Path-integral approach to diffusion in random media

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Using the path-integral method, we derive the analytical solution for the following one-dimensional diffusion in random media: $\partial P(x,t)/\partial t = D[\partial^2 P(x,t)/\partial x^2] + \lambda V(x)P(x,t)$, where V is a white-noise Gaussian potential. A quantity $\tau = (16D/9\lambda^4)^{1/3}$ is introduced for the time scale. When the diffusion time $t \ll \tau$, the behavior of the average $\langle P(x,t) \rangle$ is essentially diffusive. When $t \gg \tau$, the random potential plays a dominant role, and the average $\langle P(x,t) \rangle$ tends to $[\lambda^4 t^{5/2}/8(\pi D^3)^{1/2}]\exp[(\lambda^4 t^3/48D)(1 - x^2/2Dt)]$.

I. INTRODUCTION

Diffusion in random media, where disorder involves the presence of traps and sources, has recently received considerable attention, both for its intrinsic theoretical interest, analogous mathematically to the problem of "localization," and for its many applications in physical, chemical, and biological systems.[1-4] Examples of such systems would be chemical or physical reactions with random nucleation centers, the size of a polymer chain in a random environment,[5-7] chain reactions with random fissile distribution, or biological multiplication with random nutrient concentration.

In Ref. 3, an analytical solution was reported for the following one-dimensional diffusion equation:

$$\frac{\partial P(x,t)}{\partial t} = D\frac{\partial^2 P(x,t)}{\partial x^2} + \lambda V(x)P(x,t) \tag{1.1}$$

where λ and D are constants and $V(x)$ is a time-independent white-noise Gaussian potential characterized by

$$\langle V(x) \rangle = 0, \quad \langle V(x_1)V(x_2) \rangle = \delta(x_1 - x_2) . \tag{1.2}$$

The angular bracket $\langle \ \rangle$ denotes statistical averages. We consider the initial condition $P(x,0) = \delta(x)$. The analytical solution was derived by a path-integral method. The asymptotic expansion of the solution in $t \to \infty$ was initially somehow in error and clarified later.[3] Rosenbluth first called this to my attention. He also used a variation approach[4] to find

$$\ln[P(0,t)] \to \lambda^4 t^3/(48D) \text{ as } t \to \infty . \tag{1.3}$$

Since then, the problem has received considerable attention. Guyer and Machta related the problem to Flory theory in polymer physics.[5] Leshke and Wonneberger connected the problem to free energy of the Pekar-Frölich polaron.[6] Nattermann and Renz showed that this problem is related to polymers in random media and stochastically growing interfaces.[7-9]

In view of the importance of the issue, it is necessary to present the detailed derivation of the analytical solution for this problem. Furthermore, all former discussions were concentrated on $\langle P(0,t) \rangle$. For many applications, it is necessary to know the behavior of $\langle P(x,t) \rangle$ with $x \neq 0$.

The present paper is organized as follows. In Sec. II, we discuss the path-integral method used in this problem and derive the analytical solution for $\langle P(0,t) \rangle$. A detailed discussion about $\langle P(0,t) \rangle$ is given in Sec. III. Section IV is devoted to the study of $\langle P(x,t) \rangle$ with $x \neq 0$. We have found a time scale τ defined by

$$\tau = \left[\frac{16D}{9\lambda^4} \right]^{1/3} , \tag{1.4}$$

which plays an important role in the present problem. When $t \ll \tau$, the behavior of average $\langle P(x,t) \rangle$ is basically diffusive, the effect of random media is almost negligible. When $t \gg \tau$, the effect of random media becomes dominant, then

$$\langle P(x,t) \rangle \to \frac{\lambda^4 t^{5/2}}{8(\pi D^3)^{1/2}} \exp\left[\frac{\lambda^4 t^3}{48D} \left[1 - \frac{x^2}{2Dt} \right] \right] . \tag{1.5}$$

If we set $x = 0$ in the above result, we recover the asymptotic behavior of $\langle P(0,t) \rangle$ reported before.[3,4]

The path-integral method used in this study was initially invented for a class of disordered systems.[10] The present paper develops this method, which is expected to have applications in other problems related to disordered systems.

II. PATH-INTEGRAL METHOD

We first write $P(x,t)$ in the form

$$P(x,t) = \int_{-\infty}^{\infty} e^{-tE} g(x,E)\, dE . \tag{2.1}$$

Introduce the Hamiltonian

$$H = -D\frac{\partial^2}{\partial x^2} - \lambda V . \tag{2.2}$$

Then, from Eq. (1.1), $g(x,E)$ satisfies the equation

$$Hg = Eg . \tag{2.3}$$

The initial condition at $t = 0$, $P(x,0) = \delta(x)$, is now

$$\delta(x) = \int_{-\infty}^{\infty} g(x,E)\, dE . \tag{2.4}$$

Equations (2.3)and (2.4) imply that

$$g(x,E)=\frac{1}{\pi}\mathrm{Im}\left\langle x\left|\frac{1}{E-H-i\eta}\right|0\right\rangle_D , \tag{2.5}$$

where $\langle\ \rangle_D$ is the Dirac notation and $\eta=0^+$. We use Re or Im to denote the real part or imaginary part, respectively. Since the Hamiltonian H is real, for simplicity, we make all eigenfunctions of H real. Equation (2.5) can be written into a path integral[10]

$$g(x,E)=\frac{-2}{\pi}\mathrm{Re}\frac{\int d\phi\,\phi(x)\phi(0)\exp\left[-i\int d\xi\,\phi(\xi)(E-H-i\eta)\phi(\xi)\right]}{\int d\phi\exp\left[-i\int d\xi\,\phi(\xi)(E-H-i\eta)\phi(\xi)\right]} . \tag{2.6}$$

The replica trick[11] enables us to write $g(x,E)$ into

$$\frac{-2}{\pi}\mathrm{Re}\lim_{n\to 0}\int d\boldsymbol{\phi}\,\phi_1(x)\phi_1(0)\times\exp\left[-i\int d\xi\,\boldsymbol{\phi}\left[E+D\frac{\partial^2}{\partial\xi^2}\right]\boldsymbol{\phi}-i\lambda\int V\boldsymbol{\phi}^2d\xi\right] \tag{2.7}$$

where $\boldsymbol{\phi}$ is an n-dimensional vector. As we take a statistical average, only the term containing V needs to be considered. For a white-noise Gaussian potential, its average yields

$$\left\langle\exp\left[-i\lambda\int V\boldsymbol{\phi}^2d\xi\right]\right\rangle=\exp\left[-\frac{\lambda^2}{2}\int(\boldsymbol{\phi}^2)^2d\xi\right] . \tag{2.8}$$

We substitute Eq. (2.8) into Eq. (2.7) and introduce a transformation $\boldsymbol{\phi}\to\boldsymbol{\phi}/\sqrt{D}$. Now $\langle g(x,E)\rangle$ is given by

$$\frac{-2}{\pi D}\mathrm{Re}\lim_{n\to 0}\int d\boldsymbol{\phi}\,\phi_1(x)\phi_1(0)\times\exp\left\{-i\int d\xi\left[\boldsymbol{\phi}\left[\frac{E}{D}+\frac{\partial^2}{\partial\xi^2}\right]\boldsymbol{\phi}-\frac{i\lambda^2}{2D^2}(\boldsymbol{\phi}^2)^2\right]\right\} . \tag{2.9}$$

Following Ref. 10, Eq. (2.9) can be further simplified to

$$\langle g(x,E)\rangle=\frac{-2}{\pi D}\mathrm{Re}\int_0^\infty\int_0^\infty dr\,dr_1\phi_0(r)\phi_0(r_1)G(r,r_1,x) \tag{2.10}$$

where $G(r,r_1,t)$ is a Green's function, satisfying the equation

$$i\frac{\partial G(r,r_1,t)}{\partial t}=\left[-\frac{1}{4}\left[\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\right]+\frac{E}{D}r^2-i\frac{\lambda^2}{2D^2}r^4\right]G(r,r_1,t) , \tag{2.11}$$

and the boundary condition $G(r,r_1,0)=r_1\delta(r-r_1)$; ϕ_0 is the solution of the equation

$$-\frac{1}{4}\left[\frac{d^2\phi_0}{dr^2}-\frac{1}{r}\frac{d\phi_0}{dr}\right]+\left[\frac{E}{D}r^2-\frac{i\lambda^2}{2D^2}r^4\right]\phi_0=0 \tag{2.12}$$

with the boundary conditions $\phi_0(0)=1$ and $\phi_0(\infty)=0$.

Differential equation (2.12) can be solved exactly. Using $u=r^2/2$, we transform it to the form

$$\frac{d^2\phi_0}{du^2}+4\left[-\frac{E}{D}+i\frac{\lambda^2}{D^2}u\right]\phi_0=0 \tag{2.13}$$

which implies that ϕ_0 is an Airy function. After taking the boundary conditions into account, we represent ϕ_0 in the form

$$\phi_0=\frac{\left[1-\frac{i\lambda^2}{DE}u\right]^{1/2}H^{(1)}_{1/3}\left[\frac{4(-DE+i\lambda^2u)^{3/2}}{3i\lambda^2D}\right]}{H^{(1)}_{1/3}\left[\frac{4(-DE)^{3/2}}{3i\lambda^2D}\right]} \tag{2.14}$$

where $H^{(1)}_{1/3}$ is the Hankel function of the first kind.

We first consider $\langle P(0,t)\rangle$ since it has a simple analytical expression. The behavior of $\langle P(x,t)\rangle$ will be studied in Sec. IV. From Eq. (2.10), at $x=0$, we have

$$\langle g(0,E)\rangle=-\frac{2}{\pi D}\mathrm{Re}\int_0^\infty(\phi_0)^2du . \tag{2.15}$$

Here again $u=r^2/2$. Differentiating Eq. (2.13) with E, we get

$$\frac{d^2}{du^2}\left[\frac{\partial\phi_0}{\partial E}\right]+4\left[-\frac{E}{D}+i\frac{\lambda^2}{D^2}u\right]\frac{\partial\phi_0}{\partial E}-\frac{4}{D}\phi_0=0 . \tag{2.16}$$

Multiplying Eq. (2.16) by ϕ_0, Eq. (2.13) by $\partial\phi_0/\partial E$, and subtracting and integrating over u, we find

$$\frac{2}{D}\int_0^\infty\phi_0^2du=\frac{1}{2}\frac{\partial}{\partial E}\left[\left[\frac{d\phi_0}{du}\right]\bigg|_{u=0}\right] \tag{2.17}$$

where we have used $\phi_0(0)=1$, $\phi_0(\infty)=0$, $(\partial\phi_0/\partial E)|_{u=0}=0$, and $(\partial\phi_0/\partial E)|_{u=\infty}=0$ in the derivation. We now have

$$\langle g(0,E)\rangle=-\frac{1}{\pi}\mathrm{Re}\frac{\partial}{\partial E}\left[\left[\frac{d\phi_0}{du}\right]\bigg|_{u=0}\right] . \tag{2.18}$$

From Eq. (2.14), we have

$$\left[\frac{d\phi_0}{du}\right]\Bigg|_{u=0} = -\frac{i\lambda^2}{2DE} + \frac{2(-E)^{1/2}H_{1/3}^{(1)\prime}[4(-DE)^{3/2}/(3i\lambda^2 D)]}{\sqrt{D}\,H_{1/3}^{(1)}[4(-DE)^{3/2}/(3i\lambda^2 D)]} \; . \tag{2.19}$$

Introduce

$$Z(E) = (-E)^{1/2}H_{1/3}^{(1)}[4(-DE)^{3/2}/(3i\lambda^2 D)] \; . \tag{2.20}$$

Then, we have

$$\left[\frac{d\phi_0}{du}\right]\Bigg|_{u=0} = \frac{-i\lambda^2}{D}\frac{Z'(E)}{Z(E)} \; . \tag{2.21}$$

Hence $\langle g(0,E)\rangle$ is given by

$$\langle g(0,E)\rangle = \frac{\lambda^2}{2\pi D}\mathrm{Im}\frac{\partial}{\partial E}\left[\frac{Z'(E)}{Z(E)}\right] \; . \tag{2.22}$$

$\langle P(0,t)\rangle$ follows from Eq. (2.22),

$$\langle P(0,t)\rangle = \int_{-\infty}^{\infty}\langle g(0,E)\rangle e^{-Et}dE = \frac{\lambda^2 t}{2\pi D}\mathrm{Im}\int_{-\infty}^{\infty}\frac{Z'(E)}{Z(E)}e^{-Et}dE \; . \tag{2.23}$$

Before calculating Eq. (2.23), we note that $Z(E)$ satisfies the Airy equation,

$$Z''(E) + \frac{4DE}{\lambda^4}Z(E) = 0 \; . \tag{2.24}$$

The second independent solution for Eq. (2.24) is $Z^*(E)$ which is a complex conjugate of $Z(E)$ for E real. The Wronskian of these two solutions is a constant, given by

$$Z'(E)Z^*(E) - Z(E)Z^{*\prime}(E) = 6i/\pi \; . \tag{2.25}$$

Applying Eq. (2.25) and the following relationship:

$$\mathrm{Im}\left[\frac{Z'(E)}{Z(E)}\right] = -\frac{i}{2}\left[\frac{Z'(E)}{Z(E)} - \frac{Z^{*\prime}(E)}{Z^*(E)}\right] \tag{2.26}$$

to Eq. (2.23), we have

$$\langle P(0,t)\rangle = \frac{3\lambda^2 t}{2\pi^2 D}\int_{-\infty}^{\infty}\frac{e^{-Et}}{|Z(E)|^2}dE \; . \tag{2.27}$$

This is the solution for $\langle P(0,t)\rangle$.

III. THE RESULT OF $\langle P(0,t)\rangle$

For further study, we substitute Eq. (2.20) into (2.27) and divide the integral $\int_{-\infty}^{\infty}$ into two parts $\int_{-\infty}^{0}$ and $\int_{0}^{+\infty}$. After some algebra, we have

$$\langle P(0,t)\rangle = \frac{3\lambda^2 t}{2\pi^2 D}\int_0^{\infty}d\eta\,\eta^{-1}\left[\frac{e^{\eta t/\tau}}{|H_{1/3}^{(1)}(-i\eta^{3/2})|^2} + \frac{e^{-\eta t/\tau}}{|H_{1/3}^{(1)}(\eta^{3/2})|^2}\right] \tag{3.1}$$

where the time scale τ is in Eq. (1.4). As $\eta\to\infty$, the asymptotic behavior for the two Hankel functions in Eq. (3.1) is different:

$$|H_{1/3}^{(1)}(-i\eta^{3/2})|^2 = \frac{2}{\pi\eta^{3/2}}\exp(2\eta^{3/2}) \tag{3.2}$$

and

$$|H_{1/3}^{(1)}(\eta^{3/2})|^2 = \frac{2}{\pi\eta^{3/2}} \; . \tag{3.3}$$

The integral in Eq.(3.1) is convergent for a positive t. To verify our analytical result in Eq. (3.1), we first consider the case of a small t, $t \ll \tau$. At $t=0$, the source was at the center, then it began to diffuse through the media. When t is small, the effect from random potential is negligible; therefore the diffusion is dominant. Equation (3.1) should give us a diffusion result for $t \ll \tau$. An examination of Eq. (3.1) reveals that the main contribution to $\langle P(0,t)\rangle$ is from the second integral in Eq. (3.1) when $0 < t \ll \tau$. We then have

$$\langle P(0,t)\rangle \to \frac{3\lambda^2 t}{4\pi D}\int_0^{\infty}d\eta\, e^{-\eta t/\tau} = \frac{1}{2\sqrt{D\pi t}} \quad \text{for } t \ll \tau \; . \tag{3.4}$$

This is the well-known diffusion result which confirms our analytical solution.

When $t \gg \tau$, the main contribution to $\langle P(0,t)\rangle$ is from the first integral in Eq. (3.1). Using the saddle-point method, we have the asymptotic behavior

$$\langle P(0,t)\rangle \to \frac{\lambda^4 t^{5/2}}{8(\pi D^3)^{1/2}}\exp\left[\frac{\lambda^4 t^3}{48D}\right] \quad \text{for } t \gg \tau \; . \tag{3.5}$$

The general behavior of $\langle P(0,t)\rangle$ is obtained by numerical calculation of Eq. (3.1). The result is plotted in Fig. 1.

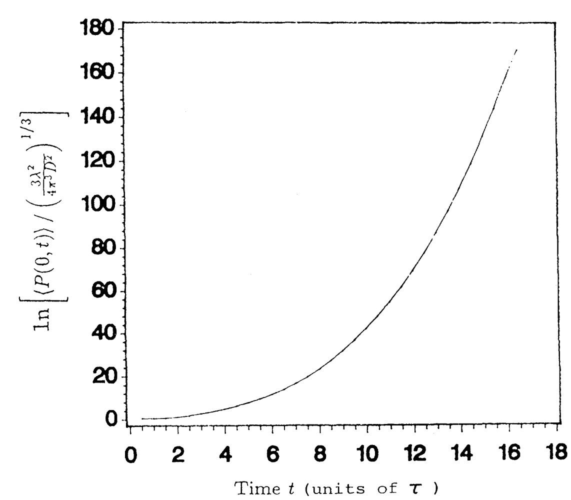

FIG. 1. The general behavior of $\langle P(0,t)\rangle$ vs time t. The time is measured in units of τ. As $t \gg \tau$, the asymptotic behavior in Eq. (3.5) gives a quite accurate result.

On the x axis, the time is measured in units of τ. On the y axis, we plot $\ln[\langle P(0,t)\rangle/(3\lambda^2/4\pi^3D^2)^{1/3}]$. The asymptotic expression in Eq. (3.5) is quite accurate for $t>\tau$. For example, when $t=6\tau$, Eq. (3.1) gives $\langle P(0,t)\rangle=101\,076.1(3\lambda^2/4\pi^3D^2)^{1/3}$, while Eq. (3.5) estimates $\langle P(0,t)\rangle=103\,537.2(3\lambda^2/4\pi^3D^2)^{1/3}$. The difference is only about 2%. This can also be seen from Fig. 1, since the curve is soon approaching $\sim t^3$.

IV. THE BEHAVIOR OF $\langle P(x,t)\rangle$

It is easy to understand that $\langle P(x,t)\rangle$ is an even function of x. In addition, since the only source was at $x=0$ when $t=0$, $\langle P(x,t)\rangle$ must have $\langle P(0,t)\rangle$ as its maximum.

To discuss the behavior of $\langle P(x,t)\rangle$, we define two operators $\hat{L}$ and $\hat{L}_0$,

$$\hat{L}=-\frac{1}{4}\left[\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\right]+\frac{E}{D}r^2-i\frac{\lambda^2}{2D^2}r^4\,,$$
$$\hat{L}_0=-\frac{1}{4}\left[\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}\right]+\frac{E}{D}r^2-i\frac{\lambda^2}{2D^2}r^4\,. \tag{4.1}$$

We note that Eq. (2.12) can be written as $\hat{L}_0\phi_0=0$. From Eq. (2.11) and the initial condition for $G(r,r_1,t)$, we have

$$\left\langle\frac{\partial^s}{\partial x^s}g(x,E)\Big|_{x=0}\right\rangle=-\frac{2}{\pi D}\mathrm{Re}(-i)^s\int_0^\infty r_1\phi_0(r_1)dr_1\int_0^\infty\phi_0(r)(\hat{L})^s\delta(r-r_1)dr\,. \tag{4.2}$$

After integrating Eq. (4.2) by parts and using the property of δ functions, we have

$$\left\langle\frac{\partial^s}{\partial x^s}g(x,E)\Big|_{x=0}\right\rangle=-\frac{2}{\pi D}\mathrm{Re}(-i)^s\int_0^\infty dr\,r\phi_0(r)\left[\hat{L}_0-\frac{1}{2r}\frac{\partial}{\partial r}\right]^s\phi_0(r)\,. \tag{4.3}$$

The Taylor expansion

$$\langle g(x,E)\rangle=\sum_{s=0}\frac{x^s}{s!}\left\langle\frac{\partial^s}{\partial x^s}g(x,E)\Big|_{x=0}\right\rangle \tag{4.4}$$

gives us

$$\langle g(x,E)\rangle=-\frac{2}{\pi D}\mathrm{Re}\int_0^\infty dr\,r\phi_0(r)\exp\left[-ix\left[\hat{L}_0-\frac{1}{2r}\frac{\partial}{\partial r}\right]\right]\phi_0(r)\,. \tag{4.5}$$

Then, $\langle P(x,t)\rangle$ is given by

$$\langle P(x,t)\rangle=-\frac{2}{\pi D}\mathrm{Re}\int_{-\infty}^\infty dE\int_0^\infty dr\,r\phi_0(r)\exp\left[-tE-ix\left[\hat{L}_0-\frac{1}{2r}\frac{\partial}{\partial r}\right]\right]\phi_0(r)\,. \tag{4.6}$$

To consider the situation where x is not too far from the center, we first calculate $\langle(\partial/\partial x)g(x,E)|_{x=0}\rangle$. From Eqs. (2.12) and (4.3), we have

$$\left\langle\frac{\partial}{\partial x}g(x,E)\Big|_{x=0}\right\rangle=-\frac{1}{2\pi D}\mathrm{Re}\left[i\int_0^\infty dr\frac{\partial\phi_0^2}{\partial r}\right]=\frac{1}{2\pi D}\mathrm{Re}(i)=0\,. \tag{4.7}$$

This also implies $[\partial\langle P(x,t)\rangle/\partial x]|_{x=0}=0$, a result consistent with the fact mentioned earlier that at a fixed t, $\langle P(x,t)\rangle$ is an even function of x and has $\langle P(0,t)\rangle$ as its maximum. For a small x, we expand $\ln\langle P(x,t)\rangle$ to x^2,

$$\ln\langle P(x,t)\rangle=\ln\langle P(0,t)\rangle+\tfrac{1}{2}x^2\left\langle\frac{\partial^2}{\partial x^2}P(x,t)\Big|_{x=0}\right\rangle\Big/\langle P(0,t)\rangle\,. \tag{4.8}$$

Define a quantity a through

$$a^2=-2\langle P(0,t)\rangle\Big/\left\langle\frac{\partial^2}{\partial x^2}P(x,t)\Big|_{x=0}\right\rangle\,. \tag{4.9}$$

Then we can write

$$\langle P(x,t)\rangle=\langle P(0,t)\rangle\exp(-x^2/a^2)\,. \tag{4.10}$$

From Eq. (4.3) and the equation $\hat{L}_0\phi_0=0$, we have

$$\left\langle\frac{\partial^2}{\partial x^2}P(x,t)\Big|_{x=0}\right\rangle=\frac{2}{\pi D}\mathrm{Re}\int_{-\infty}^\infty dE\,e^{-Et}\times\int_0^\infty\left[\frac{E}{D}-\frac{2i\lambda^2}{D^2}u\right]\times\phi_0^2du\,. \tag{4.11}$$

In view of Eqs. (2.15) and (2.23), we write the first term in Eq. (4.11) in the form of

$$\frac{2}{\pi D}\mathrm{Re}\int_{-\infty}^\infty dE\,e^{-Et}\int_0^\infty\frac{E}{D}\phi_0^2du=\frac{1}{D}\frac{\partial}{\partial t}\langle P(0,t)\rangle\,. \tag{4.12}$$

Differentiating Eq. (2.13) with λ^2, we get

$$\frac{d^2}{du^2}\left[\frac{\partial\phi_0}{\partial(\lambda^2)}\right]+4\left[-\frac{E}{D}+i\frac{\lambda^2}{D^2}u\right]\frac{\partial\phi_0}{\partial(\lambda^2)}+\frac{4i}{D}u\phi_0=0\ . \tag{4.13}$$

Multiplying Eq. (4.13) by ϕ_0, Eq. (2.13) by $\partial\phi_0/\partial(\lambda^2)$, and subtracting and integrating over u, we find

$$\frac{4i}{D^2}\int_0^\infty \phi_0^2 u\, du=\frac{\partial}{\partial(\lambda^2)}\left[\left[\frac{d\phi_0}{du}\right]\Big|_{u=0}\right]\ . \tag{4.14}$$

In view of Eq. (2.21), Eq. (4.14) reads

$$\int_0^\infty \phi_0^2 u\, du=-\frac{D}{4}\frac{\partial}{\partial(\lambda^2)}\left[\lambda^2\frac{Z'}{Z}\right] \tag{4.15}$$

where Z is defined in Eq. (2.20). Applying Eq. (2.23) to Eq. (4.15), we have the second term in Eq. (4.11),

$$-\frac{4\lambda^2}{\pi D^3}\mathrm{Re}\left[i\int_{-\infty}^{\infty}dE\, e^{-Et}\int_0^\infty \phi_0^2 u\, du\right] =-\frac{2\lambda^2}{Dt}\frac{\partial}{\partial(\lambda^2)}\langle P(0,t)\rangle\ . \tag{4.16}$$

Combining Eqs. (4.16), (4.12), and (4.9), we have

$$\frac{1}{a^2}=\left[\frac{-1}{2D}\frac{\partial}{\partial t}+\frac{\lambda^2}{Dt}\frac{\partial}{\partial(\lambda^2)}\right]\ln[\langle P(0,t)\rangle]\ . \tag{4.17}$$

As $t\ll\tau$, Eqs. (3.4), (4.17), and (4.10) give us the well-known diffusion solution

$$\langle P(x,t)\rangle\to\frac{1}{2\sqrt{D\pi t}}\exp(-x^2/4Dt)\quad \text{for } t\ll\tau\ . \tag{4.18}$$

From Eq. (3.5), we have the result for $t\gg\tau$,

$$\langle P(x,t)\rangle\to\frac{\lambda^4 t^{5/2}}{8(\pi D^3)^{1/2}}\exp\left[\frac{\lambda^4 t^3}{48D}\left[1-\frac{x^2}{2Dt}\right]\right] \quad \text{for } t\gg\tau\ . \tag{4.19}$$

This result shows that in the long time limit, the random potential plays a dominant role, but the effect of diffusion cannot be ignored. The diffusion starts from the original source and spreads to the whole space. At time t, the effective diffusion length is $\sqrt{2Dt}$. In the process of diffusion, the peaks are getting higher and higher and the total population $\int\langle P(x,t)\rangle dx$ also increases as fast as $\sim\exp(\lambda^4 t^3/48D)$.

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[1]For example, see G. H. Weiss, J. Stat. Phys. **42**, 3 (1986), and references therein; K. Kang and S. Redner, Phys. Rev. A **32**, 435 (1985); **30**, 3362 (1984).
[2]F. C. Zhang, Phys. Rev. Lett. **56**, 2113 (1986); **56**, 1980 (1987); A. Engel and W. Ebeling, *ibid.* **59**, 1979 (1987); W. Ebeling, A. Engel, B. Esser, and R. Feistel, J. Stat. Phys. **37**, 369 (1984).
[3]R. Tao, Phys. Rev. Lett. **61**, 2405 (1988); **63**, 2695 (1989).
[4]M. N. Rosenbluth, Phys. Rev. Lett. **63**, 467 (1989).
[5]R. A. Guyer and J. Machta, Phys. Rev. Lett. **64**, 494 (1990).
[6]H. Leshke and S. Wonneberger, J. Phys. A **22**, L1009 (1989).
[7]T. Nattermann and W. Renz, Phys. Rev. A **40**, 4675 (1989).
[8]M. E. Cates and R. C. Ball, J. Phys. (Paris) **49**, 2009 (1988).
[9]S. F. Edwards and M. Muthukumar, J. Chem. Phys. **89**, 2435 (1988).
[10]R. Tao and J. M. Luttinger, Phys. Rev. B **27**, 935 (1983).
[11]J. L. Cardy, J. Phys. C **11**, L321 (1978).