Statistics for the transient response of single-mode semiconductor laser gain switching

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For a gain-switched semiconductor laser we study the statistics of the time at which the laser intensity first reaches a threshold level and also the laser intensity fluctuations in the nonlinear regime. The dispersion in the maximum value of the laser intensity at the first peak of the relaxation oscillations is calculated. A simple relation between this maximum value and the passage time for each individual switch-on event is found.

I. INTRODUCTION

The statistics of laser switch-on processes have been recently considered a variety of systems and situations.[1–11] An historical perspective on the problems associated with transient fluctuations during the switch-on is given in Ref. 8. There are in general two different statistical problems to be considered. One is the statistics of the switch-on time at which laser emission is observed. This is described by the method of passage times.[1–9,11,12] This problem can generally be studied during a linear regime of laser amplification. The second problem refers to the large statistical fluctuations of the laser intensity during a later nonlinear regime. These problems might appear in a variety of forms for different types of lasers. In fact lasers have been classified from the point of view of dynamical systems in three categories (A, B, C) according to the number of effective degrees of freedom.[12] For type-A lasers (He-Ne, Ar, dye, etc.) results for passage-time (PT) statistics are well established.[1,2,4,5] The intensity fluctuations in the nonlinear regime were considered earlier[13,14] than the PT statistics. A description of these fluctuations is possible by taking a simple average over a distribution of random initial values of the laser field.[15,16] The consequence is that the time evolutions of the laser field for different switch-on events correspond to shifted versions in time of the same trajectory. The problem of PT statistics for type-B lasers (CO_2, semiconductor, etc.) has been already considered in Refs. 5–9 and 11, but the statistical properties of these systems in the nonlinear regime have not been considered in any detail until very recently.[17] The nonlinear regime of type-B lasers is characterized by the existence of large relaxation oscillations in the laser intensity whose peak value can be orders of magnitude larger than its stationary value. For type-B laser during this regime there are important differences in the statistical properties of the laser intensity in the cases of Q switching and gain switching. In the gain-switching process it is found that the laser intensity at the peaks of the relaxation oscillations takes random values in different switch-on events. In particular, the dispersion in the values of the intensity at the first and dominant maximum after switch-on is rather large. The control of this dispersion is of obvious importance in the practical applications of semiconductor lasers. This dispersion in the value of the first maximum in the laser intensity is not observed in Q-switching events.[8]

In this paper we consider the statistical properties of the transient response of single-mode semiconductor lasers. Semiconductor lasers are usually gain-switched, so we shall not consider here Q-switching processes. Our main purpose is to characterize the statistics of the laser field intensity at its maximum value occurring in the first relaxation oscillation and to relate them to the statistics of the passage times. The fact that this maximum value shows a rather large dispersion indicates, as also pointed out in Ref. 17, that the switch-on events are not shifted versions of the same time evolution. It will be shown that in gain-switching a simple relation between the maximum intensity and the passage time holds for each individual switch-on event. Moreover, if the laser is operated well above threshold, the variance of the PT distribution becomes rather small. This small statistical spread in the leading edge of laser transients allows us to establish a linear relationship between the maximum intensity and the PT for each individual switch-on event.

The paper is organized as follows. In Sec. II we introduce the stochastic rate equations on which our calculations are based and we describe the laser gain switching process. In Sec. III we revisit the problem of PT statistics as appropriate to our specific situation. The accuracy of the analytic expressions obtained for the PT statistics is checked by numerical simulations. In Sec. IV we describe the statistical fluctuations in the nonlinear regime of relaxation oscillations. In Sec. V we calculate the fluctuations of the maximum intensity and we find a simple relation between the dispersions of the passage time and the maximum intensity which depends on the operating point of the laser. The validity of this relation is also shown by numerical simulations. Our conclusions are summarized in Sec. VI.

II. STOCHASTIC RATE EQUATION DESCRIPTION OF LASER GAIN SWITCHING

Our description is based on the Langevin formulation of the rate equations for a single mode semiconductor laser. The equations for the slowly varying complex amplitude $E(s)$ of the optical field and the minority-carrier number N are[17,18]

$$d_s E = \tfrac{1}{2}[g(1+i\alpha)N - \gamma]E + (\beta N)^{1/2}\xi(s) , \tag{2.1}$$

$$d_s N = -\gamma_e N + C - gN|E|^2 , \tag{2.2}$$

where α is the linewidth enhancement factor; g gives the gain rate per carrier; γ, the inverse cavity lifetime; γ_e the carrier-recombination rate; $C \equiv \gamma_e N_0$, the injection current; β the spontaneous emission rate per carrier. The random spontaneous emission process is modeled by a noise term $\xi(s)$ taken as a complex Gaussian white noise of zero mean value and correlation

$$\langle \xi(s)\xi^*(s')\rangle = 2\delta(s-s') . \tag{2.3}$$

In these equations we assume constant values for the gain rate per carrier g and for the recombination rate γ_e.

In this paper we shall consider the statistical properties of the intensity $I=|E|^2$ and carrier number N, but we shall not address any questions concerning the statistical properties of the electric field phase. Since the statistics of I and N are independent of α, for simplicity we henceforth set $\alpha=0$.

Equations (2.1) and (2.2) predict that the laser threshold occurs for $N_0 = N_{\text{th}} \equiv (\gamma/g)$. At this point the stationary off-solution $I=|E|^2=0$, $N=N_0$ becomes unstable. For $N_0 > N_{\text{th}}$ the stable stationary state is $I_{\text{st}} = (N_0/N_{\text{st}}-1)\gamma_e/g$, $N_{\text{st}} = N_{\text{th}}$. The asymptotic approach to this stable state shows relaxation oscillations for $\gamma_e/\gamma < 4N_{\text{st}}(N_0-N_{\text{st}})/N_0^2$.

The stochastic rate equations are conveniently written in dimensionless form normalizing E and N to their steady-state values

$$z = \frac{E}{I_{\text{st}}^{1/2}}, \quad n = \frac{N}{N_{\text{st}}} . \tag{2.4}$$

Introducing a dimensionless time t

$$t = (\gamma\gamma_e)^{1/2} s \tag{2.5}$$

we find

$$\dot{z} = \frac{z}{2\epsilon}(n-1) + \sqrt{\eta n}\,\zeta(t) , \tag{2.6}$$

$$\dot{n} = -\epsilon[(1+\lambda|z|^2)n - \lambda - 1] , \tag{2.7}$$

where $\dot{z}$ and $\dot{n}$ indicate differentiation with respect to t, and

$$\begin{aligned} &\epsilon \equiv (\gamma_e/\gamma)^{1/2}, \quad \lambda \equiv N_0/N_{\text{th}} - 1 , \\ &\eta \equiv \delta/\lambda \equiv \beta/[\epsilon^2(\gamma\gamma_e)^{1/2}\lambda] . \end{aligned} \tag{2.8}$$

$\zeta(t)$ is a complex Gaussian white noise with correlation $\langle \zeta(t)\zeta^*(t')\rangle = 2\delta(t-t')$. The parameter λ gives the operating point of the laser while ϵ and δ are characteristic of the laser. For a typical semiconductor laser $\gamma_e^{-1} \approx 10^{-9}$ sec, $\gamma^{-1} \approx 10^{-12}$ sec, $\beta \approx 10^3$ sec^{-1}. Throughout this paper we take[6,9] $\epsilon = 0.036$, $\delta = 1.54 \times 10^{-4}$, and let λ vary over a wide range of values. Laser gain switching is modeled by the solution of (2.6) and (2.7) with initial conditions $n(t=0)=0$, $z(t=0)=0$, and $\lambda > 0$.

The superposition of 10 switching events for $i(t) \equiv |z(t)|^2$ and $n(t)$ is plotted in Fig. 1 as obtained from the numerical integration of (2.6) and (2.7) with 10 different sequences of random numbers associated with $\zeta(t)$. In each individual event there is a clear correspondence between the evolution of the minority carrier number and the laser light intensity. In each switch-on process, the carrier number n grows due to current injection, and attains the threshold value $n=1$ at the same time $\bar{t}$. There is a delay between $\bar{t}$ and the time at which laser intensity is amplified to observable values. For $t > \bar{t}$, n still increases up to a time T, different for each realization, at which stimulated emission rate is large enough to saturate the growth of n. Then, the laser light intensity is strongly amplified, reaching a maximum value $i_{\max}$ at times $T_{\max}$ at which n is reduced to its threshold value. For $t > T_{\max}$, n is below threshold, so the laser intensity decays to a value close to zero; meanwhile, n has passed through its first minimum, and is again at threshold when the laser light intensity reaches its minimum value. From there on, similar relaxation oscillations follow. If saturation effects in the gain parameter g or in the recombination rate γ_e were considered the oscillation damping would be much larger with only a few oscillations being observed.[18,19]

The difference between the various switching events arises from the stochastic character of the time at which recombination occurs. Essentially, the later the amplification to observable values occurs, the larger n

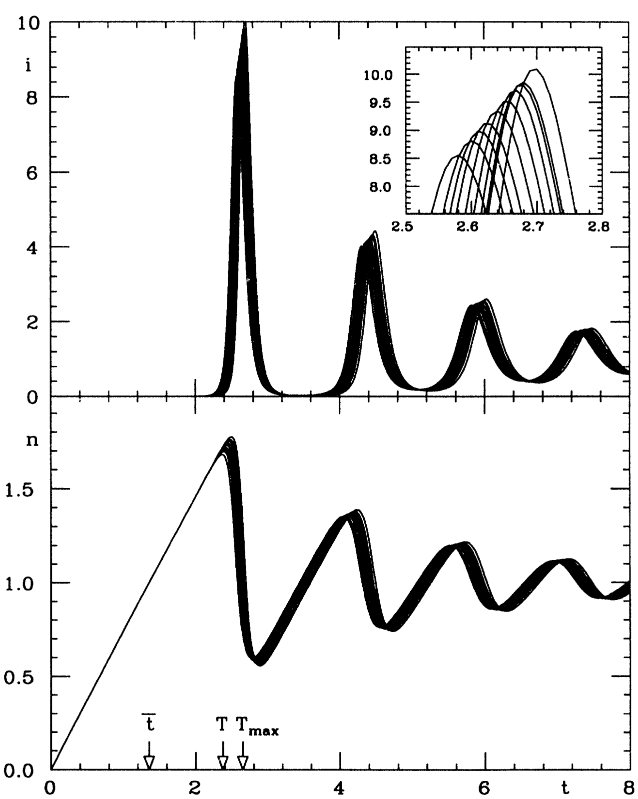

FIG. 1. Switching events as obtained from (2.6) and (2.7) for $\lambda=20$.

grows and as a consequence the larger is the peak in the laser light intensity. This fact points out that the crucial dynamics for the different switching events occurs for times t around the first maximum of n, $\bar{t} < T_- \leq t \leq T_+ < T_{\max}$. For times $t < T_- \leq T$, the laser light intensity is very small. Therefore the evolution of n is mainly deterministic and due to the current injection. In this regime, noise effects, which enter via the saturation term $(n|z|^2)$ are negligible. For times $T_{\max} > t > T_+ \geq T$ the laser electric field z and n are large, so that the evolution of z and n is again deterministic. In the regime $T_- \leq t \leq T_+$, both noise and nonlinearities are important, and small changes in the value of n give rise to large differences in $i_{\max}$.

The result of this complicated dynamics around T is that the different curves $i(t)$ are not shifted versions of the same deterministic curve and the initial spread in trajectories is not conserved along the trajectory: a spread of the order of 2–3 % in the switch-on time is converted into a dispersion of $i_{\max}$ of the order of 10%. In addition the spread in $T_{\max}$ is less than that of T. These effects do not appear in the first peak of the transient dynamics of an instantaneously Q-switched laser,[8] in this case the switch-on occurs at a random time but the value of n at this time is the same for all switching events. The consequence is that there is no dispersion in the value of $i_{\max}$. In this paper we primarily focus on the relation between the dispersion of $i_{\max}$, $\sigma_{i_{\max}}$, and the spread in the amplification time σ_T. In the insert of Fig. 1 the peak structure for different switching events is displayed in detail. The locus of the peak heights is seen to be well approximated by a straight line as a function of time.

The threshold time $\bar{t}$ can be obtained considering the linear regime for the minority carrier density in which saturation effects can be neglected in (2.7). In this regime

$$d_t n \approx -\epsilon(n-\lambda-1) \tag{2.9}$$

so that

$$n(t)=(1+\lambda)(1-e^{-\epsilon t}) \ . \tag{2.10}$$

The threshold time defined by $n(t)=1$ is given by

$$\bar{t}=\frac{1}{\epsilon}\ln\frac{1+\lambda}{\lambda} \tag{2.11}$$

which in the limit $\lambda \gg 1$ becomes

$$\bar{t}\approx\frac{1}{\epsilon\lambda} \ . \tag{2.12}$$

The time at which the amplified intensity becomes noticeable can be estimated as follows. Equation (2.6) is formally solved as

$$z(t)=h(t)e^{A(t)/2} \ , \tag{2.13}$$

where

$$A(t)=\frac{1}{\epsilon}\int_0^t dt'[n(t')-1] \tag{2.14}$$

and the stochastic process $h(t)$ is defined as

$$h(t)\equiv\int_0^t dt'\sqrt{\eta n(t')}\,\zeta(t')e^{-A(t')/2} \ . \tag{2.15}$$

Amplification becomes noticeable at a time t^* such that $A(t^*)=0$. For times of interest, $\epsilon t \ll 1$, it follows from (2.10) that $n(t)$ grows linearly in time:

$$n(t)\approx(1+\lambda)\epsilon t \ . \tag{2.16}$$

Replacing (2.16) in (2.14) we obtain $t^*=2/[(1+\lambda)\epsilon]$ which for $\lambda \gg 1$ gives from (2.12)

$$t^*\approx 2\bar{t} \ . \tag{2.17}$$

The difference between t^* and $\bar{t}$ gives a switch-on delay. It happens that $\bar{t}$ is essentially independent of spontaneous emission noise while it is clear from Fig. 1 that observable intensity is reached at random switch-on times T. The estimation above of t^* neglects noise effects which determine the time delay between $\bar{t}$ and the actual switch-on time. As a consequence t^* only gives an estimate for the switch-on time. Jitter in the switch-on process is reflected in the randomness of T. This jitter is also responsible for the dispersion of $i_{\max}$.

III. STATISTICS OF SWITCH-ON DELAY

In this section we address the calculation of the statistics of the switch-on time at which the signal becomes observable. This time is identified with the time at which i reaches a prescribed reference value i_r. Mathematically it corresponds to the passage time (PT) to reach i_r from $i=0$ under the effect of the random term $\zeta(t)$. The time T at which amplification occurs identifies the time at which saturation comes into play so that it is the upper limit of validity of the approximation (2.9). The statistics of such time can then be calculated from (2.10) and (2.13). The general idea is to solve (2.13) to obtain T as a random function of the reference value i_r. This is basically the same problem[5] of calculating the PT statistics of a type-A laser when the control parameter is swept through threshold at a finite rate. In such a situation one generally distinguishes between the cases of fast and slow sweeping. Fast sweeping corresponds to Q-switching dynamics in a type-B laser[8] while slow sweeping in a type-B laser corresponds to the case of gain-switching considered here. The slow sweeping condition corresponds here to a requirement that the final value of the amplification factor $(n-1)/2\epsilon$ in (2.6), as obtained from (2.10), be much larger than the sweeping rate $a=(1+\lambda)/2$ obtained from (2.6) and (2.16). This is guaranteed whenever $\epsilon^2\lambda^{-1} \ll 1$.

Equation (2.13) can be understood as the amplification of the time-dependent random initial condition $h(t)$. From (2.10) and (2.15), $h(t)$ is seen to be a Gaussian process whose time dependent variance $\langle |h(t)|^2\rangle$ converges to a finite value $\langle |h(\infty)|^2\rangle$. If such convergence occurs at times much smaller than PT's of interest, $h(t)$ in (2.13) can be replaced by a Gaussian random variable with the same variance as that of $h(\infty)$. Solving (2.13) for t, the statistics of t are then determined[5] as a function of this random variable which plays the role of an effective random initial condition for the laser field z at $t=0$. However we have noticed in connection with Fig. 1 that noise is only important for times $t > \bar{t}$. In fact the variance of a distribution of random initial conditions at $t=0$ is contracted by the dynamical evolution from $t=0$ to $t=\bar{t}$. A simpler and more physical approach in our situation is to consider a deterministic evolution from $t=\bar{t}$ with initial condition $n(t=\bar{t})=1$ and $z(t=\bar{t})=h(\infty)$. Solving (2.9) with this initial condition the amplification factor (2.14) becomes

$$A(t-\bar{t})=\frac{\lambda}{\epsilon}\left[t-\bar{t}-\frac{1-e^{-\epsilon(t-\bar{t})}}{\epsilon}\right] \tag{3.1}$$

and from (2.15)

$$\langle |h(t-\bar{t})|^2\rangle=2\eta\epsilon[1-e^{-A(t-\bar{t})}]+2\eta B(t-\bar{t}) , \tag{3.2}$$

where

$$B(t-\bar{t})=\int_{\bar{t}}^{t} ds\, e^{-A(s-\bar{t})} . \tag{3.3}$$

For times of interest such that $\epsilon(t-\bar{t})\ll 1$, $A(t-\bar{t})\approx(\lambda/2)(t-\bar{t})^2$, and

$$B(t-\bar{t})\approx(\pi/2\lambda)^{1/2}\mathrm{erf}[(\lambda/2)^{1/2}(t-\bar{t})] , \tag{3.4}$$

where erf(z) is the error function.[20] If in addition we consider times such that $(\lambda/2)^{1/2}(t-\bar{t})\gg 1$ the process $h(t-\bar{t})$ approaches a random variable $h(\infty)$ with variance

$$\langle |h(\infty)|^2\rangle=2\eta\epsilon+\eta(2\pi/\lambda)^{1/2} . \tag{3.5}$$

It follows from (2.13) that in this regime a reference value $i_r=|z_r|^2$ is reached at a time T given as a function of the random variable $h(\infty)$ by

$$(T-\bar{t})^2=\frac{2}{\lambda}\ln\frac{i_r}{|h(\infty)|^2} . \tag{3.6}$$

The simultaneous fulfillment of the conditions $\epsilon(T-\bar{t})\ll 1$ and $(\lambda/2)^{1/2}(T-\bar{t})\gg 1$ that guarantee the validity of (3.6) requires working with parameter values such that $\epsilon^2\lambda^{-1}\ll 1$, as it is in our case.

From Eq. (3.6) the statistics of the PT's T are obtained by calculating the generating function $W(\mu)$ as an average over the bivariate Gaussian distribution $P(h_1,h_2)$ of $h(\infty)=h_1+ih_2$:

$$\begin{aligned} W(\mu)&\equiv\langle e^{-\mu(T-\bar{t})}\rangle\\ &\equiv\int dh_1 dh_2 P(h_1,h_2)e^{-\mu(T-\bar{t})}\\ &\approx\Gamma\left[\frac{\mu}{\sqrt{2}}\lambda^{-1/2}\tau^{-1/2}+1\right]\exp(-\mu\sqrt{2}\,\lambda^{-1/2}\tau^{1/2}) , \end{aligned} \tag{3.7}$$

where

$$\tau\equiv\ln\frac{i_r}{\langle |h(\infty)|^2\rangle} \tag{3.8}$$

is a large quantity of order $\ln\lambda^{1/2}/\eta=\ln\lambda^{3/2}/\delta$ [see (2.8)] for $\epsilon^2\lambda\ll 1$. Equation (3.7) follows from an expansion in τ^{-1}. The mean PT, $\bar{T}$, and its variance σ_T are obtained from (3.7) as

$$\begin{aligned} \bar{T}-\bar{t}&\equiv\langle T\rangle-\bar{t}\\ &=\left[-\frac{d}{d\mu}\ln W(\mu)\right]_{\mu=0}\\ &=\sqrt{2}\,\lambda^{-1/2}\tau^{1/2}-\frac{1}{\sqrt{2}}\tau^{-1/2}\lambda^{-1/2}\Psi(1) \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} \sigma_T^2&\equiv\langle (T-\bar{t})^2\rangle-\langle T-\bar{t}\rangle^2\\ &=\left[\frac{d^2}{d\mu^2}\ln W(\mu)\right]_{\mu=0}\\ &=\tfrac{1}{2}\lambda^{-1}\tau^{-1}\Psi'(1) , \end{aligned} \tag{3.10}$$

where $\Psi(1)$ and $\Psi'(1)$ are the digamma function and its derivative, respectively.[20]

Equation (3.9) gives the mean delay in the observation of laser amplification. The consistency of (3.9) with the conditions $\epsilon(T-\bar{t})\ll 1$ and $(\lambda/2)^{1/2}(T-\bar{t})\gg 1$ is guaranteed if simultaneously we have that $\epsilon^2\lambda^{-1}\ll 1$ and $\lambda^{3/2}\delta^{-1}\gg 1$. In this domain of parameters the second

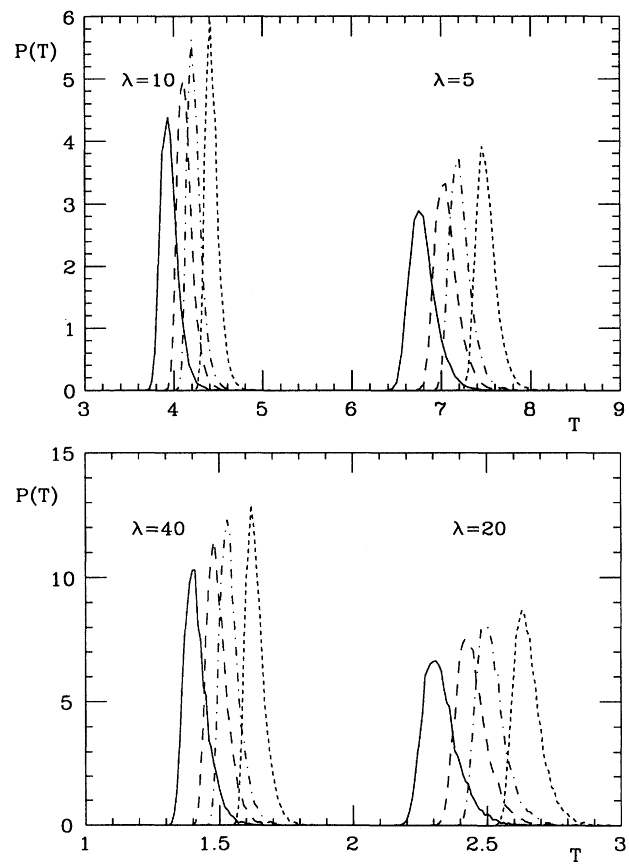

FIG. 2. Distribution of passage times for different values of λ (as indicated) and i_r(——, $i_r=0.05$, — —, $i_r=0.5$, and —·—·—·, $i_r=2$). The distribution of times $T_{\max}$ at which the maximum intensity is reached is indicated by small dashes.

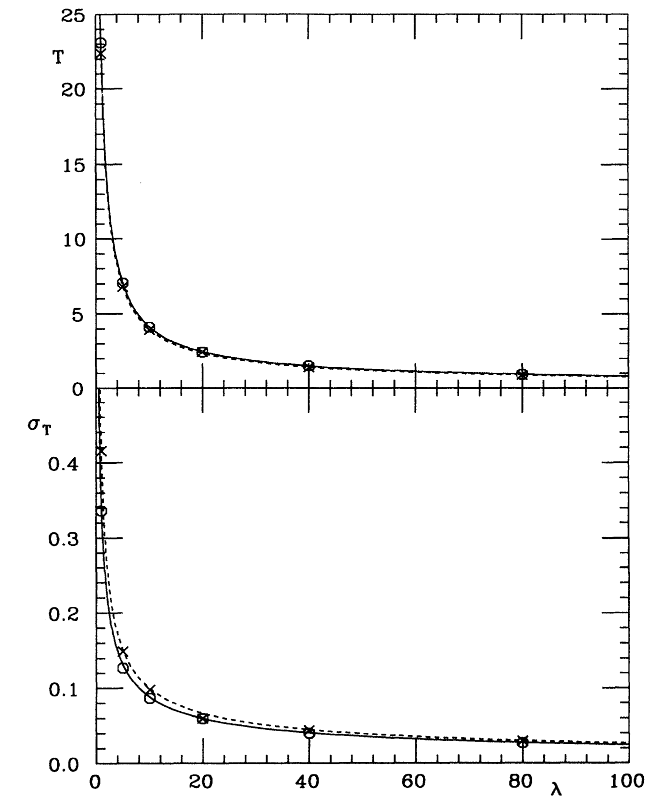

FIG. 3. Comparison of (3.9) and (3.10) with results from numerical simulations for $i_r=0.05$ (— — — theory, × simulations) and $i_r=0.5$ (—— theory, ○ simulations).

term on the right-hand side (3.9) is negligible. The variance σ_T is a small quality due to the factor τ^{-1} in (3.10) and becomes smaller with larger values of λ.

A precise test of the above results on the PT statistics has been performed by extensive numerical simulations of (2.6) and (2.7) with $z(0)=0$, $n(0)=0$ by averaging over 10^4 trajectories. The distribution of passage times for several values of λ and i_r are shown in Fig. 2. As can be seen in the figure, for a given value of λ the PT distribution becomes narrower as the reference value increases. This is due to the deterministic nonlinear evolution of z and n. Moreover, the mean PT for increasing λ is smaller and also the spread of the PT distribution decreases. Explicit results for the mean PT and its variance as obtained from these distributions are compared in Fig. 3 with our results (3.9) and (3.10) with $\bar{t}$ given by (2.11) and τ by (3.2) and (3.5). The agreement is remarkable for a broad range of values of λ of interest and different values of i_r.

IV. PHOTON AND CARRIER NUMBER FLUCTUATIONS DURING RELAXATION OSCILLATIONS

Lasers of type A are those in which the cavity decay rate is much smaller than the material linewidths[13] and as a consequence the dynamics is well characterized by a single closed equation for the slowly varying amplitude of the optical field. It is well known that the transient dynamics of those lasers is satisfactorily described by a deterministic evolution from a random initial condition so that noise along the dynamical path plays no significant role. The exception to this fact are dye lasers in which important pump noise cannot be reduced to a random initial condition.[1,2] The dynamics of semiconductor lasers (type B) cannot be reduced to a single equation since $\gamma \gg \gamma_e$ in (2.1) and (2.2). Additionally an analytic solution of the deterministic version of these two equations is not known. Still, the question arises if the stochastic dynamics defined by (2.6) and (2.7) with $n(t=0)=z(t=0)=0$ is well represented by a deterministic evolution from a random initial condition. We have seen in the previous section that this is so in the early regime. In this regime the PT statistics are well determined solving (2.6) and (2.7) deterministically with $n(t=\bar{t})=1$ and a random initial condition for z at $t=\bar{t}$: $z(t=\bar{t})=h(\infty)$. We now show evidence that this procedure remains valid in the nonlinear regime where relaxation oscillations occur[21] (Fig. 4). We remark that our conclusions are taken in a statistical sense of the ensemble average over different switch-on events. It is also important to note that although noise effects can be reduced to a random initial condition at the threshold time this does not imply that the stochastic trajectories are just shifted versions of the same deterministic curve as already explained in connection with Fig. 1. In Fig. 4 we show that there is good agreement between the time dependent ensemble averages obtained from a numerical simulation of (2.6) and (2.7) and those obtained from the numerical integration of the deterministic equations with the initial conditions just mentioned. This result reduces the problem posed by (2.6) and (2.7) to the problem of solving the coupled nonlinear deterministic equations with a random effective initial condition. Moreover, it can be seen that the agreement is better for large λ due to two factors: (i) substituting $h(\infty)$ for $h(t)$ is a better approximation for large λ [see (3.4)] and (ii) for small λ the minimum intensity after the first maximum becomes very small so that noise (which is neglected in this time interval) can be relevant for later evolution.

A noticeable fact in Fig. 4 is that the intensity fluctuations show large transient anomalous fluctuations associated with the decay of an unstable state as in lasers of type A. However these fluctuations follow the relaxation oscillations and have an interesting local minimum at the time that the mean intensity reaches a maximum in the oscillations. This minimum arises from the fact that the standard deviation of the PT distribution is appreciably smaller than the duration of the first peak.

V. STATISTICS OF THE FIRST LASER INTENSITY PEAK

In this section we characterize the statistical properties of the first peak of the laser intensity after gain switching. Results for the distributions of the times $T_{\max}$ at which the maximum occurs are shown in Fig. 2. The distribution of the intensity at the maximum $i_{\max}$ is shown in Fig.

5 as obtained from our numerical simulations. Due to the nonlinear coupling of Eqs. (2.6) and (2.7), $T_{\rm max}$ is a nonlinear function of T, as can be seen by comparing the different curves of $P(t)$ with $P(T_{\rm max})$ in Fig. 2.

Results for $\langle i_{\rm max}\rangle$ and its standard deviation $\sigma_{i_{\rm max}} \equiv (\langle i^2_{\rm max}\rangle - \langle i_{\rm max}\rangle^2)^{1/2}$ as obtained from the simulation are shown in Fig. 6. We observe a rather large dispersion in the values of $i_{\rm max}$. This dispersion results from the amplification of the spread of stochastic trajectories around the amplification time T. We now address the question of relating $\sigma_{i_{\rm max}}$ to σ_T.

Our discussions in Secs. III and IV makes it clear that a deterministic evolution from a random initial condition for z at $t=\bar{t}$ gives a good approximation to the stochastic dynamics defined by (2.6) and (2.7). We then look for a simple deterministic approximation for the dynamics in the vicinity of $T_{\rm max}$. Strictly at the maximum, $\dot{z}=0$, and n crosses threshold, $n=1$. For $n \approx 1$ saturation effects dominate so that (2.7) can be approximated by

$$\dot{n} \approx -\epsilon\lambda|z|^2 n \ . \tag{5.1}$$

Introducing a new variable u by

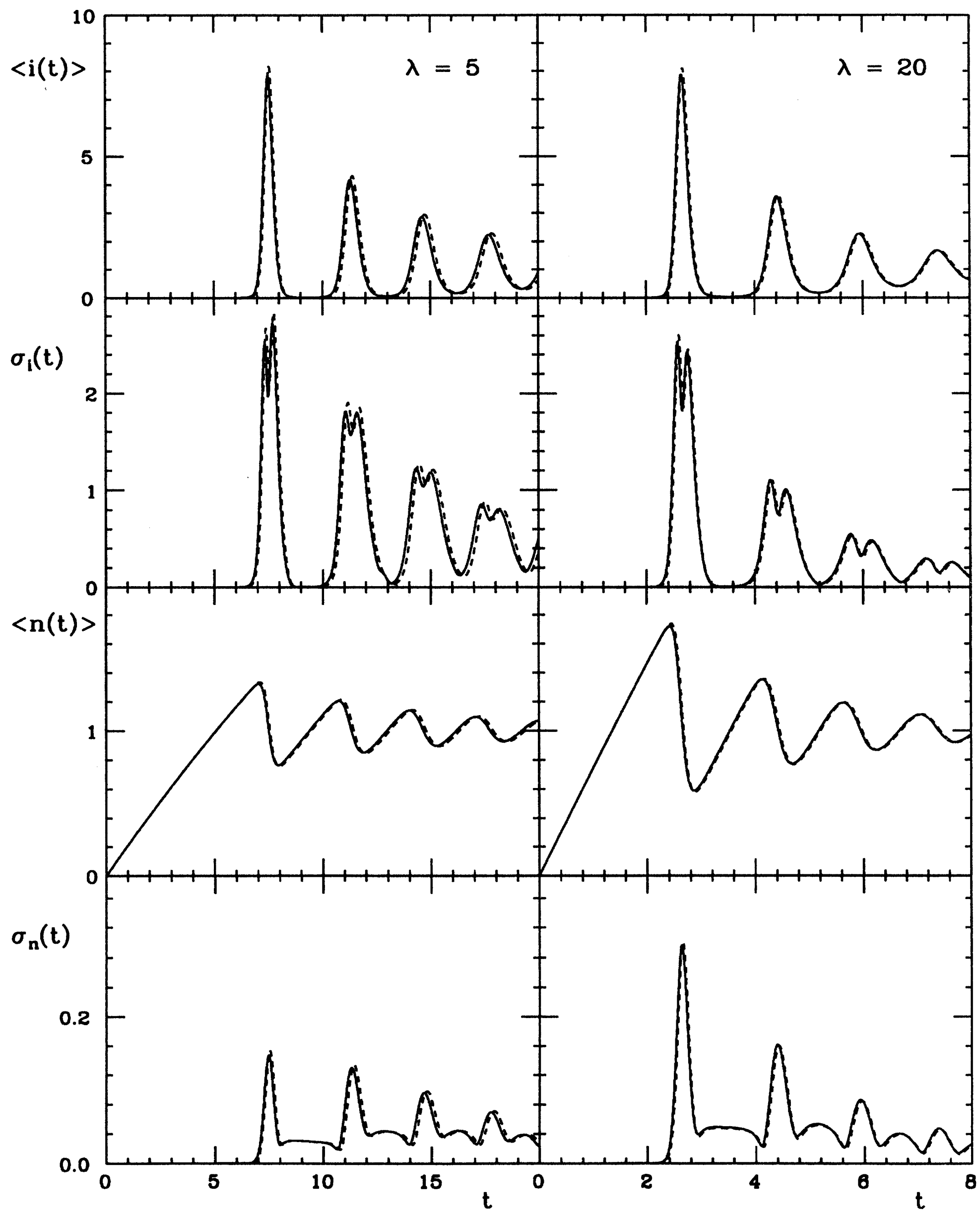

FIG. 4. Average values for i and n and associated standard deviations σ_i and σ_n for $\lambda=5$ and $\lambda=20$. The solid line corresponds to numerical integration by averaging over 10^4 trajectories. Dashed line is obtained from the deterministic solution averaged over random initial conditions as explained in the text.

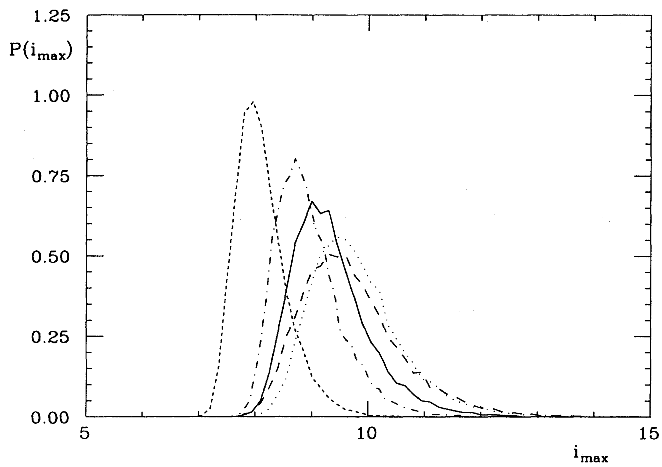

FIG. 5. Distribution of i_{max} for different values of λ:——, $\lambda=5$; $\cdots$, $\lambda=10$; ——, $\lambda=20$; —·—·—·, $\lambda=40$; — — —, $\lambda=80$, as obtained from simulations.

$$n=e^{-u} \tag{5.2}$$

it follows from (5.1) that

$$|z|^2=\frac{\dot u}{\epsilon\lambda} . \tag{5.3}$$

Replacing (5.3) in the deterministic version of (2.6) gives

$$\frac{\ddot u}{\dot u}=\frac{e^{-u}-1}{\epsilon} \tag{5.4}$$

which can be integrated once with a given initial condition $u(t_0)=u_0$, $\dot u(t_0)=\dot u_0$. Making use of (5.2) and (5.3) we obtain

$$|z|^2=|z_0|^2+\frac{n_0-n+\ln n-\ln n_0}{\epsilon^2\lambda} . \tag{5.5}$$

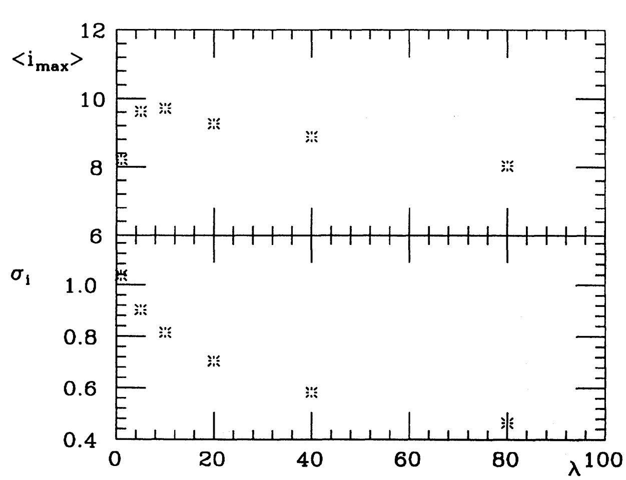

FIG. 6. Mean value of the maximum intensity and associated variance as obtained from the distributions in Fig. 5.

Since at $t=T_{max}$, $n=1$ we find

$$i_{max}=i_0+\frac{n_0-1-\ln n_0}{\epsilon^2\lambda} . \tag{5.6}$$

Equation (5.6) gives the peak intensity in terms of the value of i and n at a time $t_0\le T_{max}$. The validity of (5.1) requires that t_0 should be larger than the time T_+ introduced in Sec. II in the discussion of Fig. 1. Physically this means that i_0 and n_0 are values reached after saturation comes into play and amplification of laser light is already noticeable. The approximation is better the closer t_0 is to T_{max}. A lower bound of validity of (5.6) is $t_0=T$. On the other hand, the PT, T, also gives an upper limit of validity for the linear dynamics of n described by (2.10). Extrapolating for each stochastic trajectory (2.10) up to the PT and (5.6) down to the PT we obtain

$$i_{max}=i_0+\frac{(1+\lambda)(1-e^{-\epsilon T})-1-\ln[(1+\lambda)(1-e^{-\epsilon T})]}{\epsilon^2\lambda} \tag{5.7}$$

which relates the peak values of the intensity to the passage time T. From (3.6) we have T as a function of the random variable $h(\infty)$, and as a consequence (5.7) and (3.6) determine the statistics of i_{max} as a transformation of the bivariate Gaussian variable $h(\infty)$. This relation is rather cumbersome. However, simpler relations hold to a very good approximation. Indeed, we now show that (5.7) gives rise to an approximate linear dependence of i_{max} on the passage time with known slope. Our results (3.9) and (3.10) indicate that the dispersion of passage times around the mean value T is rather small so that (5.7) should be safely approximated by

$$\begin{aligned} i_{max}&=i_0+i_{max}(T=\bar T)+\left[\frac{di_{max}}{dT}\right]_{T=\bar T}(T-\bar T)\\ &=i^0_{max}+a(\lambda)T , \end{aligned} \tag{5.8}$$

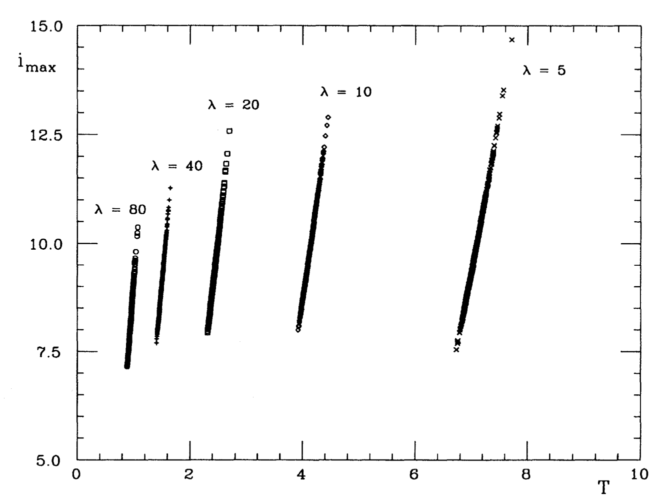

FIG. 7. i_{max} vs the passage time ($i_r=0.5$) for each trajectory and different values of λ as indicated.

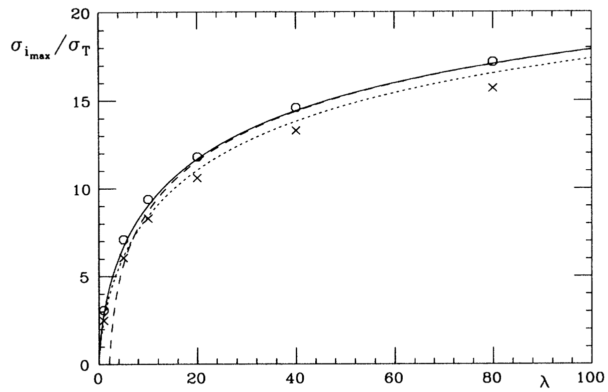

FIG. 8. Ratio $\sigma_{i_{max}}/\sigma_T$ vs λ as given by $a(\lambda)$ in (5.9) for $i_r=0.5$ (——) and $i_r=0.05$ ($\cdots$) and obtained from simulation: ○ ($i_r=0.5$) and $\times$($i_r=0.05$). The dashed line corresponds to $a(\lambda)$ given by the approximation (5.10) for $i_r=0.5$.

where the slope $a(\lambda)$ can be calculated from (5.7). We obtain[22]

$$a(\lambda)=\frac{1}{\epsilon\lambda}\left[(1+\lambda)e^{-\epsilon\overline{T}}-\frac{1}{e^{\epsilon\overline{T}}-1}\right]. \tag{5.9}$$

A direct test of the linear relation (5.8) for each stochastic trajectory is given in Fig. 7 where $i_{\max}$ is plotted vs the passage time T as obtained from our numerical simulations for several values of λ. The best fit to straight lines and the corresponding values obtained from (5.9) using the values of T obtained from (3.9) are compared in Table I. The agreement is very good and becomes better well above threshold. For large values of λ the variance of the PT distribution σ_T becomes very small [see (3.10) and Fig. 3] and the approximation (5.8) is then better justified.

A further important consequence of (5.8) is that $a(\lambda)$ gives the ratio of $\sigma_{i_{\max}}$ to σ_T so that the variance of the peak intensity $\sigma_{i_{\max}}$ can be obtained from the variance of the PT distribution given the knowledge of $a(\lambda)$. A comparison of $a(\lambda)$ with the ratio $\sigma_{i_{\max}}/\sigma_T$ as obtained from simulations is shown in Fig. 8, where the slope $a(\lambda)$ is calculated from (5.9) and (3.9). The agreement is very good for $i_r=0.5$. The matching process leading to (5.7) implies that the reference value i_r cannot be chosen too small as explicitly shown in the figure. Considering times of interest $\epsilon\overline{T}\ll 1$, (5.9) can be approximated as

$$a(\lambda)\approx\frac{1}{\epsilon\lambda}\left[-\frac{1}{\epsilon\overline{T}}+\tfrac{3}{2}+\lambda-(\tfrac{13}{12}+\lambda)\epsilon\overline{T}\right]. \tag{5.10}$$

This approximation is very accurate for large values of λ as is shown in Fig. 8.

TABLE I. Values of the slope $a(\lambda)$ for different values of λ obtained from a least-squares fit to numerical simulation data (see Fig. 7) and the corresponding theoretical values from (5.9) and (3.9).

λ	$a(\lambda)$ (best fit)	$a(\lambda)$ (theory)
5	6.99	6.68
10	9.29	9.04
20	11.80	11.68
40	14.61	14.43
80	17.22	17.09

VI. SUMMARY AND CONCLUSIONS

We have shown that a linear relationship between $i_{\max}$ and the passage time T holds for a broad range of values of λ. This permits us to obtain the statistics of $i_{\max}$ in terms of the statistics of the PT. Our results are based on a quasideterministic approximation for a type-B laser which properly describes the whole time evolution of the system. In the linear regime this approximation allows the calculation of the PT statistics. The validity of our results is supported by simulations of the stochastic process.

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