Transverse laser patterns. II. Variational principle for pattern selection, spatial multistability, and laser hydrodynamics

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(Received 12 July 1990)

We show that the stationary transverse patterns, formed by the interaction of the modes of a frequency-degenerate family, are selected by the laser according to a specific variational principle. We discuss the phenomenon of spatial multistability, which consists in the coexistence of two or more different configurations for the same values of the control parameters. We establish a general connection between laser physics and hydrodynamics, by reformulating the dynamical equations of the laser in the form of hydrodynamical equations for a compressible fluid, similar to the law of mass conservation and to the Bernoulli equation. This result provides a general framework for the discussion of the relations between optical and hydrodynamical turbulence.

I. INTRODUCTION

In a companion paper,[1] to be referred to hereafter as I, we analyze the stationary transverse patterns that emerge from the competition among the transverse cavity modes belonging to a frequency-degenerate family with $2p+l=q$ fixed, where p and l are the radial and the angular indices of the modes, respectively. We showed that by varying the control parameters of the system one meets several different patterns, which were calculated theoretically and observed experimentally in a Na_2 laser. Most of these patterns exhibit the presence of a crystal of phase singularities, which have a structure quite similar to the vortices that are familiar, for example in hydrodynamics and superconductivity.

In this paper (II), we address some questions which concern mainly the general principles that govern the phenomena of transverse spatiotemporal pattern formation and dynamics in lasers.[2] First of all, it is highly desirable to find some general laws which rule the selection of the spatial patterns, at least in the case of the states with a stationary output. In this paper we give the solution to this problem for the special case of a frequency-degenerate family. Precisely, we formulate a variational principle, which governs the selection of the stationary patterns for any value of q; we define a generalized free-energy functional V of the electric field, such that the stable stationary configurations correspond to the local minima of V. Hence the intensities and the relative phases of the modes in the stationary solutions are fixed by this minimality requirement; the transitions from one pattern to another, which are observed by varying the control parameters, are also governed by the generalized free energy.

The second part of this paper focuses on an interesting aspect that emerges from the results of I, namely, the phenomenon of spatial multistability. Precisely, under appropriate conditions, we find that two or more stationary solutions coexist for the same values of the parameters. The simplest example of spatial bistability was demonstrated in Ref. 3, in the case where $q=1$, in which only two modes compete with each other and give rise to a bistability between two patterns which differ in their transverse field configuration, but not in their intensity distribution. Here we show the possibility of multistable behavior, also among patterns which differ in their intensity configuration. This phenomenon is potentially very interesting in the perspective of applications to the field of optical information processes, associative memories, and pattern recognition. At variance from the standard optical multistability, in which the system is able to produce outputs of different intensity under the same parametric conditions, in the case of the spatial multistability the outputs differ basically for the transverse configuration of the field.

The aim of the last part of the paper is, on the one hand, to generalize some of our results beyond the case of a frequency-degenerate family of modes and, on the other hand to substantiate the connection between lasers and hydrodynamics which was elaborated in Sec. IV of I in the discussion of the vortex nature of phase singularities. Hence we start from the set of dynamical equations, recently derived in Ref. 4, that govern the competition of the complete set of transverse modes which correspond to a given value of the longitudinal index of the modes. In the limit of adiabatic elimination of the atomic variables we reformulate the laser equations in a way that enables us to establish a general connection with the case of hydrodynamics. Precisely, we reshape the dynamical equations of the laser in the form of "laser hydrodynamical equations" for a compressible fluid, similar to the law of mass conservation and to the Bernoulli equation. This result can help to clarify the connection between optical and hydrodynamical turbulence.

In Sec. II we formulate the variational principle for the selection of stationary patterns in a frequency-degenerate family of modes; this principle is generalized in Sec. III to include the possibility of a coherent signal injected into the cavity. Section IV is devoted to the topic of spatial multistability. In Sec. V we remind the model of Ref. 4 and discuss the issue of cooperative frequency locking. In Sec. VI we derive the hydrodynamical equations for the laser.

II. THE CASE OF A FREQUENCY-DEGENERATE FAMILY OF MODES: DYNAMICAL EQUATIONS AND VARIATIONAL PRINCIPLE FOR PATTERN SELECTION

Throughout this paper we will use the same notations of I and therefore we will not repeat here their meaning. However, let us recall the equations which govern the dynamics of the frequency-degenerate family of modes.

The slowly varying envelope of the electric field is written in the form

$$F(\rho,\varphi,t)=\sum_{p,l,i}{}' f_{pli}(t)A_{pli}(\rho,\varphi)\ , \tag{2.1}$$

or in the form

$$F(\rho,\varphi,t)=\sum_{p,l,i}{}' g_{pli}(t)B_{pli}(\rho,\varphi)\ , \tag{2.2}$$

where $A_{pli}(\rho,\varphi)$ and $B_{pli}(\rho,\varphi)$ are the Gauss-Laguerre and the Gauss doughnut modes, respectively, described in Sec. II of I. The prime indicates that the sum is restricted to the modes of the degenerate family with $2p+l=q$. The modal amplitudes f_{pli} obey the time evolution equations

$$\frac{df_{pli}}{dt}=-k\left[(1-i\Delta)f_{pli}-2C\int_0^{2\pi}d\varphi\int_0^{\infty}d\rho\,\rho\,A_{pli}(\rho,\varphi)P(\rho,\varphi,t)\right]\ , \tag{2.3a}$$

which are coupled with the atomic equations

$$\frac{\partial P}{\partial t}=\gamma_\perp[F(\rho,\varphi,t)D(\rho,\varphi,t)-(1+i\Delta)P(\rho,\varphi t)]\ , \tag{2.3b}$$

$$\frac{\partial D}{\partial t}=-\gamma_\parallel\{\mathrm{Re}[F^*(\rho,\varphi,t)P(\rho,\varphi t)]+D(\rho,\varphi,t)-\chi(\rho)\}\ . \tag{2.3c}$$

The modal amplitudes g_{pli} obey a set of equations identical to Eqs. (2.3a), but with the replacement of A_{pli} by B^*_{pli}.

At steady state the amplitudes f_{pli} obey the equations

$$f_{pli}=2C\sum_{p',l',i'}{}'\int_0^{2\pi}d\rho\,\rho\frac{A_{pli}(\rho,\varphi)A_{p'l'i'}(\rho,\varphi)}{1+\Delta^2+|F(\rho,\varphi)|^2}\chi(\rho)f_{p'l'i'}\ ; \tag{2.4}$$

in the case of the modes g_{pli}, one has a set of stationary equations identical to Eqs. (2.4) provided one replaces $A_{pli}(\rho,\varphi)$ and $A_{p'l'i'}(\rho,\varphi)$ by $B^*_{pli}(\rho,\varphi)$ and $B_{p'l'i'}(\rho,\varphi)$, respectively.

We note that if one multiplies both sides of Eqs. (2.4) by f^*_{pli}, perform the sum over the modes of the family, take into account Eq. (2.1) and the identity

$$\sum_{p,l,i}{}'|f_{pli}|^2=\int_0^{2\pi}d\varphi\int_0^{\infty}d\rho\,\rho|F(\rho,\varphi)|^2\ , \tag{2.5}$$

which follows from the orthonormality of the modes A_{pli}, one obtains the equation

$$\int_0^{2\pi}d\varphi\int_0^{\infty}d\rho\,\rho|F(\rho,\varphi)|^2=\int_0^{2\pi}d\varphi\int_0^{\infty}d\rho\,\rho\frac{2C\chi(\rho)|F(\rho,\varphi)|^2}{1+\Delta^2+|F(\rho,\varphi)|^2}\ , \tag{2.6}$$

which generalizes in a straightforward way the well-known steady-state equation of the plane-wave theory. In Sec. VI we will show that Eq. (2.6) remains valid even beyond the case of a degenerate family. Let us now introduce the functional

$$V=\int_0^{2\pi}d\varphi\int_0^{\infty}d\rho\,\rho\{|F(\rho,\varphi)|^2-2C\chi(\rho)\ln[1+\Delta^2+|F(\rho,\varphi)|^2]\}\ , \tag{2.7}$$

where it is understood that the field F is given by Eq. (2.1) or (2.2). Thus, V is in fact a function of the mode amplitudes f_{pli} and of their complex conjugates f^*_{pli} (or, equivalently, of the mode amplitudes g_{pli} and of their complex conjugates g^*_{pli}) with $2p+l=q$, $i=1,2$.

By using Eqs. (2.1) and the orthonormality of the modes A_{pli}, one verifies easily that the stationary equations (2.4) can be written in the form

$$0=\frac{\partial V}{\partial f^*_{pli}}\ . \tag{2.8}$$

Similarly, the complex conjugates of Eqs. (2.4) read

$$0=\frac{\partial V}{\partial f_{pli}} . \quad (2.8')$$

In the same way, the steady-state equations for the amplitudes g_{pli} can be reshaped in the form

$$0=\frac{\partial V}{\partial g^*_{pli}}, \quad 0=\frac{\partial V}{\partial g_{pli}} . \quad (2.9)$$

Hence the stationary solutions correpond to the stationary points of the functional V, which therefore plays the role of a generalized free energy in this system, which lies far from thermal equilibrium.[5] A necessary condition for the stability of a stationary solution is that it corresponds to a minimum of V; this condition becomes sufficient in the good cavity limit $k \ll \gamma_\parallel, \gamma_\perp$ in which the atomic variables can be adiabatically eliminated by setting $\partial P/\partial t = \partial D/\partial t = 0$ in Eqs. (2.3b) and (2.3c) so that one obtains the set of equations for the modal amplitudes

$$\frac{df_{pli}}{d\tau}=-(1-i\Delta)\frac{\partial V}{\partial f^*_{pli}} \quad (2.10)$$

together with their complex conjugates; the normalized time τ is equal to κt. In the case $k \ll \gamma_\parallel$, $\gamma_\perp$ one can exclude the possibility of instabilities which lead to the onset of spontaneous oscillations because V is a monotonically decreasing function of time. As a matter of fact, we have

$$\frac{dV}{d\tau}=\sum_{p,l,i}{}' \left[\frac{\partial V}{\partial f_{pli}}\frac{df_{pli}}{d\tau}+\frac{\partial V}{\partial f^*_{pli}}\frac{df^*_{pli}}{d\tau} \right] ,$$

so that, by using Eqs. (2.10) and their complex conjugates, we obtain

$$\frac{dV}{d\tau}=-2\sum_{p,l,i}{}' \left| \frac{\partial V}{\partial f_{pli}} \right|^2 \le 0 . \quad (2.11)$$

The absence of oscillations arises from the fact that all the interacting modes have the same frequency, so that frequency competition is completely eliminated.

In the case of a frequency-degenerate family of modes, as we are considering here, the phenomenon of cooperative frequency locking is in principle not necessary to generate stationary transverse patterns, simply because the modes have by definition an equal frequency. It must be kept in mind, however, that in real lasers, due to the residual asymmetries of the cavity, the modes of the family are in general not exactly degenerate in frequency; hence the cooperative frequency locking plays a role in synchronizing the modal frequencies. In addition, the locking process fixes the relative phases of the modes in the stationary patterns, which are selected according to the variational principle dictated by the generalized free energy V. As a matter fact, this principle determines the values of the moduli $|f_{pli}|$ or $|g_{pli}|$ of the modal amplitudes in the steady states, as well as their relative phases. In order to find the stationary states, it is essential to allow for *complex* values of the amplitudes f_{pli}, because in all multimode stationary solutions at least one of the relative phases is different from zero.

A simple example of application of the variational principle is obtained if one introduces the cubic approximation, valid near the laser threshold, which consists in replacing

$$\frac{1}{1+\Delta^2+|F(\rho,\varphi)|^2} \to \frac{1}{1+\Delta^2}-\frac{|F(\rho,\varphi)|^2}{(1+\Delta^2)^2} \quad (2.12)$$

in Eq. (2.6) and, correspondingly, in approximating the ln term in the potential (2.7) as follows:

$$\ln(1+\Delta^2+|F(\rho,\varphi)|^2) \to \ln(1+\Delta^2)+\frac{|F(\rho,\varphi)|^2}{(1+\Delta^2)^2}-\frac{1}{2}\frac{|F(\rho,\varphi)|^4}{(1+\Delta^2)^4} . \quad (2.12')$$

The functional V in the cubic approximation is analytically calculated in Ref. 6 for the case $2p+l=2$ as a function of $g_1, |g_2|, |g_3|, \theta_2+\theta_3$, where θ_2 and θ_3 are the phases of the amplitudes g_2 and g_3 corresponding to the two doughnut modes of the family (the phase of g_1 is set equal to zero by definiteness). The expression of V shows immediately that it has a minimum for $\theta_2+\theta_3=\pi$; as we have shown in Sec. III B of I, this result remains true also beyond the cubic approximation, because all the stable multimode stationary patterns of the case $2p+l=2$ have $\theta_2+\theta_3=\pi$, as the numerical analysis shows.

It is important to observe the symmetry properties of the generalized free energy (2.7). It is clear that the value V does not change if one performs the following operations:

(a) phase conjugation

$$F \to F^* ; \quad (2.13a)$$

(b) parity transformation

$$\varphi \to -\varphi ; \quad (2.13b)$$

(c) rotation

$$\varphi \to \varphi+\varphi_0 . \quad (2.13c)$$

This implies that if $F(\rho,\varphi)$ is a stable stationary state, all the configurations obtained by performing one of the operations (2.13) correspond to other stable stationary states, for the same values of the parameters.

It must be noted that there is no translational symmetry even in the case of a flat pump profile $\chi(\rho)=1$ (i.e., $\psi \to \infty$), despite the fact that in this case the potential V does not change under this operation. As a matter of fact, by performing a translation, one obtains a configuration of the field that can no longer be expressed as a linear combination of the modes of the frequency-degenerate family.

In the resonant case $\Delta=0$, the generalized free energy V can be easily extended to include the case of radially dependent losses that can simulate the presence of an aperture (compare Ref. 36 in I). As a matter of fact, the term

$$\int_0^{2\pi} d\varphi \int_0^\infty d\rho\, \rho |F(\rho,\varphi)|^2$$

in Eq. (2.7) must be replaced by

$$\int_0^{2\pi} d\varphi \int_0^\infty d\rho\, \rho\, \zeta(\rho) |F(\rho,\varphi)|^2 , \quad (2.14)$$

where the function $\zeta(\rho)$ describes the radial distribution of the losses.

We note that the potential V is directly a functional of the field F, so that its expression (2.7) does not depend on the particular choice of Gauss-Laguerre or Gauss-Hermite modes as the basis.

The presence of the generalized free energy V becomes especially interesting if one includes noise in the description, as is done in Ref. 6. As a matter of fact, if the noise is additive one can immediately obtain the stationary distribution for the modal amplitudes.[6]

III. THE CASE OF AN INJECTED SIGNAL

Throughout this paper, with the exception of this section, we assume that the laser is running freely. Here, however, we show briefly that our treatment can be easily generalized to the case in which the system is driven by an external coherent field (laser with injected signal). Let us assume for the sake of simplicity (a) the resonant condition $\Delta=0$, (b) that the input field has a frequency ω_0 equal to that of the laser, and (c) that it has the transverse configuration of a mode $(\bar{p},\bar{l},\bar{i})$ belonging to the frequency-degenerate family.

In the right-hand side of the dynamical equations (2.3a) one must add the term

$$y\delta_{p\bar{p}}\delta_{l\bar{l}}\delta_{i\bar{i}} , \tag{3.1}$$

where y is the normalized amplitude of the incident field, which is assumed real for definitess. Equations (2.3b) and (2.3c) remain unchanged.

Correspondingly, in the expression of the generalized free energy (2.7) one must add the contribution

$$y\left[\int_0^{2\pi}d\varphi\int_0^{\infty}d\rho\,\rho\, A_{\bar{p}\bar{l}\bar{i}}(\rho,\varphi)F(\rho,\varphi)+\text{c.c.}\right] , \tag{3.2}$$

which, using Eq. (2.1) and the orthonormality of the modes A_{pli}, becomes equal to

$$y(f_{\bar{p}\bar{l}\bar{i}}+f^*_{\bar{p}\bar{l}\bar{i}}) . \tag{3.3}$$

With this inclusion, the steady-state equations have still the form (2.8) also in the presence of the injected field.

We note that the same considerations hold if we consider, instead of an amplifying medium, a passive absorbing medium, as in the standard configuration of optical bistability.[7] In this case we must add the injected field and set $\chi(\rho)=-1$ in the equations.

In both the active and passive cases, the injected signal introduces a breaking of the cylindrical symmetry if the mode $\bar{p}\,\bar{l}\,\bar{i}$ is not symmetric.

IV. SPATIAL MULTISTABILITY

A very interesting phenomenon that emerges from our investigations is that there are regions of the parameter space of the system in which two or more stable stationary states coexist. This means that for such values of the parameters the system displays two or more distinct attractors in its phase space, and according to the initial conditions the system approaches one or the other attractor.

The simplest example of this phenomenon is provided by the case $2p+l=1$, in which the two doughnut solutions $p=0$, $l=1$, $i=1,2$ coexist for all values of the parameters for which they are above threshold.[1] The two doughnut solutions have the same transverse intensity configuration, but different field configurations, which are obtained from each other by performing the transformation (2.13b). By using the astigmatic detection technique,[3] one transforms different *field* patterns, which are obtained from each other by the transformation (2.13b), into distinct *intensity* patterns, so that one can distinguish, for example, the two doughnut configurations. For instance, in the case $2p+l=1$, the two doughnut configurations are transformed in the Gauss-Hermite modes TEM_{10} and TEM_{01} respectively.

In the case $2p+l=2$, there is a region of the parameter space where the four-hole (4H) pattern coexists with the oval, and another domain where the 4H coexists with the doughnut patterns. With respect to Fig. 5 of I, the 4H-oval domain lies in region 4H-O, while the 4H-doughnut region is region D-4H.

In both cases, one has for the first time the phenomenon of bistability between different *intensity* patterns. However, from the viewpoint of the field, one has an effective situation of tristability, because for the doughnut and oval configurations using the astigmatic detection technique one can distinguish the two different field configurations connected by the transformation (2.13b). Of course, in presence of cylindrical symmetry, all patterns can be arbitrarily rotated around the origin, and therefore we do not consider as different the configurations that are obtained from one another by rotation. If, however, one introduces into the system a small rotational asymmetry, one can fix the position of the 4H pattern, for example, by setting one of the sides of the square, formed by its four phase singularities, parallel to the axis of the astigmatic lens. In this case, one has the possibility of two different stationary field configurations for the 4H pattern, which can be obtained from each other by the parity transformation (2.13b). Thus, by using the astigmatic technique one can distinguish four different field patterns, which produces a case of *tetrastability*.

In the case $2p+l=3$ the numerical analysis of Sec. III C of I shows the existence of several domain of multistability between different intensity patterns (see Figs. 15 and 16 of I). In particular, in the narrow shaded region of Fig. 16 four different intensity patterns coexist, with 3, 5, 7, and 9 phase singularities, respectively.

It must be noted that the phenomenon of spatial multistability is basically different from what is usually meant by optical multistability. As a matter of fact, in ordinary optical multistability one has that the system under identical conditions is able to produce outputs of completely different intensity. In the case of spatial multistability, on the other hand, the system produces—again under identical conditions—outputs that differ in their transverse configuration much more than in their total intensity.

We believe that the phenomenon of spatial multistability can find useful applications in such fields as, for example, optical information processing. For instance, one can conceive of the construction of optical associative memories based on the principles elaborated here; this would represent a concrete realization of the general scheme recently formulated by Haken and collaborators.[8,9] This point will be elaborated in a separate publication.

V. BEYOND THE CASE OF A FREQUENCY-DEGENERATE FAMILY

We drop now the assumption that the envelope F is given by a superposition of the modes of a frequency-degenerate family. Precisely, we now assume that the reference frequency is the mode-pulled frequency $\bar{\omega}_0$ given by formula (2.5) of I with $\omega_q=\omega_0$, where ω_0 is the frequency of the fundamental mode $p=l=0$. In Ref. 4, by introducing an appropriate set of assumptions, we derived the following time evolution equation for the envelope $F(\rho,\varphi,t)$:

$$\frac{\partial F}{\partial t}=-k\left[\left[1-i\Delta'-i\frac{a}{2}(\tfrac{1}{4}\nabla_\perp^2-\rho^2+1)\right]\times F(\rho,\varphi,t)-2CP(\rho,\varphi,t)\right] , \tag{5.1}$$

where $\nabla_\perp^2$ is the transverse Laplacian

$$\nabla_\perp^2=\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} , \tag{5.2}$$

a is the frequency separation between the modes $p=0$, $l=0$, and $p=1$, $l=0$, measured in units of the cavity linewidth k, and Δ' is given by Eq. (2.9) of I with $q=0$.

$$\Delta'=\frac{\bar{\omega}_0-\omega_0}{k}=\frac{\omega_a-\bar{\omega}_0}{\gamma_\perp} . \tag{5.3}$$

Equation (5.1a) must be coupled with the atomic equations

$$\frac{\partial P}{\partial t}=\gamma_\perp[F(\rho,\varphi,t)D(\rho,\varphi,t)-(1+i\Delta')P(\rho,\varphi,t)] , \tag{5.1'}$$

$$\frac{\partial D}{\partial t}=-\gamma_\parallel\{\mathrm{Re}[F^*(\rho,\varphi,t)P(\rho,\varphi,t)]+D(\rho,\varphi,t)-\chi(\rho)\} . \tag{5.1''}$$

Equations (5.1) include all the modes of the cavity which correspond to a given fixed value of the longitudinal index of the modes. By introducing the expansion

$$F(\rho,\varphi,t)=\sum_{p,l,i}f_{pli}(t)A_{pli}(\rho,\varphi) , \tag{5.4}$$

where the sum is extended to *all* the values of the indices p and l ($p,l=0,1,2,3\ldots$), and taking into account the identity

$$(\tfrac{1}{4}\nabla_\perp^2-\rho^2+1)A_{pli}=-(2p+l)A_{pli} , \tag{5.5}$$

Eq. (5.1) can be reformulated as the following set of equations for the modal amplitudes f_{pli}:

$$\frac{df_{pli}}{dt}=-k\left[(1-i\Delta'+ia_{pl})f_{pli}-2C\int_0^{2\pi}d\varphi\int_0^{\infty}d\rho\,\rho\,A_{pli}(\rho,\varphi)P(\rho,\varphi,t)\right] , \tag{5.6}$$

where

$$a_{pl}=\frac{a}{2}(2p+l)=\frac{\omega_{pl}-\omega_0}{k} , \tag{5.7}$$

and ω_{pl} denotes the frequency of the modes A_{pli}, which does not depend on the index i.

A. Stationary-intensity solutions and cooperative frequency locking

The stationary-intensity solutions are governed by the phenomenon of cooperative frequency locking.[10] They have the form

$$F(\rho,\varphi,t)=\exp(-i\delta t)F_S(\rho,\varphi) , \tag{5.8a}$$

$$P(\rho,\varphi,t)=\exp(-i\delta t)P_S(\rho,\varphi) , \tag{5.8b}$$

$$D(\rho,\varphi,t)=D_S(\rho,\varphi) , \tag{5.8c}$$

where δ denotes the frequency offset between the common oscillation frequency ω_S, cooperatively selected by the modes in the locking process and the reference frequency $\bar{\omega}_0$. By following the same procedure outlined in Sec. VII of Ref. 10, and using Eqs. (5.4), (5.5), and (5.7), one can derive from Eqs. (5.1) the formula

$$\delta=\frac{k\gamma_\perp}{k+\gamma_\perp}\frac{\sum_{p,l,i}a_{pl}|f_{pli}|^2}{\sum_{p,l,i}|f_{pli}|^2} , \tag{5.9}$$

which, using Eq. (2.5) of I (with ω_q replaced by ω_0) and Eq. (5.7) can be rephrased as a formula which gives the common oscillation frequency ω_S in the cooperatively frequency-locked state:

$$\omega_S=\bar{\omega}_0+\delta=\frac{\sum_{p,l,i}\bar{\omega}_{pli}|f_{pli}|^2}{\sum_{p,l,i}|f_{pli}|^2} , \tag{5.10}$$

with

$$\bar{\omega}_{pli}=\frac{k\omega_a+\gamma_\perp\omega_{pl}}{k+\gamma_\perp} . \tag{5.11}$$

Equation (5.10) states that the common oscillation frequency is given by the average of the frequencies of the modes [pulled by the atomic line according to the mode-pulling formula (5.11)] weighted over the distribution $|f_{pli}|^2$ of the intensities of the various modes. Equation (5.10) generalizes to the case of a cavity with spherical mirrors the result obtained in Ref. 10 for a Cartesian cavity geometry.

B. Connection with the case of a frequency-degenerate family

If we consider only the amplitudes f_{pli} with $2p+l=q$ fixed, the frequency locking is automatically assured because all the modes have the same frequency ω_q.

From Eqs. (5.10) and (5.7) the common frequency of oscillation of the modes is now

$$\omega_S=\omega_0+\delta=\overline{\omega}_q=\frac{k\omega_a+\gamma_\perp\omega_q}{k+\gamma_\perp} . \tag{5.12}$$

It is easy to verify in this case that the set of equations (5.6) and (5.1′) and (5.1″) is completely equivalent to the equations (2.3) that we used to describe the frequency-degenerate family.

In fact, when we consider only the modes with $2p+l=q$, the quantity ka_{pl} represents the difference $\omega_q-\omega_0$ [see Eq. (5.7)]. Hence, using the definition of Δ [given by Eq. (2.9) of I] and Eqs. (5.3) and (5.12), we obtain

$$\Delta'-a_{pl}=\frac{\overline{\omega}_0-\omega_0}{k}-\frac{\omega_q-\omega_0}{k}=\frac{\overline{\omega}_q-\delta-\omega_q}{k}=\Delta-\frac{\delta}{k} \tag{5.13}$$

and

$$\Delta'=\frac{\omega_a-\overline{\omega}_0}{\gamma_\perp}=\frac{\omega_a-\overline{\omega}_q+\delta}{\gamma_\perp}=\Delta+\frac{\delta}{\gamma_\perp} . \tag{5.14}$$

If we now make the change of variables

$$F(\rho,\varphi,t)=\exp(-i\delta t)F'(\rho,\varphi,t) , \tag{5.15a}$$

$$P(\rho,\varphi,t)=\exp(-i\delta t)P'(\rho,\varphi,t) , \tag{5.15b}$$

we observe, with the help of Eqs. (5.13) and (5.14), that the equations for F', P', and D derived from Eqs. (5.6) and (5.1′) and (5.1″) are identical to Eqs. (2.3).

VI. LASER HYDRODYNAMICS

In this section, we perform a first step in the direction of generalizing the results of this paper and of I beyond the case of a frequency-degenerate set of modes. In doing that, we will reshape the laser equations in a form which is reminescent of the structure of the hydrodynamical equations. This formulation will allow us to derive a Bernoulli-type equation which governs the motion of the phase singularities in the transverse plane, similar to the case of a compressible fluid with vortices.

Even if the following analysis can be extended to the general case, we now perform the adiabatic elimination of the atomic variables, which is valid in the "good cavity" limit $k\ll\gamma_\parallel,\gamma_\perp$. We set the time derivatives equal to zero in the atomic equations (2.3b) and (2.3c); thus by inserting Eq. (2.11) of I (with Δ replaced by Δ') into Eq. (5.1), we obtain

$$\frac{\partial F}{\partial\tau}=-[1-i\Delta'-i\frac{a}{2}(\tfrac{1}{4}\nabla_\perp^2-\rho^2+1)]F+2C\chi(\rho)\frac{(1-i\Delta')F}{1+(\Delta')^2+|F|^2} , \tag{6.1}$$

where we have introduced the normalized time

$$\tau=kt . \tag{6.2}$$

In order to reformulate Eq. (6.1) in the form of hydrodynamical equations,[11] we introduce the quantities [compare Eqs. (2.19), (2.20), and (4.21) of I]

$$\sigma=|F|^2,\quad \mathbf{v}=\frac{a}{4}\nabla\Phi , \tag{6.3}$$

where σ plays the role of the density of a fluid in motion and $\mathbf{v}$ that of the velocity field, consistently with the analogy of the laser with a fluid developed in Sec. IV of I [note, however, that σ and $\mathbf{v}$ defined by Eq. (6.3) are adimensional]. Clearly the vector field $\mathbf{v}$ is orthogonal to the equiphase lines. With some algebraic manipulations, Eq. (6.1) and its complex conjugate can be cast in the form

$$\frac{\partial\sigma}{\partial\tau}+\nabla\cdot(\sigma\mathbf{v})=-2\sigma\left[1-\frac{2C\chi}{1+(\Delta')^2+\sigma}\right] , \tag{6.4}$$

$$\frac{\partial\Phi}{\partial\tau}=-\frac{a}{8}(\nabla\Phi)^2+\frac{a}{8\sigma^{1/2}}\nabla^2\sigma^{1/2}+\Delta'+\frac{a}{2}(1-\rho^2)-\frac{2C\chi\Delta'}{1+(\Delta')^2+\sigma} , \tag{6.5}$$

where the spatial derivatives are performed with respect to the spatial coordinates normalized to the beam waist w. We recall that it is convenient to consider the vectors in the full three-dimensional space, taking into account that σ and $\mathbf{v}$ do not depend on the longitudinal coordinate z.

Next, let us show the connection with the case of hydrodynamics. Let us consider the Euler equation for the velocity field $\mathbf{v}$ of the fluid

$$\frac{\partial\mathbf{v}}{\partial t}=-(\mathbf{v}\cdot\nabla)\mathbf{v}+\nabla\mathcal{P}=-(\nabla\times\mathbf{v})\times\mathbf{v}-\tfrac{1}{2}\nabla\mathbf{v}^2+\nabla\mathcal{P} , \tag{6.6}$$

where $\mathcal{P}$ is the pressure term. In the case that $\mathbf{v}=\nabla\eta$, with η a smooth function over the transverse plane (see Sec. IV of I), Eq. (6.6) can be integrated (Bernoulli's theorem) to give the following equation for η:

$$\frac{\partial\eta}{\partial t}+\tfrac{1}{2}(\nabla\eta)^2-\mathcal{P}=\mu(t)\quad\text{(Bernoulli's equation)} , \tag{6.7}$$

with μ an arbitrary time-dependent function, as a consequence of the fact that $\mathbf{w}=\nabla\times\mathbf{v}=\mathbf{0}$ over the transverse plane. Such a result still holds also for the singular velocity field described by Eq. (4.15) of I, which is a pseudogradient field due to the presence of the contribution $\nabla\beta$. In this case η is simply replaced by $(q/2\pi)\Phi=\eta+(q/2\pi)\beta$ [see Eqs. (4.19) and (4.20) of I], which is multivalued due to β. Then we have (by setting $\mu=0$)

$$\frac{\partial\Phi}{\partial t}=-\frac{q}{4\pi}(\nabla\Phi)^2+\mathcal{P}\frac{2\pi}{q} , \tag{6.8}$$

which holds in all points in the transverse plane with the exception of the singularity points with nonzero vorticity $\mathbf{r}=\mathbf{r}_j$. Notice that the present case of fluid with vorticity confined to a discrete set of points is the simplest possible, since no information is required about the interaction between the fluid and the vortex cores, i.e., the regions where $w\neq\mathbf{0}$. Models where the vorticity field is distributed over finite regions instead of points involve further equations that take into account the dynamics of the degrees of freedom describing the vortex-core inner structure.

Now, recalling that the velocity field is coupled to the mass density scalar field σ via the equation

$$\frac{\partial\sigma}{\partial t}+\nabla\cdot(\sigma\mathbf{v})=0 , \tag{6.9}$$

we can compare Eqs. (6.4) and (6.5) with Eqs. (6.9) and (6.8) respectively. The optical equation (6.4) differs from (6.9) because of the nonvanishing right-hand side which is originated by the real part of $1-2CP$, that plays the role of a dissipative term in the Maxwell equation (5.1). This prevents the quantity

$$\int_0^{2\pi}d\phi\int_0^{\infty}d\rho\,\rho\sigma , \tag{6.10}$$

which represents the total mass, to be a conserved quantity for the optical fluid dynamics as in the hydrodynamical case. At steady state, by integrating both members of Eq. (6.4) all over the transverse plane, we obtain for (6.10) the expression

$$\int_0^{2\pi}d\varphi\int_0^{\infty}d\rho\,\rho\sigma=\int_0^{2\pi}d\varphi\int_0^{\infty}d\rho\,\rho\frac{2C\chi\sigma}{1+(\Delta')^2+\sigma} , \tag{6.11}$$

which generalizes the steady-state equation (2.6) obtained in the case of a frequency-degenerate family.

On the other hand, we see that Eq. (6.5) has the same form of the Bernoulli equation (6.8) provided one introduces the identifications

$$\mathcal{P}=\frac{a^2}{32\sigma^{1/2}}\nabla^2\sigma^{1/2}+\frac{a}{4}\left[\Delta'+\frac{a}{2}(1-\rho^2)-\frac{2C\chi\Delta'}{1+(\Delta')^2+\sigma}\right] \tag{6.12}$$

and

$$q=\frac{\pi}{2}a , \tag{6.13}$$

where $\mathcal{P}$ represents the pressure term for the optical fluid, which turns out to be σ dependent. Hence the velocity potential Φ no longer evolves independently of σ and Eq. (6.8) couples with Eq. (6.9) via the pressure $\mathcal{P}(\sigma,\nabla^2\sigma)$.

We hope that the analysis of this section will provide a general framework for the discussion of the relation between optical and hydrodynamical turbulence.

ACKNOWLEDGMENTS

This work was carried out in the framework of the ESPRIT Basic Research Action on "Transverse Optical Patterns."

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