Two-photon-loss model of intracavity second-harmonic generation

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The input-output theory, originally developed for a linear coupling between cavity mode a and a bath, is extended to a coupling that is quadratic in the cavity mode operator a. We then apply this to second-harmonic generation, in a cavity that is resonant at the fundamental frequency and has an active laser medium working at the fundamental frequency. We find intensity squeezing of 50% below the shot-noise level in the output light at the second-harmonic frequency, whether or not the cavity is resonant at the second-harmonic frequency. To compare the singly resonant cavity to the doubly resonant cavity, we develop in an appendix a general technique for extracting a single mode from a continuum.

I. INTRODUCTION

The input-output theory of Gardiner and Collett[1] has been applied to many problems in quantum optics where a cavity mode is coupled linearly to a continuum of output modes; for example, the degenerate parametric amplifier,[2,3] four-wave mixing[4] and two-photon transitions[5] inside a cavity, and squeezed-reservoir lasers.[6] In this article we extend the standard theory to a coupling which is quadratic in the cavity mode operator a. An immediate application of this lies in second-harmonic generation intracavity with an active laser medium, which has been the subject of much recent research.[7–11] The cavity is generally resonant at or close to the laser frequency, but experimentally it is difficult to make the cavity resonant at the second-harmonic frequency also. Thus the singly resonant cavity is the more commonly employed configuration, but the doubly resonant cavity has been concentrated on theoretically to enable straightforward calculation of the second-harmonic output. Intensity squeezing of 50% below the shot-noise level has been found[8] for the doubly resonant cavity; we wish to see how much can be achieved with the singly resonant cavity.

To compare the two systems we will redescribe the continuum at the second-harmonic frequency in a way which allows the two-photon loss model to also describe the doubly resonant case. This is the subject of the Appendix to this paper.

II. QUADRATIC COUPLING TO A BATH

Consider the Hamiltonian

$$H=H_{\text{system}}+H_{\text{bath},2}+H_{\text{int},2} \ , \tag{2.1}$$

where the 2 is to remind us that the system is quadratically coupled to the bath, and

$$H_{\text{bath},2}=\hbar\int_{-\infty}^{\infty} d\omega\, b^{\dagger}(\omega)b(\omega) \ , \tag{2.2}$$

$$H_{\text{int},2}=i\hbar\int_{-\infty}^{\infty} d\omega\, \kappa(\omega)[b^{\dagger}(\omega)a^2-b(\omega)a^{\dagger 2}] \ . \tag{2.3}$$

a and $b(\omega)$ are the boson operators for a single mode of the system and a continuum of bath modes, respectively. Thus a single mode a of the system interacts quadratically with a continuum of bath modes $b(\omega)$, the coupling constant being $\kappa(\omega)$. We make the approximation that $\kappa(\omega)$ is independent of frequency ω (a Markov approximation), and define μ, the two-photon-loss rate, by

$$\mu\equiv 2\pi\kappa^2 \ . \tag{2.4}$$

We define "input" and "output" fields $b_{\text{in}}(t)$ and $b_{\text{out}}(t)$ in the usual way,[1] and obtain Langevin equations for a with two-photon damping

$$\dot{a}=-\frac{i}{\hbar}[a,H_{\text{system}}]-\mu a^{\dagger}a^2+2\sqrt{\mu}a^{\dagger}b_{\text{in}} \tag{2.5}$$

$$=-\frac{i}{\hbar}[a,H_{\text{system}}]-\mu a^{\dagger}a^2-2\sqrt{\mu}a^{\dagger}b_{\text{out}} \ . \tag{2.6}$$

Following the methods of Ref. 1 we can investigate the statistics of the output field b_{out} and relate these to the statistics of the system mode a. We find that, for instance (given vacuum or coherent input field b_{in}),

$$\langle b_{\text{out}}(t),b_{\text{out}}(t')\rangle=\mu T\langle a^2(t),a^2(t')\rangle \ , \tag{2.7}$$

where T is a time-ordering operator, by which annihilation operators are ordered with time increasing from right to left, and creation operators with time increasing from left to right, and for any operators A and B

$$\langle A,B\rangle\equiv\langle AB\rangle-\langle A\rangle\langle B\rangle \ . \tag{2.8}$$

The statistics of the output field are related to the statistics of the system operator to which they are coupled, which in this case is a^2 instead of the more usual a. In Sec. III we will be interested in intensity fluctuations. Defining the output intensity operator

$$I_{\text{out}}(t)\equiv b_{\text{out}}^{\dagger}(t)b_{\text{out}}(t) \ , \tag{2.9}$$

we can show that for a vacuum input field

$$\langle I_{\text{out}}(t),I_{\text{out}}(t')\rangle=\mu^2T\langle :I^2(t),I^2(t'):\rangle \ , \tag{2.10}$$

where : : denotes normal ordering, and

$$I(t)\equiv a^{\dagger}(t)a(t) \ , \tag{2.11}$$

the internal photon number operator for mode a.

III. INTRACAVITY SECOND-HARMONIC GENERATION IN A SINGLY RESONANT CAVITY

We are interested here in second-harmonic generation intracavity with an active laser medium working at the "fundamental" frequency ω_1. The cavity is constructed to be resonant at frequency ω_1, but not at the second-harmonic frequency $2\omega_1$. Experimentally this is a far simpler system to realize than a cavity resonant at both ω_1 and $2\omega_1$. Upconverted photons are emitted directly into the bath field at the second-harmonic frequency. The Hamiltonian for the system is then of the form of Eq. (2.1) with H_{system} defined by

$$H_{\text{system}} = \hbar\omega a^\dagger a + H_{\text{laser}} + H_{\text{bath},1} + H_{\text{int},1} , \tag{3.1}$$

where H_{laser} represents the active laser medium driving mode a at the fundamental frequency, and $H_{\text{int},1}$ describes the linear coupling of mode a to the bath represented by $H_{\text{bath},1}$. $H_{\text{bath},2}$ and $H_{\text{int},2}$ are then defined by Eqs. (2.2) and (2.3), with the identification of $\kappa(\omega)$ as the coupling between cavity mode a at the fundamental frequency and the continuum of modes $b(\omega)$ around the second-harmonic frequency.

Walls, Collett, and Lane[8] have considered the statistics of mode a, using Louisell's[12] model of the laser, and a more conventional interpretation of the quadratic coupling as a two-photon absorber. Equation (2.10) tells us that we will be able to use these results to calculate the statistics of the output field at the higher frequency.

The phase of mode a diffuses freely, but decouples from the intensity. The steady-state internal intensity in mode a is

$$i^s = \frac{1}{2X}\{[(X+1)^2 + 4X(C-1)]^{1/2} - (X+1)\} , \tag{3.2}$$

where

$$i^s \equiv \frac{\langle I \rangle}{n_s} . \tag{3.3}$$

n_s and C are laser parameters; n_s is the saturation photon number, and C is the upper level pump rate (normalized so that $C=1$ corresponds to threshold, with the second-harmonic generating crystal not present). X is a scaled nonlinearity, defined by

$$X \equiv \frac{n_s \mu}{\gamma_1} , \tag{3.4}$$

where γ_1 is the linear amplitude loss rate at the fundamental frequency (and μ is the two-photon-loss rate, as above). The normally ordered spectrum of intensity fluctuations for the internal mode a, defined by

$$:S_I(\omega): \equiv \int_{-\infty}^{\infty} d\tau \, T\langle :I(t+\tau), I(t): \rangle e^{-i\omega\tau} , \tag{3.5}$$

was calculated using a standard linearization procedure to be

$$:S_I(\omega): = \frac{\langle I \rangle}{2\gamma_1} \frac{2(1+Xi^s)/(1+i^s) - Xi^s}{\{[i^s/(1+i^s)](1+X+2Xi^s)\}^2 + (\omega/2\gamma_1)^2} . \tag{3.6}$$

To obtain the spectrum of the output intensity around the fundamental frequency we need only multiply (3.6) by $4\gamma_1^2$. Only modest levels of squeezing (at most 12.5% below the shot-noise level) were found.

We now use the techniques outlined in Sec. II to calculate the spectrum of the output intensity at the second-harmonic frequency:

$$:S_{I_{\text{out}}}(\omega): \equiv \int_{-\infty}^{\infty} d\tau \, T\langle :I_{\text{out}}(t+\tau), I_{\text{out}}(t): \rangle e^{-i\omega\tau} , \tag{3.7}$$

where

$$I_{\text{out}} \equiv b_{\text{out}}^\dagger b_{\text{out}} . \tag{3.8}$$

We can use the relation (2.10) between correlation functions to write

$$:S_{I_{\text{out}}}(\omega): = \mu^2 :S_{I^2}(\omega): , \tag{3.9}$$

where

$$:S_{I^2}(\omega): \equiv \int_{-\infty}^{\infty} d\tau \, T\langle :I^2(t+\tau), I^2(t): \rangle e^{-i\omega\tau} \tag{3.10}$$

$$\simeq 4\langle I \rangle^2 :S_I(\omega): \tag{3.11}$$

with the last step being valid provided $I \gg 1$. In this limit $\langle I_{\text{out}} \rangle = \mu \langle I^2 \rangle \simeq \mu \langle I \rangle^2$, so

$$\frac{:S_{I_{\text{out}}}(\omega):}{\langle I_{\text{out}} \rangle} = 4\gamma_1 X i^s \frac{:S_I(\omega):}{\langle I \rangle} \tag{3.12}$$

$$= 2Xi^s \frac{2(1+Xi^s)/(1+i^s) - Xi^s}{\{[i^s/(1+i^s)](1+X+2Xi^s)\}^2 + (\omega/2\gamma_1)^2} , \tag{3.13}$$

which has its minimum value of $-\frac{1}{2}$ (i.e., squeezing of 50% below the shot-noise level) at zero frequency ($\omega=0$), when

$$i^s \gg 1 \tag{3.14}$$

and

$$Xi^s \gg 1 . \tag{3.15}$$

The first of these conditions guarantees that the laser is operating well above threshold, and hence producing photons with Poissonian statistics; the second condition implies that nearly all of these photons are upconverted.

IV. THE DOUBLY RESONANT CAVITY

Let us consider again the coupling between our single cavity mode a at frequency ω_1 and the continuum of modes $b(\omega)$ at frequency $2\omega_1$, in the Markov approximation:

$$H_{\rm int}=i\hbar\kappa\int_{-\infty}^{\infty}d\omega[b^\dagger(\omega)a^2-b(\omega)a^{\dagger 2}] \ . \tag{4.1}$$

The "bare" coupling constant κ is a property only of the crystal used and the elements of the cavity at the fundamental frequency.

If we now make the cavity resonant at frequency $2\omega_1$, with cavity loss rate γ_2, we are selecting a single mode B from the continuum of modes $b(\omega)$, where, in a rotating frame,

$$B=\left[\frac{\gamma_2}{\pi}\right]^{1/2}\int_{-\infty}^{\infty}d\omega\frac{b(\omega)}{\gamma_2-i\omega} \ . \tag{4.2}$$

We can then write [from (A21)]

$$b(\omega)=\left[\frac{\gamma_2}{\pi}\right]^{1/2}\frac{B}{\gamma_2+i\omega}+b'(\omega) \ . \tag{4.3}$$

The coupling to the other modes contained in $b'(\omega)$ is assumed to be very weak, so that we may ignore these modes; substituting (4.3) into (4.1) we obtain

$$H_{\rm int}=i\hbar B^\dagger a^2\kappa\int_{-\infty}^{\infty}d\omega\frac{\sqrt{\gamma_2/\pi}}{\gamma_2-i\omega}+\text{c.c.} \tag{4.4}$$

$$=\frac{i\hbar}{2}\chi B^\dagger a^2+\text{c.c.} \ , \tag{4.5}$$

the usual Hamiltonian for second-harmonic generation, where the second-harmonic coupling constant χ is

$$\chi=2\kappa\sqrt{\pi\gamma_2} \ . \tag{4.6}$$

V. COMPARISON BETWEEN THE OUTPUT INTENSITY SPECTRA

In this section we compare the intensity spectra of the output light at the second-harmonic frequency, for the singly and doubly resonant cavities. The spectrum for the singly resonant cavity is given in Eq. (3.13). Intracavity second-harmonic generation with a *doubly* resonant cavity has been analyzed by Walls, Collett, and Lane.[8] Above a critical internal second-harmonic intensity the phase difference between the two modes becomes bistable; below this point the zero-frequency ($\omega=0$) component of the spectrum of the output light at the second harmonic turns out to be *identical* to that in (3.13), with the replacement of X by X' where

$$X'\equiv\frac{\chi^2 n_s}{2\gamma_1\gamma_2} \ , \tag{5.1}$$

where γ_2 is the amplitude cavity loss rate at the second-harmonic frequency, and χ is the normal second-harmonic coupling constant, as defined in Eq. (4.5) above. On first inspection it seems as though X' depends on γ_2, whereas X does not. However, Sec. IV shows us that in fact

$$X'=X=\frac{n_s(2\pi\kappa^2)}{\gamma_1} \ ; \tag{5.2}$$

that is, the two scaled nonlinearities are in fact equal, and depend only on n_s, κ, and γ_1, which are all properties of the cavity at the fundamental frequency only.

Thus the zero-frequency component of the output spectrum obtained, for a given crystal and given properties of the cavity at the fundamental frequency, is the same whether the cavity is resonant at the second-harmonic frequency or not. At nonzero frequencies the spectrum for the doubly resonant cavity does differ from that for the singly resonant cavity. In the region in which the doubly resonant system exhibits bistability,[8] the spectra for the two systems are completely different. However, the intensity squeezing for that system is not improved in the bistable region.[13] Thus, if intensity squeezing of 50% below the shot-noise level is desired, our results show that the singly resonant cavity, which is easier to realize experimentally, is just as good as the doubly resonant cavity.

VI. CONCLUSION

In this paper we have applied the input-output theory of Gardiner and Collett[1] to a case where we have a quadratic coupling to a bath rather than the usual linear coupling. We have then used this technique to examine intracavity second-harmonic generation in a singly resonant cavity. We obtained intensity squeezing of 50% below the shot-noise level in the output intensity at the second-harmonic frequency. By selecting a single mode from the continuum at the second-harmonic frequency we were able to derive the usual Hamiltonian for second-harmonic generation in a *doubly* resonant cavity. We finally made use of this derivation to show the surprising result that, below the bistable region observed in the doubly resonant cavity,[8] the zero-frequency fluctuations in the light output at the second-harmonic frequency were in fact independent of the properties of the cavity at the second harmonic. The singly resonant cavity is the easier system to realize experimentally, and hence is to be preferred if the intensity squeezing is to be observed.

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APPENDIX: GENERAL TECHNIQUE FOR EXTRACTING AN ISOLATED MODE FROM A CONTINUUM

The problem we wish to address is this: given a system described by a continuum of mode operators $b(\omega)$, but which possesses an isolated mode B of particular interest, how can we redescribe the system in terms of a orthonormal set formed by this one mode B and a new continuum $\bar{b}(\omega)$? This is effectively the converse of the problem treated by Fano.[14]

Let the isolated mode be given by

$$B=\int_{-\infty}^{\infty}d\omega\,\beta(\omega)b(\omega) \ , \tag{A1}$$

where $[B,B^\dagger]=\int_{-\infty}^{\infty}d\omega|\beta(\omega)|^2=1$, and the new continuum by

$$\bar{b}(\omega)=b(\omega)+\int_{-\infty}^{\infty}d\omega' f(\omega,\omega')b(\omega') \ . \tag{A2}$$

Our problem is then: given $\beta(\omega)$, find $f(\omega,\omega')$. The orthonormality conditions that must be satisfied are

$$[\bar{b}(\omega),B^{\dagger}]=0 \ , \tag{A3}$$

$$[\bar{b}(\omega),\bar{b}^{\dagger}(\omega')]=\delta(\omega-\omega') \ . \tag{A4}$$

The first of these requires

$$\beta^*(\omega)+\int_{-\infty}^{\infty}d\omega' f(\omega,\omega')\beta^*(\omega')=0 \ , \tag{A5}$$

which is automatically satisfied by writing

$$f(\omega,\omega')=-\frac{\beta^*(\omega)\beta(\omega')g(\omega,\omega')}{\int_{-\infty}^{\infty}d\varpi\, g(\omega,\varpi)|\beta(\varpi)|^2} \tag{A6}$$

for some $g(\omega,\omega')$. The second requires

$$f(\omega,\omega')+f^*(\omega',\omega)+\int_{-\infty}^{\infty}d\varpi\, f(\omega,\varpi)f^*(\omega',\varpi)=0 \tag{A7}$$

and hence

$$g(\omega,\omega')g^*(\omega',\varpi)+g^*(\omega',\omega)g(\omega,\varpi)-g(\omega,\varpi)g^*(\omega',\varpi)=0 \ . \tag{A8}$$

We wish to find the most general form of $g(\omega,\omega')$ satisfying this condition. Applying Eq. (A8) for two arbitrary values ω_1 and ω_2 of ϖ, and combining to eliminate $g^*(\omega',\omega)$, gives

$$g(\omega,\omega')=\frac{g(\omega,\omega_1)g(\omega,\omega_2)[g^*(\omega',\omega_2)-g^*(\omega',\omega_1)]}{g^*(\omega',\omega_2)g(\omega,\omega_1)-g^*(\omega',\omega_1)g(\omega,\omega_2)} \tag{A9}$$

$$=\frac{c(\omega)}{h(\omega)-h^*(\omega')} \ , \tag{A10}$$

where

$$c(\omega)=\frac{2g(\omega,\omega_1)g(\omega,\omega_2)}{g(\omega,\omega_1)-g(\omega,\omega_2)} \ , \tag{A11}$$

$$h(\omega)=\frac{g(\omega,\omega_1)+g(\omega,\omega_2)}{g(\omega,\omega_1)-g(\omega,\omega_2)} \ . \tag{A12}$$

Choosing $g(\omega,\omega')$ of the form given by (A10) now satisfies Eq. (A8) in general, for arbitrary $c(\omega)$ and for any $h(\omega)$ satisfying

$$h(\omega)-h^*(\omega)=0 \ . \tag{A13}$$

Substituting back into Eq. (A6), $c(\omega)$ cancels and we are left with

$$f(\omega,\omega')=-\frac{\beta^*(\omega)\beta(\omega')}{n(\omega)[h(\omega)-h(\omega')]} \ , \tag{A14}$$

where

$$n(\omega)=\int_{-\infty}^{\infty}d\varpi\frac{|\beta(\varpi)|^2}{h(\omega)-h(\varpi)} \tag{A15}$$

and $h(\omega)$ is an arbitrary real function. In practice, the most sensible choice is likely to be $h(\omega)=\omega$ for a rotating-wave system (with mode operators for all frequencies, positive and negative) and $h(\omega)=\omega^2$ for a system with positive-frequency mode operators only. The integral may be taken in either direction around the pole, as long as the choice is adhered to consistently.

The inverse transform to Eqs. (A1) and (A2) may readily be confirmed to be

$$b(\omega)=\bar{b}(\omega)+\beta^*(\omega)B+\int_{-\infty}^{\infty}d\omega' f^*(\omega',\omega)\bar{b}(\omega') \ . \tag{A16}$$

Applying these results to the single damped cavity mode of Sec. IV, we have

$$\beta(\omega)=\left[\frac{\gamma_2}{\pi}\right]^{1/2}\frac{1}{\gamma_2-i\omega} \ . \tag{A17}$$

Choosing $h(\omega)=\omega$, we have

$$n(\omega)=\frac{\gamma_2}{\pi}\int_{-\infty}^{\infty}d\varpi\frac{1}{\gamma_2^2+\varpi^2}\frac{1}{\omega-\varpi\pm i\epsilon} \tag{A18}$$

$$=\frac{1}{\omega\mp i\gamma_2} \tag{A19}$$

(where the infinitesimal ϵ performs its usual function of selecting direction of integration around the pole). The upper sign is the sensible choice, giving

$$f(\omega,\omega')=i\frac{\gamma_2}{\pi(\gamma_2-i\omega')}\frac{1}{\omega-\omega'+i\epsilon} \tag{A20}$$

and hence

$$b(\omega)=\bar{b}(\omega)+\left[\frac{\gamma_2}{\pi}\right]^{1/2}\frac{B}{\gamma_2+i\omega}-\frac{i\gamma_2}{\pi(\gamma_2+i\omega)}\int_{-\infty}^{\infty}d\omega'\frac{\bar{b}(\omega')}{\omega'-\omega-i\epsilon} \ , \tag{A21}$$

which is the required result.

[1]C. W. Gardiner and M. J. Collett, Phys. Rev. A **31**, 3761 (1985).
[2]M. J. Collett and D. F. Walls, Phys. Rev. A **32**, 2887 (1985).
[3]P. D. Drummond and M. D. Reid, Phys. Rev. A **37**, 1806 (1988).
[4]M. D. Reid and D. F. Walls, Phys. Rev. A **34**, 4929 (1986).
[5]C. A. Blockley and D. F. Walls, Opt. Commun. **79**, 241 (1990).
[6]M. A. M. Marte, H. Ritsch, and D. F. Walls, Phys. Rev. A **38**, 3577 (1988).
[7]V. N. Gorbachev and and E. S. Polzik, Zh. Eksp. Teor. Fiz. **96**, 1984 (1989) [Sov. Phys.—JETP **69**, 1119 (1989)].
[8]D. F. Walls, M. J. Collett, and A. S. Lane, Phys. Rev. A **42**, 4366 (1990).
[9]P. Garcia-Fernandez, L. A. Lugiato, F. J. Bermejo, and P. Galatoa, Quantum Opt. **2**, 365 (1990).
[10]R. B. Levien, M. J. Collett, and D. F. Walls, Quantum Opt. **2**,

365 (1990).
[11]A. Sizmann, R. Schack, and A. Schenzle, Europhys. Lett. **13**, 109 (1990).
[12]W. H. Louisell, *Quantum Statistical Properties of Radiation* (Wiley, New York, 1973).
[13]R. B. Levien, M. Sc. thesis, University of Auckland, Auckland, 1990 (unpublished).
[14]U. Fano, Phys. Rev. **124**, 1866 (1961).