ with an effective threshold value $x_{\rm th}^2 = a/b$.

In order to obtain analytical results for the escape-time distribution we analyze the process $\Omega(\tau)$ defined in (3.13). For $k = 0$ we have

$$\Omega(\tau) = \int_0^\tau ds\, W^2(s) = \tau^2\Omega, \tag{4.3}$$

where Ω is a random variable defined as

$$\Omega \equiv \int_0^1 W^2(s)ds . \tag{4.4}$$

Using (3.4) and with the definition $y(t^*) = 0$ in (3.12) we obtain for the escape time

$$t^* = (2b\varepsilon\Omega)^{-1/2} . \tag{4.5}$$

This result implies the existence of scaling properties in a strict sense: The time dependence of $\Omega(\tau)$ can be scaled out so that t^* can be given as a function of a random variable Ω.

For $k \gg 1$, the stochastic process $h_R(\tau)$ defined in (3.15) can be approximated as $h_R(\infty) = (b\varepsilon)^{1/4}h(\infty)$ in (4.2). The random variable $h_R(\infty)$ is Gaussian with zero mean and variance:

$$\langle h_R^2(\infty)\rangle = \frac{1}{k} . \tag{4.6}$$

In this limit (3.16) gives

$$\Omega(\tau) \simeq \frac{1}{2k}(e^{2k\tau} - 1)h_R^2(\infty) . \tag{4.7}$$

Substituting (4.7) into (3.12) and going back to the original variable t, we have for $e^{2at} \gg 1$

$$t^* = \frac{1}{2a}\ln\frac{a}{\varepsilon b h^2(\infty)} . \tag{4.8}$$

This coincides with (4.1) with $x_{\rm th}^2 = a/b$. We find again a scaling property since the time dependence of $\Omega(\tau)$ has been scaled out and t^* is given as a function of the random variable $h(\infty)$. This property only holds here in a long-time regime $at \gg 1$.

In both cases considered, $k = 0$ and $k \gg 1$, we have

$$\Omega(\tau) = f(\tau)\nu, \tag{4.9}$$

where ν is a random variable [either Ω for $k = 0$ or $h_R(\infty)$ for $k \gg 1$]. For an intermediate value of k scaling does not exist because a relation of the form (4.9) does not hold.

A similar discussion about scaling can be given in terms of the generating function $G_\Omega(\lambda,\tau)$. For $k = 0$, (3.17) give

$$G_\Omega(\lambda,\tau) = \left[\cosh(2\tau\sqrt{\lambda})\right]^{-1/2} . \tag{4.10}$$

This result coincides with the one obtained in Ref. 8. For $k \gg 1$, we have from (3.17)

$$G_\Omega(\lambda,\tau) = \left[1 + \lambda\left(\frac{e^{2k\tau} - 1 - 2k\tau}{k^2}\right)\right]^{-1/2} . \tag{4.11}$$

In both cases

$$G_\Omega(\lambda,\tau) = F(\lambda f(\tau)) . \tag{4.12}$$

This equation has the same content as (4.9), F being the generating function of the random variable ν. In the case $k = 0$ the generating function for Ω is[8]

$$F(\lambda) = \left[\cosh(2\sqrt{\lambda})\right]^{-1/2} . \tag{4.13}$$

Substituting $f(\tau) = \tau^2$ [from (4.3)] we recover (4.10) from (4.12) and (4.13). On the other hand, the generating function for $h_R^2(\infty)$ is given by

$$F(\lambda) = [1 + 2\lambda/k]^{-1/2} . \tag{4.14}$$

From (4.7), $f(\tau) = (e^{-2k\tau} - 1)/(2k)$, and substituting into (4.12) we obtain an expression that, in the limit $k \to \infty$, coincides with (4.11).

The general form (4.12) evidences scaling behavior. This is found here in two limits: For $k \gg 1$ we have the usual scaling based on initial Gaussian statistics. For $k = 0$ we have an "exceptional" scaling due to being at the critical point $a = 0$. For intermediate values of k there is no scaling. However, for a given value of k, and τ large enough, the generating function (3.17) is effectively zero for values of λ such that

$$\lambda \gg k^2/2 . \tag{4.15}$$

In the range of values of λ in which $G_\Omega(\lambda,\tau)$ is nonzero,

the generating function approaches the scaling form (4.11) with increasing τ (see Fig. 6). The net results is that while for $k = 0$ there is scaling for all times, scaling is postponed to long times for $k \neq 0$: There is an asymptotic final regime in which Gaussian scaling effectively sets in. This scaling is postponed to very long times if k is small. The problem is that for k small enough the asymptotic regime of Gaussian scaling is beyond the interesting time regime in which the system escapes from $x = 0$.

The lack of scaling for intermediate values of k makes impossible, in principle, the calculation of the escape-time statistics by expressing t^* as a function of a random variable as it would follow from $y_R(\tau^*) = 0$ and a scaling form (4.9) for $\Omega(\tau)$. A possible way out is through a scalinglike ansatz for $\Omega(\tau)$: For intermediate values of k we propose the following approximation for $\Omega(\tau)$:

$$\Omega(\tau) \approx \frac{\langle\Omega(\tau)\rangle}{\langle\Omega(\tau_m)\rangle}\Omega(\tau_m), \tag{4.16}$$

where τ_m is a parameter to be determined. In the limiting cases $k = 0$, $k \to \infty$, (4.8) holds. As a consequence the approximation (4.16) gives the correct results (4.3) and (4.7) for any arbitrary value of τ_m. In general (4.16) is of the form (4.9) with $\Omega(\tau_m)$ playing the role of the random variable ν. The statistical properties of $\Omega(\tau_m)$ are known through the generating function (3.17). The time-dependent function $\langle\Omega(\tau)\rangle$ is explicitly known since it can be exactly calculated from (3.13)

$$\langle\Omega(\tau)\rangle = \frac{e^{2k\tau} - 1 - 2k\tau}{k^2} . \tag{4.17}$$

The free parameter τ_m is here determined by the requirement

$$< y_R(\tau_m) >= 1 - 2\langle\Omega(\tau_m)\rangle = 0 , \tag{4.18}$$

which together with (4.17) fixes our choice of the value of τ_m.

From (4.16), (3.12), and the identification $y_R(\tau^*) = 0$, the escape time can be obtained by numerical inversion as a function of the random variable $\Omega(\tau_m)$. In order to check the reliability of the statistical properties of τ^* obtained in this way we need to compare the statistical properties of $\Omega(\tau)$ given by (3.17) and those which follow from the ansatz (4.16). A way to do this is to check if statistical properties of $\Omega(\tau_m)$, which follow from (4.16), are consistent with the exact result (3.17) for $\tau = \tau_m$: The ansatz (4.16) gives for $\tau = \tau^*$

$$\Omega(\tau^*) = \frac{\Omega(\tau_m)\langle\Omega(\tau^*)\rangle}{\langle\Omega(\tau_m)\rangle}. \tag{4.19}$$

The definition $y_R(\tau^*) = 0$ implies that $\Omega(\tau^*) = \frac{1}{2}$ and (4.18) implies that $\langle\Omega(\tau_m)\rangle = \frac{1}{2}$. These two results together with (4.17) and (4.19) give

$$\Omega(\tau_m) \approx \frac{k^2}{4}\left(e^{2k\tau^*} - 1 - 2k\tau^*\right)^{-1} . \tag{4.20}$$

Therefore the statistical properties for $\Omega(\tau_m)$ implied by the ansatz (4.16) are given by the generating function

$$\left\langle \exp\left[-\frac{\lambda k^2}{4}\left(e^{2k\tau^*} - 1 - 2k\tau^*\right)^{-1}\right]\right\rangle . \tag{4.21}$$

This is compared in Fig. 7 with the exact result (3.17) for $\tau = \tau_m$. For $k = 0$ the agreement is perfect since

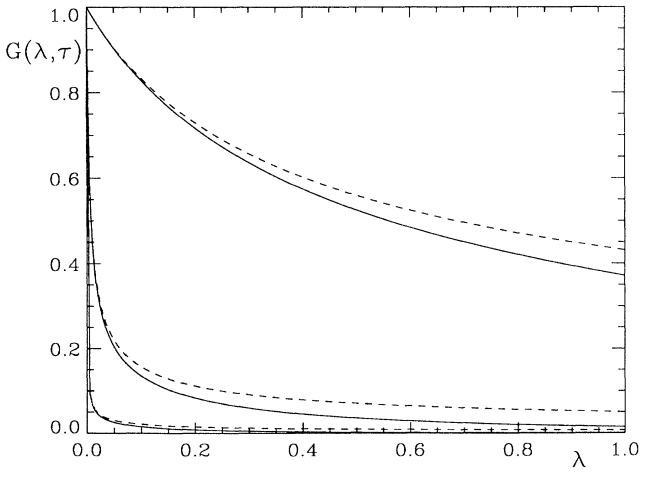

FIG. 6. Gaussian scaling of the generating function $G_\Omega(\lambda,\tau)$. Solid lines correspond to $G_\Omega(\lambda,\tau)$ for $k = 1$ given by (3.17). Dashed lines correspond to the Gaussian-scaled $G_\Omega(\lambda,\tau)$ given by (4.10). From top to bottom, $\tau = 1, 3, 5$.

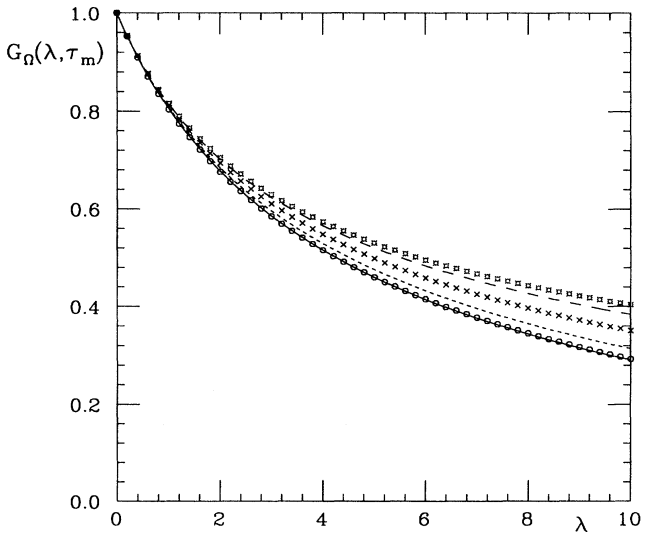

FIG. 7. Comparison between (4.20) with τ^* as the time at which $y_R(\tau^*) = 0$ and $G_\Omega(\lambda,\tau_m)$ given by (3.17), for different values of k. Results for (4.20) obtained by numerical simulations are plotted for $k = 0.01$ (solid line), $k = 1$ (short-dashed line), and $k = 10$ (long-dashed line). Circles, crosses, and squares correspond to $G_\Omega(\lambda,\tau_m)$ for the same values of k, respectively.

scaling holds exactly. There exists Gaussian scaling for $k \to \infty$, so that for large k our approximation works very well, and already for $k = 10$ the result is good. The worst situation concerning the lack of scaling is for $k \approx 1$. Even in that case the agreement in Fig. 7 is rather satisfactory. Such agreement gives support to the approximation (4.16) as a scaling interpolation approximation for intermediate values of k.

Following the procedure of Ref. 8, Sec. II B, the moments of the FPT distribution $< t^n >$ can be calculated numerically from the ansatz (4.16). An advantage of this approximated method is that it only involves one numerical integration, while the standard FPT technique requires $2n$ integrations. As a final remark we note that transient fluctuations for an arbitrary value of k can be evaluated from the FPT statistics following the ideas of Sec. III of Ref. 8, but using the deterministic evolution starting at x^* at time t^* instead of a step-function approximation. For small k the deterministic evolution $x(t)$ is a steplike function at $t = t^*$, so that the results for $a = 0$ are recovered. In this procedure the moments $\langle x^n(t) \rangle$ are determined by a transformation of the statistical properties of the escape time, so that a form of scaling is recovered.

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APPENDIX A

In this appendix we present a derivation of the result (3.17) for the generating function $G_\Omega(\lambda, \tau)$. We start from a general calculation for a generating function of the form

$$G(\lambda,\tau) = \left\langle \exp\left[-\lambda \int_0^\tau f^2(\tau') \left(\int_0^{\tau'} g(\tau'')\eta(\tau'')d\tau'' \right)^2 d\tau' \right] \right\rangle , \tag{A1}$$

where the average is to be understood over the ensemble of realizations of the white noise $\eta(\tau)$. In our case $f(\tau) = e^{k\tau}$ and $g(\tau) = e^{-k\tau}$. The first step in the calculation of (A1) is to discretize the time as $\tau = ns$. Then the noise $\eta(\tau)d\tau = dW(\tau)$ will be also discretized as a Gaussian random number ΔW_k with

$$< \Delta W_k >= 0 , \quad < \Delta W_k \Delta W_l >= 2s\delta_{kl} . \tag{A2}$$

Then

$$\int_0^\tau f^2(\tau') \left(\int_0^{\tau'} g(\tau'')\eta(\tau'')d\tau'' \right)^2 d\tau' \approx s \sum_{k=0}^{n-1} f^2(\tau_k) \left(\sum_{l=0}^{k-1} g(\tau_l)\Delta W_l \right)^2 . \tag{A3}$$

In order to perform the average we introduce auxiliary variables $x_i, i = 0, 1, ..., n-1$ and make use of the well-known properties of Gaussian integrals:

$$\exp\left[-\lambda s \sum_{k=0}^{n-1} f^2(\tau_k) \left(\sum_{l=0}^{k-1} g(\tau_l)\Delta W_l \right)^2 \right]$$

$$= \int_{-\infty}^{\infty} dx_0 \frac{1}{\sqrt{2\pi}} \cdots \int_{-\infty}^{\infty} dx_{n-1} \frac{1}{\sqrt{2\pi}} \exp\left[-\tfrac{1}{2} \sum_{k=0}^{n-1} x_k^2 + i\sqrt{2\lambda s} \sum_{k=0}^{n-1} f(\tau_k) \left(\sum_{l=0}^{k-1} g(\tau_l)\Delta W_l \right) x_k \right] . \tag{A4}$$

The double sum of the exponent can be written as

$$\sum_{k=0}^{n-1} \sum_{l=0}^{k-1} f(\tau_k) x_k g(\tau_l)\Delta W_l = \sum_{l=0}^{n-2} A_l g(\tau_l)\Delta W_l , \tag{A5}$$

where

$$A_l \equiv \sum_{k=l+1}^{n-1} f(\tau_k) x_k . \tag{A6}$$

Substituting (A5) and (A4) into (A1), we have a discretized generating function

$$G(\lambda, ns) = \int_{-\infty}^{\infty} dx_0 \frac{1}{\sqrt{2\pi}} \cdots \int_{-\infty}^{\infty} dx_{n-1} \frac{1}{\sqrt{2\pi}} \exp\left(-\tfrac{1}{2} \sum_{k=0}^{n-1} x_k^2 \right) \left\langle \exp\left(i\sqrt{2\lambda s} \sum_{l=0}^{n-2} A_l g(\tau_l)\Delta W_l \right) \right\rangle . \tag{A7}$$

The average that appears in (A7) can be performed by expanding the exponential in series, averaging term by term and summing up the final series. We obtain

$$\left\langle \exp\left(i\sqrt{2\lambda s}\sum_{l=0}^{n-2} A_l g(\tau_l)\Delta W_l \right) \right\rangle = \exp\left(i2\lambda s^2 \sum_{l=0}^{n-2} A_l^2 g^2(\tau_l) \right) . \quad \text{(A8)}$$

By substituting (A8) into (A7), the resulting multiple integral can be evaluated step by step. Retaining in each step of integration the leading contribution (i.e., neglecting terms of order s^4 with respect to terms of order s^2) we obtain, after n integrations,

$$G(\lambda, ns) = \left[\prod_{k=1}^{n-1} \left(1 + 4\lambda s^2 f^2(\tau_k) \sum_{l=0}^{k-1} g^2(\tau_l) \right) \right]^{-1/2} . \quad \text{(A9)}$$

Keeping only terms up to s^2, this is equivalent to

$$G(\lambda, ns) = \left(1 + 4\lambda s^2 \sum_{k=1}^{n-1} f^2(\tau_k) \sum_{l=0}^{k-1} g^2(\tau_l) \right)^{-1/2} . \quad \text{(A10)}$$

The argument of the square root of (A10) can be obtained by the following recurrence relations for the auxiliary quantities u_n and v_n:

$$u_{n+1} = u_n + \sqrt{4\lambda} s f^2(\tau_n) v_n, \qquad v_{n+1} = v_n + \sqrt{4\lambda} s g^2(\tau_n) u_n, \quad \text{(A11)}$$

with initial conditions $u_0 = 1$, $v_0 = 0$. Then, by neglecting higher-order terms in s^2 it is possible to prove by induction that

$$G(\lambda, ns) = u_n^{-1/2}. \quad \text{(A12)}$$

In the large-n limit, i.e., for vanishingly small s, we get a set of differential equations

$$\frac{du(\tau)}{d\tau} = \sqrt{4\lambda} f^2(\tau) v(\tau), \qquad \frac{dv(\tau)}{d\tau} = \sqrt{4\lambda} g^2(\tau) u(\tau), \quad \text{(A13)}$$

to be solved with initial conditions $u(0) = 1$, $v(0) = 0$. These equations are equivalent to

$$d_\tau^2 u - 2\frac{f'(\tau)}{f(\tau)} d_\tau u - 4\lambda f^2(\tau) g^2(\tau) = 0, \quad \text{(A14)}$$

with $u(0) = 1$, $u'(0) = 0$. For our special forms of $f(\tau)$ and $g(\tau)$ (A14) reduces to

$$d_\tau^2 u - 2k d_\tau u + 4\lambda = 0 . \quad \text{(A15)}$$

Solving (A15) with the quoted initial condition we obtain the final result (3.17) with

$$G(\lambda, \tau) = \lim_{\substack{n\to\infty \\ s\to 0}} G(\lambda, ns) = u^{-1/2}(\tau). \quad \text{(A16)}$$

APPENDIX B

Here we derive the generating function for the process $\phi(\tau)$ given in (3.19). Starting from the definition for $G_\phi(\lambda, \tau)$ and using (3.18) we have

$$G_\phi(\lambda, \tau) \equiv \left\langle e^{-\lambda\phi(\tau)} \right\rangle = \left\langle y_R^{-\lambda/2}(\tau) \right\rangle . \quad \text{(B1)}$$

From the definition of the Γ function for an arbitrary c,

$$\Gamma(z) = c^z \int_0^\infty u^{z-1} e^{-cu} du , \quad \text{(B2)}$$

we obtain

$$y_R^{-\lambda/2}(\tau) = \frac{1}{\Gamma(\lambda/2)} \int_0^\infty u^{\lambda/2-1} e^{-u y_R(\tau)} du = \frac{1}{\Gamma(\lambda/2)} \int_0^\infty u^{\lambda/2-1} e^{-u[1-2\Omega(\tau)]} du, \quad \text{(B3)}$$

where we have used (3.12). Then (B1) reduces to

$$G_\phi(\lambda, \tau) = \frac{1}{\Gamma(\lambda/2)} \int_0^\infty u^{\lambda/2-1} e^{-u} \left\langle e^{2u\Omega(\tau)} \right\rangle du . \quad \text{(B4)}$$

Taking into account the definition of $G_\Omega(\lambda, \tau)$ we get (3.19).

The average of the process $\phi(\tau)$ can be derived in a direct way. From (3.18) and (3.12)

$$\langle \phi(\tau) \rangle = -\tfrac{1}{2} \langle \ln[1 - 2\Omega(\tau)] \rangle . \quad \text{(B5)}$$

Expanding the logarithm we have

$$-\ln[1 - 2\Omega(\tau)] = \sum_{k=1}^{\infty} \frac{(k-1)!}{k!} [2\Omega(\tau)]^k = \sum_{k=1}^{\infty} \frac{[2\Omega(\tau)]^k}{k!} \int_0^\infty u^{k-1} e^{-u} du = \int_0^\infty \frac{1}{u} e^{-u} \left(e^{2u\Omega(\tau)} - 1 \right) du , \quad \text{(B6)}$$

so that

$$\langle \phi(\tau) \rangle = \tfrac{1}{2} \int_0^\infty \frac{1}{u} e^{-u} \left[G_\Omega(-2u, \tau) - 1 \right] du, \quad \text{(B7)}$$

which coincides with (3.20).

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