Quantum theory of a noninversion laser with injected atomic coherence

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A quantum theory of a Λ-type three-level single-mode quantum-beat laser with injected atomic coherences and with an external microwave field, which coherently drives the lower two nearly degenerate levels, is studied. A master equation for the field-density operator and the photon statistics is derived. The laser operation is analyzed in terms of the coefficients of the Fokker-Planck equation of the laser field. Noninversion lasing is found in both cases: without and with injected atomic coherences. At the same time, in the latter case large quantum-noise reduction is found, and for particular initial parameter choices the laser field is very near to a coherent state with exactly Poissonian photon-number distribution and near-Poissonian phase distribution.

I. INTRODUCTION

The semiclassical and quantum theories of the laser were developed more than 20 years ago and subsequently reached a very high level of sophistication.[1-3] The underlying physics is now believed to be well understood. Some of the key concepts of the theoretical description are population inversion and laser threshold. In the models considered in all of the above theories it is vital to establish population inversion since the gain is proportional to the population difference between the upper and lower levels of the lasing transition. The interpretation is obvious: emission is proportional to the upper-level population, whereas absorption is proportional to the lower-level population. The net effect of these two elementary processes is the gain. On the other hand, the cavity losses determine the laser threshold: obviously, in order to establish steady-state oscillation, the gain has to balance the loss.

Recent studies, however, on the possibility of amplification in a noninverted medium with very closely spaced upper (lower) levels and a single lower (upper) level pointed to the crucial role of the atomic coherence in the lasing process, especially when compared to conventional laser theory which involves incoherent pumping. Kocharovskaya and Khanin[4] predicted amplification of ultrashort pulses in an active medium consisting of three-level atoms with two nearly degenerate lower levels. Arkhipkin and Heller[5] showed the possibility of amplification in the case in which a single upper level is submerged in a continuum of field-induced autoionizing states (which act as a second degenerate upper level). In closely related works, Harris considered cw amplification with nearly degenerate upper autoionizing levels decaying to the same continuum,[6(a)] and also investigated the effect of transient response on the dynamics.[6(b)] A more quantitative study was presented by Lyras *et al.*[7] Other related works have recently appeared concerning different systems, transient effects, and novel methods to produce nonabsorbing resonances,[8(a)] as well as a novel interpretation based on dressed atomic states.[8(b)] Scully, Zhu, and Gavrielides[9] pointed to the crucial role played by atomic coherence between the degenerate levels. The source of noninversion lasing is that in the case of upper-level degeneracy the emissions from the two levels add coherently (constructive interference), whereas with degenerate lower levels the absorption amplitudes from the two levels subtract coherently (destructive interference). That is, nonreciprocity between absorption and emission takes place in such a way that emission dominates over absorption. In this context, it should be noted that a similar effect due to recoil splitting of Doppler-broadened emission and absorption spectra was suggested[10] some time ago. Scully, Zhu, and Gavrieldes[9] suggested microwave coupling of the nearly degenerate levels in order to introduce the necessary coherence into the system. In closely related works noninversion laser with atomic coherences has recently been studied[11] even in the collision dominated regime.[12] In fact, such coherences have long since been investigated in Raman-type processes,[13-15] and experimentally demonstrated[16] in connection with studies on optical pumping. It should also be mentioned at this point that the noninversion lasing effect may have far reaching practical consequences. It is increasingly difficult to establish inversion on transitions in the high-frequency part of the spectrum (e.g., vuv, x ray, etc.). A similar difficulty is encountered in two-photon lasers. Instead of trying to pump the system harder to reach inversion it may be more practical to create appropriate coherence between nearly degenerate levels.

We have already mentioned that the concept of inversion plays a central role in incoherently pumped lasers. It is, however, not the inversion that drives the laser but the atomic dipole moment, associated with the lasing transition, which is the source of radiation. In incoherently pumped lasers the laser field itself induces a dipole moment and it turns out to be proportional to the population difference between the levels. In coherently pumped devices, however, the active atoms are prepared in a coherent superposition of the lasing levels and have a finite dipole moment even without the field (injected coherence versus the induced coherence of incoherently pumped lasers). The gain expression of such devices is very different from that of incoherently pumped lasers. Indeed, in closely related studies on correlated-emission laser schemes (coherently pumped lasers or lasers with injected coherence)[17–19] aimed at the reduction of quantum noise it was found, as a by-product, that noninversion lasing was possible, e.g., in the two-photon correlated-emission laser[18] and even in a single-photon laser with injected atomic coherence.[19] In these cases, however, it is not the coherence between closely spaced upper or lower levels that leads to noninversion gain but rather the atomic coherence between upper and lower levels of the lasing transition. It should be noted that the main feature of these correlated-emission laser schemes is the significant amount of quantum-noise quenching and even squeezing under appropriate conditions.

Motivated by the above arguments that point to the crucial role of atomic coherence in achieving noninversion lasing and quantum-noise quenching, in this paper we suggest a new type of single-mode laser where the active medium consists of three-level atoms in Λ configuration with atomic coherence between all levels. In the special case when there is coherence only between the two nearly degenerate lower levels, our system reduces to previously suggested noninversion laser schemes[9] and the present theory proves that noninversion lasing persists in an all order treatment. Previous studies were limited to linear treatment only. In the other special case when there is coherence between the upper and one lower lasing level this system reproduces the noise-quenching features of the single-photon laser with injected coherence in the highly nonlinear regime.[19] The introduction of split lower levels into this latter system, with coherence between them, gives the flexibility of simultaneously optimizing the gain that arises from the coherences between the upper and lower lasing level (long coherences) and suppressing the loss due to absorption by the lower levels via the coherence between split lower levels (short coherence). This optimization requires a fine balance between the coherences but the payoffs are twofold. Firstly, it turns out that under optimum conditions, the gain of the present system is so high that it immediately enters the nonlinear regime or, in other words, the effective threshold for laser operation is zero and the system does not even have a linear regime. Secondly, the conditions for maximum gain coincide with those of minimum noise and the generated field is very near to an ideal pure coherent state. The effect, in fact, is a striking manifestation for an active system of the Fano-type interferences.[20]

The paper is organized as follows. In Sec. II we present the Hamiltonian model of the Λ-type system, in Sec. III the solution of the corresponding Schrödinger equation is provided. In Sec. IV we derive the master equation of the previously described model. In Sec. V by converting the master equation into a Fokker-Planck equation we obtain the diffusion and drift coefficients for the photon number and phase, and study the steady-state operation.[21] We show here that noninversion lasing is accompanied with reduced photon number and phase noise due to the coherent superposition of atomic states and the laser field is very near to an ideal coherent state at the threshold. Section VI is concerned with the discussion of the results.

II. MODEL

In this section we derive the interaction Hamiltonian of the coupled atom-field system after two transformations in a second interaction picture. Introducing particular assumptions concerning the cavity modes and the detunings between the frequencies of the modes and transitions, the obtained interaction matrix will be used in Sec. III for the investigation of the time evolution of the physical problem under consideration.

We consider a system of Λ-type three-level atoms as shown in Fig. 1, having one upper level $|a\rangle$ with energy $\hbar\omega_a$, and two lower ones $|b\rangle$ and $|c\rangle$ with energies $\hbar\omega_b$ and $\hbar\omega_c$, respectively. The $|a\rangle \Longrightarrow |b\rangle$, and $|a\rangle \Longrightarrow |c\rangle$ transitions are assumed to be dipole allowed. The two lower levels $|b\rangle$ and $|c\rangle$, are strongly coupled by a (classical) external microwave field, characterized by a Rabi frequency $\mathcal{V}$ and phase ϕ. Also, the upper level $|a\rangle$ and lower levels $|b\rangle$ and $|c\rangle$ are assumed to be in a coherent superposition due to injected coherence, so that the ρ_{ab} and ρ_{ac} elements of the atomic density matrix are different from zero.

The Hamiltonian for the field and one active atom is given in the Schrödinger picture as

$$H = H_0 + V \,, \tag{1}$$

where

$$H_0 = \sum_{i=a,b,c} \hbar\omega_i |i\rangle\langle i| + \hbar\Omega_1(a_1^\dagger a_1 + \tfrac{1}{2}) + \hbar\Omega_2(a_2^\dagger a_2 + \tfrac{1}{2}) \tag{2}$$

and

$$V = \hbar g_1 a_1 |a\rangle\langle b| + \hbar g_2 a_2 |a\rangle\langle c| - \tfrac{1}{2}\hbar\mathcal{V} e^{i(\Omega_3 t + \phi)} |b\rangle\langle c| + \text{H.c.} \tag{3}$$

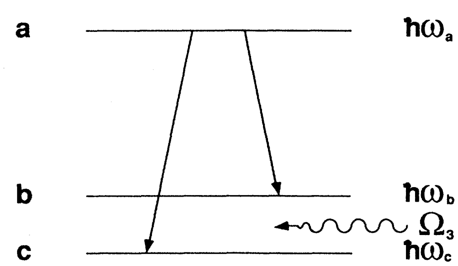

FIG. 1. Scheme of a Λ-type three-level quantum-beat laser having one upper level a and two closely spaced lower levels b and c, with energies $\hbar\omega_a$, $\hbar\omega_b$, and $\hbar\omega_c$, respectively. The allowed atomic transitions and the external microwave field coupling the lower two levels are denoted by arrows.

Here Ω_1, Ω_2 are the cavity-mode frequencies, a_1 ($a_1^\dagger$), a_2 ($a_2^\dagger$) are the annihilation (creation) operators in these modes, respectively, g_1, g_2 are the coupling constants for the transitions $|a\rangle \Longrightarrow |b\rangle$, and $|a\rangle \Longrightarrow |c\rangle$. Ω_3 and ϕ are the frequency and phase of the external microwave field, driving the $|b\rangle \Longrightarrow |c\rangle$ transition, which is treated semiclassically. It is convenient to work in the interaction picture, defined as

$$V_I = \exp\left[\frac{i}{\hbar} H_0 t\right] V \exp\left[-\frac{i}{\hbar} H_0 t\right] . \tag{4}$$

It can be shown that

$$V_{\mathrm{I}} = V_1 + V_2 , \tag{5}$$

where

$$V_1 = -\tfrac{1}{2}\hbar\mathcal{V} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \exp(i\Delta_3 t - i\phi) \\ 0 & \exp(-i\Delta_3 t + i\phi) & 0 \end{bmatrix} \tag{6}$$

and

$$V_2 = \hbar \begin{bmatrix} 0 & g_1 a_1 \exp(i\Delta_1 t) & g_2 a_2 \exp(i\Delta_2 t) \\ g_1 a_1^\dagger \exp(-i\Delta_1 t) & 0 & 0 \\ g_2 a_2^\dagger \exp(-i\Delta_2 t) & 0 & 0 \end{bmatrix} . \tag{7}$$

Here

$$\Delta_1 = \omega_{ab} - \Omega_1 = \omega_a - \omega_b - \Omega_1 ,$$
$$\Delta_2 = \omega_{ac} - \Omega_2 = \omega_a - \omega_c - \Omega_2 ,$$
$$\Delta_3 = \omega_{bc} - \Omega_3 = \omega_b - \omega_c - \Omega_3$$

are the detunings. We assume that the driving field is resonant,

$$\Delta_3 = 0, \quad \text{and futhermore,} \quad \Delta_1 = -\Delta_2 = \tfrac{1}{2}\mathcal{V} , \tag{8}$$

and

$$a_1 = a_2 =: a, \quad g_1 = g_2 =: g . \tag{9}$$

Using a second interaction picture defined as

$$V_{\mathrm{II}} = \exp\left[\frac{i}{\hbar} V_1 t\right] V_2 \exp\left[-\frac{i}{\hbar} V_1 t\right] , \tag{10}$$

and applying a rotating-wave approximation, where we neglect the rapidly varying terms, $\exp[i(\Delta_1 + \frac{1}{2}\mathcal{V})t]$ and $\exp[i(\Delta_2 - \frac{1}{2}\mathcal{V})t]$, and retain the slowly varying ones, $\exp[i(\Delta_1 - \frac{1}{2}\mathcal{V})t]$ and $\exp[i(\Delta_2 + \frac{1}{2}\mathcal{V})t]$, the interaction matrix has the following form:

$$V_{\mathrm{II}} = \hbar g \begin{bmatrix} 0 & -i \exp\left[i\frac{\phi}{2}\right] \sin\frac{\phi}{2} a & \exp\left[-i\frac{\phi}{2}\right] \cos\frac{\phi}{2} a \\ i \exp\left[-i\frac{\phi}{2}\right] \sin\frac{\phi}{2} a^\dagger & 0 & 0 \\ \exp\left[i\frac{\phi}{2}\right] \cos\frac{\phi}{2} a^\dagger & 0 & 0 \end{bmatrix} . \tag{11}$$

III. SOLUTION OF THE MODEL

Based on the obtained Hamiltonian (11) we proceed further with the investigation of the time evolution of the Λ system coupled to a single cavity mode under the particular set of conditions (8) and (9). From the wave functions obtained from the time-dependent Schrödinger equation we will derive a master equation for the field-density matrix and an equation for the steady-state photon distribution in Sec. IV.

The Schrödinger equation in the second interaction picture (10) can be written as

$$i\hbar\dot{\psi} = V_{\mathrm{II}}\psi , \tag{12}$$

where ψ is a column vector with components ψ_a, ψ_b, ψ_c. Equation (12) written in components reads

$$i\dot{\psi}_a=-ig\exp\left[i\frac{\phi}{2}\right]\sin\frac{\phi}{2}a\psi_b+g\exp\left[-i\frac{\phi}{2}\right]\cos\frac{\phi}{2}a\psi_c-i\frac{\gamma}{2}\psi_a ,$$
$$i\dot{\psi}_b=ig\exp\left[-i\frac{\phi}{2}\right]\sin\frac{\phi}{2}a^\dagger\psi_a-i\frac{\gamma}{2}\psi_b , \quad (13)$$
$$i\dot{\psi}_c=g\exp\left[i\frac{\phi}{2}\right]\cos\frac{\phi}{2}a^\dagger\psi_a-i\frac{\gamma}{2}\psi_c .$$

Here γ is a decay constant for the levels a,b,c (for simplicity, the same for all levels). With the substitutions

$$\psi_a=\exp\left[-\frac{\gamma}{2}(t-t_0)\right]\psi'_a ,$$
$$\psi_b=\exp\left[-i\frac{\phi}{2}\right]\exp\left[-\frac{\gamma}{2}(t-t_0)\right]\psi'_b , \quad (14)$$
$$\psi_c=\exp\left[i\frac{\phi}{2}\right]\exp\left[-\frac{\gamma}{2}(t-t_0)\right]\psi'_c ,$$

the components of the new wave function ψ' satisfy the following equations:

$$i\dot{\psi}'_a=-ig\sin\frac{\phi}{2}a\psi'_b+g\cos\frac{\phi}{2}a\psi'_c ,$$
$$i\dot{\psi}'_b=ig\sin\frac{\phi}{2}a^\dagger\psi'_a , \quad (15)$$
$$i\dot{\psi}'_c=g\cos\frac{\phi}{2}a^\dagger\psi'_a .$$

Introducing

$$\psi'_1=\cos\frac{\phi}{2}\psi'_b-i\sin\frac{\phi}{2}\psi'_c ,$$
$$\psi'_2=\cos\frac{\phi}{2}\psi'_c-i\sin\frac{\phi}{2}\psi'_b , \quad (16)$$

Eq. (15) can be written as

$$i\dot{\psi}'_1=0 ,$$
$$i\dot{\psi}'_2=ga^\dagger\psi'_a , \quad (17)$$
$$i\dot{\psi}'_a=ga\psi'_2 .$$

Since the relation between ψ and ψ' is written in Eq. (14) from the solution of Eq. (17) for ψ', the solution of Eqs. (12) and (13) is the following:

$$\psi_a(t)=\exp\left[-\frac{\gamma}{2}(t-t_0)\right]\left[\cos[g(aa^\dagger)^{1/2}(t-t_0)]\psi_a(t_0)-\exp\left[i\frac{\phi}{2}\right]\sin\frac{\phi}{2}\sin[g(aa^\dagger)^{1/2}(t-t_0)](aa^\dagger)^{-1/2}a\psi_b(t_0)\right.$$
$$\left.-i\exp\left[-i\frac{\phi}{2}\right]\cos\frac{\phi}{2}\sin[g(aa^\dagger)^{1/2}(t-t_0)](aa^\dagger)^{-1/2}a\psi_c(t_0)\right] ,$$
$$\psi_b(t)=\exp\left[-\frac{\gamma}{2}(t-t_0)\right]\left[\exp\left[-i\frac{\phi}{2}\right]\sin\frac{\phi}{2}a^\dagger(aa^\dagger)^{-1/2}\sin[g(aa^\dagger)^{1/2}(t-t_0)]\psi_a(t_0)\right.$$
$$+\left[\sin^2\frac{\phi}{2}\cos[g(a^\dagger a)^{1/2}(t-t_0)]+\cos^2\frac{\phi}{2}\right]\psi_b(t_0)$$
$$\left.+i\exp(-i\phi)\sin\frac{\phi}{2}\cos\frac{\phi}{2}\{\cos[g(a^\dagger a)^{1/2}(t-t_0)]-1\}\psi_c(t_0)\right] , \quad (18)$$
$$\psi_c(t)=\exp\left[-\frac{\gamma}{2}(t-t_0)\right]\left[-i\exp\left[i\frac{\phi}{2}\right]\cos\frac{\phi}{2}a^\dagger(aa^\dagger)^{-1/2}\sin[g(aa^\dagger)^{1/2}(t-t_0)]\psi_a(t_0)\right.$$
$$-i\exp(i\phi)\sin\frac{\phi}{2}\cos\frac{\phi}{2}\{\cos[g(a^\dagger a)^{1/2}(t-t_0)]-1\}\psi_b(t_0)$$
$$\left.+\left[\cos^2\frac{\phi}{2}\cos[g(a^\dagger a)^{1/2}(t-t_0)]+\sin^2\frac{\phi}{2}\right]\psi_c(t_0)\right] .$$

We shall use these solutions in Sec. IV to obtain a master equation for the field-density operator and the steady-state photon distribution.

IV. MASTER EQUATION AND PHOTON STATISTICS

The density matrix of the three-level Λ atom and the one-mode field described by the Hamiltonian (1)–(3) satisfies the following equation of motion in the second interaction picture determined by Eq. (10):

$$\dot{\rho}=-\frac{i}{\hbar}[V_{\rm II},\rho] . \quad (19)$$

Although, formally, this equation is identical to the one neglecting atomic decay, the spirit in which we employ it is very different, as will be clear from the considerations below. When we take the trace of this equation over the atomic variables (in order to obtain an equation for the reduced density operator of the field only [see Eq. (20) below], then in the right-hand side in the matrix elements appearing in Eq. (22) we effectively replace the atomic variables by their steady-state values (adiabatic elimination of atomic variables), which is justified by the much faster relaxation rate of the atomic variables than that of the field ($\gamma \gg \gamma_c$). In doing so, we make use of Eqs. (13). That is, in the resulting master equation for the density operator of the field only, Eq. (28), the atomic relaxation process is fully accounted for.

We introduce the reduced density operator ρ_F for the field only as

$$\rho_F = \mathrm{Tr}_{\mathrm{atom}}\rho \; , \tag{20}$$

where $\mathrm{Tr}_{\mathrm{atom}}$ stands for tracing of ρ over the atom. Consequently, the equation of motion of the field-density operator ρ_F can be obtained from Eq. (19) in the form

$$\dot{\rho}_F = \mathrm{Tr}_{\mathrm{atom}}\dot{\rho} = -\frac{i}{h}\mathrm{Tr}_{\mathrm{atom}}[V_{\mathrm{II}},\rho] \; . \tag{21}$$

Using the expression (11) for V_{II}

$$\dot{\rho}_F = -g\sin\frac{\phi}{2}\left[\exp\left[i\frac{\phi}{2}\right][a,\rho_{ba}] - \exp\left[-i\frac{\phi}{2}\right][a^\dagger,\rho_{ab}]\right] - ig\cos\frac{\phi}{2}\left[\exp\left[-i\frac{\phi}{2}\right][a,\rho_{ca}] - \exp\left[i\frac{\phi}{2}\right][a^\dagger,\rho_{ac}]\right] + \mathcal{L}_c\rho_F \; , \tag{22}$$

where $\mathcal{L}_c\rho_F$ describes the effect of field loss due to cavity damping. Its explicit form is given in Eq. (27).

To obtain an expression for ρ_{ab}, ρ_{ac}, and their Hermitian conjugates we first calculate the contribution of one atom injected at time t_0 with arbitrary initial condition into the cavity and then sum the contribution of all atoms injected at random times between $t-1/\gamma$ and t (i.e., $t-1/\gamma < t_0 < t$) at rate r. This means that the atom-field interaction is considered on a time scale shorter than the atomic lifetime $1/\gamma$. In this way,

$$\rho_{ab} = r\int_{t-1/\gamma}^{t} dt_0 \psi_a(t,t_0)\psi_b^\dagger(t,t_0) \tag{23}$$

and

$$\rho_{ac} = r\int_{t-1/\gamma}^{t} dt_0 \psi_a(t,t_0)\psi_c^\dagger(t,t_0) \; . \tag{24}$$

Substituting the expressions for ψ_a, ψ_b, ψ_c from Eq. (19) into Eqs. (23) and (24),

$$\begin{aligned}
\rho_{ab} = r\int_{t-1/\gamma}^{t} & dt_0 \exp[-\gamma(t-t_0)] \\
\times \Bigg[& \exp\left[i\frac{\phi}{2}\right]\sin\frac{\phi}{2}\cos[g(aa^\dagger)^{1/2}(t-t_0)]\rho_{aa}(t_0)\rho_F(t_0)\sin[g(aa^\dagger)^{1/2}(t-t_0)](aa^\dagger)^{-1/2}a \\
& -\exp\left[i\frac{\phi}{2}\right]\sin\frac{\phi}{2}\sin[g(aa^\dagger)^{1/2}(t-t_0)](aa^\dagger)^{-1/2}a \\
& \quad \times \rho_{bb}(t_0)\rho_F(t_0)\left[\sin^2\frac{\phi}{2}\cos[g(a^\dagger a)^{1/2}(t-t_0)] + \cos^2\frac{\phi}{2}\right] \\
& -\exp\left[i\frac{\phi}{2}\right]\sin\frac{\phi}{2}\cos^2\frac{\phi}{2}\sin[g(aa^\dagger)^{1/2}(t-t_0)](aa^\dagger)^{-1/2}a\;\rho_{cc}(t_0)\rho_F(t_0)\{\cos[g(a^\dagger a)^{1/2}(t-t_0)]-1\} \\
& +\cos[g(aa^\dagger)^{1/2}(t-t_0)]\rho_{ab}(t_0)\rho_F(t_0)\left[\sin^2\frac{\phi}{2}\cos[g(a^\dagger a)^{1/2}(t-t_0)] + \cos^2\frac{\phi}{2}\right] \\
& -\exp(i\phi)\sin^2\frac{\phi}{2}\sin[g(aa^\dagger)^{1/2}(t-t_0)](aa^\dagger)^{-1/2}a \\
& \quad \times \rho_{ba}(t_0)\rho_F(t_0)\sin[g(aa^\dagger)^{1/2}(t-t_0)](aa^\dagger)^{-1/2}a - i\exp(i\phi)\sin\frac{\phi}{2}\cos\frac{\phi}{2}\cos[g(aa^\dagger)^{1/2}(t-t_0)] \\
& \qquad \times \rho_{ac}(t_0)\rho_F(t_0)\{\cos[g(a^\dagger a)^{1/2}(t-t_0)]-1\} \\
& -i\sin\frac{\phi}{2}\cos\frac{\phi}{2}\sin[g(aa^\dagger)^{1/2}(t-t_0)](aa^\dagger)^{-1/2}a\rho_{ca}(t_0)\rho_F(t_0)\sin[g(aa^\dagger)^{1/2}(t-t_0)](aa^\dagger)^{-1/2}a \\
& +i\exp\left[3i\frac{\phi}{2}\right]\sin^2\frac{\phi}{2}\cos\frac{\phi}{2}\sin[g(aa^\dagger)^{1/2}(t-t_0)](aa^\dagger)^{-1/2}a\;\rho_{bc}(t_0)\rho_F(t)\{\cos[g(a^\dagger a)^{1/2}(t-t_0)]-1\} \\
& -i\exp\left[-i\frac{\phi}{2}\right]\cos\frac{\phi}{2}\sin[g(aa^\dagger)^{1/2}(t-t_0)](aa^\dagger)^{-1/2}a \\
& \quad \times \rho_{cb}(t_0)\rho_F(t_0)\left[\sin^2\frac{\phi}{2}\cos[g(a^\dagger a)^{1/2}(t-t_0)] + \cos^2\frac{\phi}{2}\right]\Bigg] \; .
\end{aligned} \tag{25}$$

In the same way

$$\rho_{ac}=r\int_{t-1/\gamma}^{t}dt_0\exp[-\gamma(t-t_0)]$$
$$\times\Bigg[i\exp\left[-i\frac{\phi}{2}\right]\cos\frac{\phi}{2}\cos[g(aa^\dagger)^{1/2}(t-t_0)]\rho_{aa}(t_0)\rho_F(t_0)\sin[g(aa^\dagger)^{1/2}(t-t_0)](aa^\dagger)^{-1/2}a$$
$$-i\exp\left[-i\frac{\phi}{2}\right]\sin^2\frac{\phi}{2}\cos\frac{\phi}{2}\sin[g(aa^\dagger)^{1/2}(t-t_0)](aa^\dagger)^{-1/2}a\rho_{bb}(t_0)\rho_F(t_0)\{\cos[g(a^\dagger a)^{1/2}(t-t_0)]-1\}$$
$$-i\cos\frac{\phi}{2}\exp\left[-i\frac{\phi}{2}\right]\sin[g(aa^\dagger)^{1/2}(t-t_0)](aa^\dagger)^{-1/2}a$$
$$\times\rho_{cc}(t_0)\rho_F(t_0)\left[\cos^2\frac{\phi}{2}\cos[g(a^\dagger a)^{1/2}(t-t_0)]+\sin^2\frac{\phi}{2}\right]$$
$$+i\exp(-i\phi)\sin\frac{\phi}{2}\cos\frac{\phi}{2}\cos[g(aa^\dagger)^{1/2}(t-t_0)]\rho_{ab}(t_0)\rho_F(t_0)\{\cos[g(a^\dagger a)^{1/2}(t-t_0)]-1\}$$
$$-i\sin\frac{\phi}{2}\cos\frac{\phi}{2}\sin[g(aa^\dagger)^{1/2}(t-t_0)](aa^\dagger)^{-1/2}a\rho_{ba}(t_0)\rho_F(t_0)\sin[g(aa^\dagger)^{1/2}(t-t_0)](aa^\dagger)^{-1/2}a$$
$$+\cos[g(aa^\dagger)^{1/2}(t-t_0)]\rho_{ac}(t_0)\rho_F(t_0)\left[\cos^2\frac{\phi}{2}\cos[g(a^\dagger a)^{1/2}(t-t_0)]+\sin^2\frac{\phi}{2}\right]$$
$$+\exp(-i\phi)\cos^2\frac{\phi}{2}\sin[g(aa^\dagger)^{1/2}(t-t_0)](aa^\dagger)^{-1/2}a\rho_{ca}(t_0)\rho_F(t_0)\sin[g(aa^\dagger)^{1/2}(t-t_0)](aa^\dagger)^{-1/2}a$$
$$-\exp\left[i\frac{\phi}{2}\right]\sin\frac{\phi}{2}[g(aa^\dagger)^{1/2}(t-t_0)](aa^\dagger)^{-1/2}a\rho_{bc}(t_0)\rho_F(t_0)\left[\cos^2\frac{\phi}{2}\cos[g(a^\dagger a)^{1/2}(t-t_0)]+\sin^2\frac{\phi}{2}\right]$$
$$+\exp\left[-3i\frac{\phi}{2}\right]\sin\frac{\phi}{2}\cos^2\frac{\phi}{2}\sin[g(aa^\dagger)^{1/2}(t-t_0)](aa^\dagger)^{-1/2}a$$
$$\times\rho_{cb}(t_0)\rho_F(t_0)\{\cos[g(a^\dagger a)^{1/2}(t-t_0)]-1\}\Bigg]\ . \tag{26}$$

Since the dynamics of the field is governed by the cavity lifetime $1/\gamma_c$, which is much longer then the atomic lifetime $1/\gamma$, ρ_F does not change appreciably during the integration time interval, and thus $\rho_F(t_0)$ in Eqs. (25) and (26) can be approximated by $\rho_F(t)$.

When $t-t_0>1/\gamma$ (i.e., $t-1/\gamma>t_0$) the contribution to the integral is negligible due to the exponential damping factor. This means that the lower limit of the integration can be extended to $-\infty$.

After performing these steps we can substitute ρ_{ab}, ρ_{ac}, and their Hermitian conjugates into the equation of motion (22). We still need the loss term for Eq. (22), which can be specified in the usual manner,[1,17,19]

$$\mathcal{L}_c\rho_F=-\frac{\gamma_c}{2}(a^\dagger a\rho_F+\rho_F a^\dagger a-2a\rho_F a^\dagger)\ . \tag{27}$$

Now taking the n, n' matrix element of the equation of motion of ρ_F it is easy to carry out the time integration, and we obtain the following master equation for the matrix elements of the field-density operator:

$$(\dot\rho_F)_{n,n'}=\Bigg[-\frac{\bar\alpha}{2}\rho_{n,n'}\rho_{aa}\left[n+1+n'+1+\frac{\beta}{4\bar\alpha}(n-n')^2\right]+\bar\alpha\rho_{n+1,n'+1}M\sqrt{(n+1)(n'+1)}$$
$$-\eta\rho_{n,n'+1}R\sqrt{n'+1}\left[1-\frac{\beta}{4\bar\alpha}(n-n')\right]-\eta\rho_{n+1,n'}R^\dagger\sqrt{n+1}\left[1+\frac{\beta}{4\bar\alpha}(n-n')\right]\Bigg]N_{n,n'}^{-1}$$
$$+\Bigg[\bar\alpha\rho_{n-1,n'-1}\rho_{aa}\sqrt{nn'}-\frac{\bar\alpha}{2}\rho_{n,n'}M\left[n+n'+\frac{\beta}{4\bar\alpha}(n-n')^2\right]$$
$$+\eta\rho_{n-1,n'}R\sqrt{n}\left[1+\frac{\beta}{4\bar\alpha}(n-n')\right]+\eta\rho_{n,n'-1}R^\dagger\sqrt{n'}\left[1-\frac{\beta}{4\bar\alpha}(n-n')\right]\Bigg]N_{n-1,n'-1}^{-1}$$
$$-\frac{\gamma_c}{2}[\rho_{n,n'}(n+n')-\rho_{n+1,n'+1}2\sqrt{(n+1)(n'+1)}]\ , \tag{28}$$
$$\bar\alpha=2r\frac{g^2}{\gamma^2},\quad \beta=8r\frac{g^4}{\gamma^4}\ ,\quad \eta=r\frac{g}{\gamma}\ ,$$

$\bar{\alpha}$ and β are the linear-gain coefficient and saturation parameter, respectively,

$$M=\tfrac{1}{2}[(1-\cos\phi)\rho_{bb}+(1-\cos\phi)\rho_{cc}-i\sin\phi(\rho_{bc}e^{i\phi}-\rho_{cb}e^{-i\phi})] , \tag{29}$$

$$R=\rho_{ab}\sin\frac{\phi}{2}\exp\left[-i\frac{\phi}{2}\right]-i\rho_{ac}\cos\frac{\phi}{2}\exp\left[i\frac{\phi}{2}\right] , \tag{30}$$

$$N_{n,n'}=1+\frac{\beta}{2\bar{\alpha}}(n+1+n'+1)+\left[\frac{\beta}{4\bar{\alpha}}\right]^2(n-n')^2 . \tag{31}$$

It can be seen that Eq. (28) reduces to the well-known field master equation[19] of a one-mode two-level laser, if $R=0$ (i.e., no atomic coherence between the upper and the lower levels), and the "effective population" of the lower two levels M is equal to ρ_{bb}.

Photon statistics

The equation of motion of the diagonal elements of the field-density matrix is obtained from Eq. (28) by setting $n=n'=:n$. We have an equation for the steady-state photon distribution if the time derivative $\rho_{n,n}$ is equal to zero,

$$\begin{aligned}\dot{\rho}_{n,n}=&-[\bar{\alpha}(\rho_{n,n}\rho_{aa}-\rho_{n+1,n+1}M)(n+1)+\eta(\rho_{n,n+1}R+\rho_{n+1,n}R^{\dagger})\sqrt{n+1}]N_{n,n}^{-1}\\&+[\bar{\alpha}(\rho_{n-1,n-1}\rho_{aa}-\rho_{n,n}M)n+\eta(\rho_{n-1,n}R+\rho_{n,n-1}R^{\dagger})\sqrt{n}]N_{n-1,n-1}^{-1}-\gamma_c[\rho_{n,n}n-\rho_{n+1,n+1}(n+1)]=0 .\end{aligned} \tag{32}$$

The differences between Eq. (32) and the equation of motion of $\rho_{n,n}$ of a usual two-level laser are in the variable M, and the terms connected with the off-diagonal elements of the field-density matrix and the initial atomic coherences again.

V. FOKKER-PLANCK EQUATION

In this section we employ the Glauber-Sudarshan P representation for the field-density matrix and transform its equation of motion (22) into a Fokker-Planck equation. Calculating the steady-state drift and diffusion coefficients of the Fokker-Planck equation the characteristics of the lasing system considered in this paper can be studied. Substituting the Glauber representation form of the field-density matrix

$$\rho_F(t)=\int d^2\alpha\, P(\alpha,\alpha^*,t)|\alpha\rangle\langle\alpha| \tag{33}$$

into Eqs. (25) and (26) for ρ_{ab} and ρ_{ac} calculating the equation of motion (22) (assuming that the mean photon number is large and 1 can be neglected compared to $\alpha\alpha^*$), we obtain the following equation of motion for $P(\alpha,\alpha^*,t)$:

$$\begin{aligned}\frac{\partial P}{\partial t}=&\left[-\frac{\bar{\alpha}}{2}(\rho_{aa}-M)\left[\frac{\partial}{\partial\alpha}\alpha+\frac{\partial}{\partial\alpha^*}\alpha^*\right]+\bar{\alpha}\rho_{aa}\frac{\partial^2}{\partial\alpha\partial\alpha^*}-\frac{\beta}{8}(\rho_{aa}+M)\left[\frac{\partial}{\partial\alpha}\alpha-\frac{\partial}{\partial\alpha^*}\alpha^*\right]^2\right]N^{-1}P\\&-\left[\eta R\left\{\frac{\partial}{\partial\alpha}+\frac{\beta}{4\bar{\alpha}}\left[2\left[\frac{\partial}{\partial\alpha}\alpha-\frac{\partial}{\partial\alpha^*}\alpha^*\right]\alpha^*-\frac{\partial}{\partial\alpha}\left[\frac{\partial}{\partial\alpha}\alpha-\frac{\partial}{\partial\alpha^*}\alpha^*\right]\right]\right\}N^{-1}P+\text{c.c.}\right]+\frac{\gamma_c}{2}\left[\frac{\partial}{\partial\alpha}\alpha+\frac{\partial}{\partial\alpha^*}\alpha^*\right]P ,\end{aligned} \tag{34}$$

where

$$P=P(\alpha,\alpha^*,t)$$

and

$$N=1+\frac{\beta}{2\bar{\alpha}}\left[2\alpha\alpha^*-\frac{\partial}{\partial\alpha}\alpha-\frac{\partial}{\partial\alpha^*}\alpha^*\right]+\left[\frac{\beta}{4\bar{\alpha}}\right]^2\left[\frac{\partial}{\partial\alpha}-\frac{\partial}{\partial\alpha^*}\alpha^*\right]^2 . \tag{35}$$

Introducing I and θ, the intensity and the phase of the field instead of α, where $\alpha=\sqrt{I}e^{i\theta}$, and expanding the equation of motion of (34) $P(I,\theta,t)$ up to second order in the derivatives, we arrive at a Fokker-Planck equation expressed in terms of I and θ,

$$\frac{\partial P}{\partial t}=\left[-\frac{\partial}{\partial I}d_{\rm I}-\frac{\partial}{\partial\theta}d_\theta+\frac{\partial^2}{\partial I^2}D_{\rm II}+\frac{\partial^2}{\partial\theta^2}D_{\theta\theta}+2\frac{\partial^2}{\partial I\partial\theta}D_{{\rm I}\theta}\right]P , \tag{36}$$

where

$$P=P(I,\theta,t) .$$

d_θ and $d_{\rm I}$, $D_{\theta\theta}$ and $D_{\rm II}$, and $D_{{\rm I}\theta}$ are the phase and intensity drift, the phase and intensity diffusion, and the crossdiffusion coefficients, respectively.

The coefficients of the Fokker-Planck equation written in terms of intensity I and phase θ are the following (from now we use simply α and γ instead of $\bar{\alpha}$ and γ_c):

$$d_\theta=\frac{\alpha}{2i}Z\left[\frac{\beta}{\alpha}I\right]^{-1/2}, \quad (37)$$

$$d_{\rm I}=\frac{\alpha I}{1+\frac{\beta}{\alpha}I}\left[\rho_{aa}-M+S\left[\frac{\beta}{\alpha}I\right]^{-1/2}-\frac{\gamma}{\alpha}\left[1+\frac{\beta}{\alpha}I\right]\right], \quad (38)$$

$$D_{\theta\theta}=\frac{\beta}{4\left[1+\frac{\beta}{\alpha}I\right]\frac{\beta}{\alpha}I}\left[\rho_{aa}+\tfrac{1}{2}(\rho_{aa}+M)\frac{\beta}{\alpha}I-\tfrac{1}{2}S\left[\frac{\beta}{\alpha}I\right]^{1/2}\right], \quad (39)$$

$$D_{\rm II}=\frac{\alpha I}{\left[1+\frac{\beta}{\alpha}I\right]^2}\left[\rho_{aa}+M\frac{\beta}{\alpha}I-S\left[\frac{\beta}{\alpha}I\right]^{1/2}\right], \quad (40)$$

$$D_{{\rm I}\theta}=\frac{i\alpha}{8\left[1+\frac{\beta}{\alpha}I\right]}Z\left[\frac{\beta}{\alpha}I\right]^{1/2}, \quad (41)$$

where

$$\rho_{ij}=|\rho_{ij}|\exp(i\varphi_{ij})=\rho_{ji}^* ,$$

that is,

$$|\rho_{ij}|=|\rho_{ji}|$$

and

$$\varphi_{ab}+\varphi_{bc}=\varphi_{ac} ,$$

$$M=\tfrac{1}{2}[(1-\cos\phi)\rho_{bb}+(1+\cos\phi)\rho_{cc}+2|\rho_{bc}|\sin\phi\sin(\phi+\varphi_{bc})] ,$$

$$S=2\,{\rm Re}(R e^{-i\theta})=|\rho_{ab}|[-(1-\cos\phi)\sin(\theta-\varphi_{ab})+\sin\phi\cos(\theta-\varphi_{ab})]+|\rho_{ac}|[-(1+\cos\phi)\sin(\theta-\varphi_{ac})+\sin\phi\cos(\theta-\varphi_{ac})] ,$$

$$Z=2i\,{\rm Im}(R e^{-i\theta})=-i\{|\rho_{ab}|[(1-\cos\phi)\cos(\theta-\varphi_{ab})+\sin\phi\sin(\theta-\varphi_{ab})]+|\rho_{ac}|[(1+\cos\phi)\cos(\theta-\varphi_{ac})+\sin\phi\sin(\theta-\varphi_{ac})]\} .$$

Note that all the Fokker-Planck coefficients are phase (θ)-sensitive via S or Z.

It can be seen from Eq. (36),

$$\frac{d}{dt}\langle I\rangle=\langle d_{\rm I}\rangle , \quad (42)$$

$$\frac{d}{dt}\langle \theta\rangle=\langle d_\theta\rangle , \quad (43)$$

which means, that in steady state $d_{\rm I}=0$ and $d_\theta=0$. We obtain the steady-state intensity $\langle I\rangle$ from the solution of the equation $d_{\rm I}=0$, and the steady-state phase $\langle\theta\rangle$ from $d_\theta=0$; that is, there is phase-locking in the system. The actual values of the diffusion coefficients in steady state are determined by the steady-state intensity $\langle I\rangle$ and phase $\langle\theta\rangle$.

In the P representation the photon number variance is[21]

$$\langle(\Delta\hat n)^2\rangle=\langle:(\Delta\hat n)^2:\rangle+\langle\hat n\rangle=\langle(\delta I)^2\rangle+\langle I\rangle , \quad (44)$$

and the phase variance is

$$\langle(\Delta\theta)^2\rangle=\langle:(\Delta\theta)^2:\rangle+\frac{1}{4\langle\hat n\rangle}=\langle(\delta\theta)^2\rangle+\frac{1}{4\langle I\rangle} , \quad (45)$$

where $\hat n=a^\dagger a$ and $\delta I=I-\langle I\rangle$, $\delta\theta=\theta-\langle\theta\rangle$. From Eq. (36) we find the equation of motion for the normally ordered photon-number variance and phase variance[21]

$$\frac{d}{dt}\langle(\delta I)^2\rangle=2\langle d_{\rm I}\delta I\rangle+2\langle D_{\rm II}\rangle , \quad (46)$$

$$\frac{d}{dt}\langle(\delta\theta)^2\rangle=2\langle d_\theta\delta\theta\rangle+2\langle D_{\theta\theta}\rangle . \quad (47)$$

Expanding $d_{\rm I}$ and $D_{\rm II}$ in terms of δI around the steady state $\langle I\rangle=n_0$, we find that the total steady-state photon-number variance is given by

$$\langle(\Delta\hat n)^2\rangle=n_0+\left[\frac{D_{\rm II}}{|\partial d_{\rm I}/\partial I|}\right]_{{\rm I}=\langle {\rm I}\rangle,\theta=\langle\theta\rangle} . \quad (48)$$

In the same way, after expanding d_θ and $D_{\theta\theta}$ in terms of $\delta\theta$ around $\langle\theta\rangle=\theta_0$ we find the phase variance to be

$$\langle(\Delta\theta)^2\rangle=\frac{1}{4n_0}+\left[\frac{D_{\theta\theta}}{|\partial d_\theta/\partial\theta|}\right]_{{\rm I}=\langle {\rm I}\rangle\theta=\langle\theta\rangle} . \quad (49)$$

In the following we study the cases of incoherent and coherent pumping to see the effect of the injected coherence.

A. Incoherent pumping

There is no injected atomic coherence in this case; that is, $S=0$ and $Z=0$. It can be seen from Eqs. (37)–(41) that none of the Fokker-Planck coefficients depends on the phase θ, i.e., there is no phase locking in this system. We obtain the mean photon number $\langle I\rangle=n_0$ from $d_{\rm I}(n_0)=0$,

$$n_0=\frac{\alpha}{\gamma}\frac{\alpha(\rho_{aa}-M)-\gamma}{\beta} , \quad (50)$$

which is not zero (there is laser operation) if $\rho_{aa}-M>\gamma/\alpha$, i.e., the population is inverted. The steady-state diffusion coefficients from Eqs. (39) and (40) are

$$D_{\theta\theta}(n_0)=\frac{\alpha(\rho_{aa}+M)+\gamma}{8n_0}\ , \tag{51}$$

$$D_{\rm II}(n_0)=\frac{\gamma}{\alpha}\frac{(\gamma+\alpha M)n_0}{\rho_{aa}-M}\ . \tag{52}$$

Calculating the photon-number variance from Eq. (48),

$$\langle(\Delta\hat{n})^2\rangle=\frac{\alpha\rho_{aa}}{\alpha(\rho_{aa}-M)-\gamma}n_0\ . \tag{53}$$

It can be seen that depending on the effective population M of the lower levels we can find different kinds of laser operations of the system under consideration. If the phase of the microwave field $\phi=\pi$ or $\phi=0$, we have the operation of an ordinary two-level laser because $M=\rho_{bb}$ or $M=\rho_{cc}$, respectively. Choosing the initial population of one of the lower levels zero at an appropriate phase ϕ, we can obtain the noninversion laser operation: if $\phi=\pi$ and $\rho_{bb}=0$ or $\phi=0$ and $\rho_{cc}=0$, then $M=0$. In this case

$$n_0=\frac{\alpha}{\gamma}\frac{\alpha\rho_{aa}-\gamma}{\beta}\ , \tag{54}$$

$$D_{\theta\theta}=\frac{\alpha\rho_{aa}+\gamma}{8n_0}\ , \tag{55}$$

$$D_{\rm II}=\frac{\gamma}{\alpha}\frac{\gamma n_0}{\rho_{aa}}\ , \tag{56}$$

$$\langle(\Delta\hat{n})^2\rangle=\frac{\alpha\rho_{aa}}{\alpha\rho_{aa}-\gamma}n_0\ . \tag{57}$$

That is, there is no need for population inversion between the upper and lower levels [like in Eq. (50)], because at an appropriate phase ϕ of the microwave field the effective population M can be zero even if the population of one of the lower levels is not zero; the only requirement to obtain laser operation is $\rho_{aa}>\gamma/\alpha$ [from Eq. (54)]. At the same time comparing Eqs. (55) and (56) to Eqs. (51) and (52) we find that the diffusion coefficients of intensity and phase have decreased (i.e., there is noise quieting) in the noninversion laser case.

This was the case of an ordinary noninversion laser without injected atomic coherence, but with an external microwave field. Now we proceed to a system, where injected coherence is present to reduce the noise further.

B. Coherent pumping

There is injected atomic coherence, thus $S\neq0$ and $Z\neq0$. It can be shown, that

$$S^2-Z^2=4\rho_{aa}M\ .$$

At a steady-state phase θ_0, when $d_\theta=0$, that is $Z=0$,

$$S^2=4\rho_{aa}M\ .$$

In this case the Fokker-Planck coefficients have the following forms:

$$d_\theta(\theta_0)=0\ , \tag{58}$$

$$d_{\rm I}(\theta_0)=\frac{\alpha I}{1+\frac{\beta}{\alpha}I}\left[\rho_{aa}-M+\left[\frac{2\rho_{aa}M}{\frac{\beta}{\alpha}I}\right]^{1/2}-\frac{\gamma}{\alpha}\left[1+\frac{\beta}{\alpha}I\right]\right]\ , \tag{59}$$

$$D_{\theta\theta}(\theta_0)=\frac{\beta}{4\left[1+\frac{\beta}{\alpha}I\right]\frac{\beta}{\alpha}I}\left[\rho_{aa}+\tfrac{1}{2}(\rho_{aa}+M)\frac{\beta}{\alpha}I-\left[\rho_{aa}M\frac{\beta}{\alpha}I\right]^{1/2}\right]\ , \tag{60}$$

$$D_{\rm II}(\theta_0)=\frac{\alpha I}{\left[1+\frac{\beta}{\alpha}I\right]^2}\left[\rho_{aa}^{1/2}-\left[M\frac{\beta}{\alpha}I\right]^{1/2}\right]^2\ , \tag{61}$$

$$D_{{\rm I}\theta}(\theta_0)=0\ .$$

It can be seen from Eq. (61), that

$$D_{\rm II}(\theta_0)\geq0\ ,$$

and since

$$D_{\theta\theta}=\frac{1}{8I^2}\left[D_{\rm II}\left[1+\frac{\beta}{\alpha}I\right]+\alpha\rho_{aa}I\right]$$

[see Eqs. (39) and (40)], thus

$$D_{\theta\theta}(\theta_0)>0\ .$$

In steady state (where the phase is locked to $\langle\theta\rangle=\theta_0$ due to the injected atomic coherence) the photon number $\langle n\rangle=n_0$ satisfies the following equation derived from $d_{\rm I}(\theta_0,n_0)=0$ using Eq. (59):

$$\left[\frac{\beta}{\alpha}n_0\right]^{3/2}+\left[1-\frac{\alpha}{\gamma}(\rho_{aa}-M)\right]\left[\frac{\beta}{\alpha}n_0\right]^{1/2}-2\frac{\alpha}{\gamma}(\rho_{aa}M)^{1/2}=0\ . \tag{62}$$

We note that the third term of Eq. (62), which stems from the nonzero atomic coherence $S=2(\rho_{aa}M)^{1/2}\neq0$, acts as a driving force, and there is no need for population inversion $\rho_{aa}-M>\gamma/\alpha$ for laser operation. At the same time since this term is determined by the effective population M, manipulating the value of M, we inevitably change the value of the third term (i.e., the injected coherence) in Eq. (62), too. Setting M equal to zero (i.e., $\phi=\pi$ or 0 and $\rho_{bb}=0$ or $\rho_{cc}=0$, respectively), we reobtain the incoherently pumped noninversion laser case of Sec. V A because, at the steady-state value of the phase, the disappearance of the injected atomic coherence (i.e., $S=0$) is a direct consequence of the zero effective population. This means that we cannot use the method of setting M equal to zero to obtain coherently pumped noninversion laser operation, but retaining $M\neq0$ (i.e., the nonzero "driving force" term), due to the coherent pumping we still have noninversion lasing. Next we investigate the conditions for the minimal noise and maximal laser intensity.

Let us see the case when

$$D_{\mathrm{II}}(\theta_0)=0 \tag{63}$$

minimal, and consequently

$$D_{\theta\theta}(\theta_0)=\frac{\alpha\rho_{aa}}{8I} . \tag{64}$$

In this case

$$\rho_{aa}=M\frac{\beta}{\alpha}I . \tag{65}$$

Substituting this into Eq. (62) we obtain the intensity and phase in steady state,

$$M=\frac{\gamma}{\alpha} , \tag{66}$$

and consequently

$$\rho_{aa}=\frac{\beta\gamma}{\alpha^2}n_0 . \tag{67}$$

Introducing the following relation between the population of the lower levels,

$$\rho_{bb}=m\rho_{cc}, \quad m\in[0,\infty] \tag{68}$$

and using Eq. (66),

$$M=\tfrac{1}{2}[(1-\cos\phi)\rho_{bb}+(1+\cos\phi)\rho_{cc}+2|\rho_{bc}|\sin\phi\sin(\phi+\varphi_{bc})]=\frac{\gamma}{\alpha} , \tag{69}$$

and the fact that

$$\rho_{aa}+\rho_{bb}+\rho_{cc}=1 , \tag{70}$$

where ρ_{aa} satisfies Eq. (67), we obtain for n_0 the following general formula:

$$n_0=\frac{\alpha}{\gamma}\frac{\alpha[m(1-\cos\phi)+1+\cos\phi+2m^{1/2}\sin\phi\sin(\phi+\varphi_{bc})]-2\gamma(m+1)}{\beta[m(1-\cos\phi)+1+\cos\phi+2m^{1/2}\sin\phi\sin(\phi+\varphi_{bc})]} . \tag{71}$$

We disregard the $\phi=\pi$, $m=0$ and the $\phi=0, 1/m=0$ pairs because M could not be γ/α in these two cases. It can be seen that for $\phi=\pi$ and $m=0$ ($\rho_{bb}=0$) or $\phi=0$ and $1/m=0$ ($\rho_{cc}=0$) we get back the noninversion laser system of Sec. V A. Taking $\sin\phi=0$, for $\phi=0$

$$n_0=\frac{\alpha}{\gamma}\frac{\alpha-\gamma(m+1)}{\beta} \quad (1/m\neq 0) , \tag{72}$$

and for $\phi=\pi$

$$n_0=\frac{\alpha}{\gamma}\frac{\alpha m-\gamma(m+1)}{\beta m} \quad (m\neq 0) . \tag{73}$$

It can be seen that just in the opposite case of the incoherent noninversion laser, if we set $\phi=0$ and $m=0$ ($1/m\neq 0$), or $\phi=\pi$ and $1/m=0$ ($m\neq 0$) [$M=\rho_{cc}$ (or ρ_{bb})$=\gamma/\alpha\neq 0$, ρ_{bb} (or ρ_{cc})$=0$, $\rho_{aa}=1-\gamma/\alpha$], that is, we put our laser system into a "two-level operation regime," then we obtain the following steady-state photon number:

$$n_0=\frac{\alpha}{\gamma}\frac{\alpha-\gamma}{\beta} , \tag{74}$$

and we find from Eqs. (63) and (64)

$$D_{\mathrm{II}}(\theta_0,n_0)=0 , \tag{75}$$

$$D_{\theta\theta}(\theta_0,n_0)=\frac{\alpha-\gamma}{8n_0}=\frac{\beta\gamma}{8\alpha} , \tag{76}$$

which are exactly the same as in the two-level laser with injected atomic coherence.[19] Note that there is still no need for population inversion, because if $\rho_{aa}=1-\gamma/\alpha<M=\gamma/\alpha$, the photon number is positive (i.e., the laser operates) in the $1<\alpha/\gamma<2$ interval. This means that at complete quenching of the intensity noise and large reduction of the phase noise we have noninversion lasing as well. The photon-number variance in steady state is

$$\langle(\Delta\hat{n})^2\rangle=n_0 , \tag{77}$$

that is, the photon-number distribution is *exactly* Poissonian. Since

$$\left|\frac{\partial}{\partial\theta}d_\theta\right|_{\theta_0,n_0}=\gamma , \tag{78}$$

thus we find the phase variance to be

$$\langle(\Delta\theta)^2\rangle=\frac{1+\alpha/\gamma}{8n_0} . \tag{79}$$

When $\alpha/\gamma\cong 1$, the phase variance is approximately the same as that of the coherent state,

$$\frac{1}{4n_0} .$$

Finally we find

$$\langle(\Delta\hat{n})^2\rangle\langle(\Delta\theta)^2\rangle=\frac{1}{8}\left[1+\frac{\alpha}{\gamma}\right] , \tag{80}$$

which (at $\alpha/\gamma\cong 1$) is approximately equal to $\frac{1}{4}$, the quantum limit of the minimum uncertainty product. That is, the laser field is in a near-coherent state.

If we take a typical "three-level operation" case, when $m=1$ [that is, for $\phi=0$ (or π) $M=\rho_{cc}=\rho_{bb}=\gamma/\alpha\neq 0$, $\rho_{aa}=1-2(\gamma/\alpha)$] in Eqs. (72) and (73), then the steady-state photon number is

$$n_0=\frac{\alpha}{\gamma}\frac{\alpha-2\gamma}{\beta} , \tag{81}$$

when we still have noninversion lasing, if $2<\alpha/\gamma<3$. The photon-number distribution is still exactly Poissonian: $\langle(\Delta\hat{n})^2\rangle=n_0$ and the phase variance is

$\langle(\Delta\phi)^2\rangle=(\alpha/\gamma)/8n_0$. This means that the laser field is in the same near-coherent state as in the previous "two-level" case. Overall, even if all three levels are populated, we still have noninversion laser operation in the range of $2<\alpha/\gamma<3$, at the noise characteristics of a near-coherent state.

Thus, it is shown that even if we do not set M equal to zero (because in that case we would get back to the incoherent noninversion laser operation: see Sec. V A), we can reach the noninversion laser operation, where the laser field can be found very near to an ideal coherent state, due to the coherent pumping, if $M=\gamma/\alpha$. Different values of the parameters of the phase of the microwave field ϕ and the ratio of the population of the lower levels $m=\rho_{bb}/\rho_{cc}$ for the same $M=\gamma/\alpha$ result in different operation thresholds.

VI. DISCUSSION AND SUMMARY

We studied the interaction of a three-level atom in the Λ configuration interacting with one mode of the quantized radiation field, where a strong classical coupling is applied between the lower two (closely spaced) levels via an external microwave field, and these lower levels are coupled to the upper one via interacting with the field mode. Starting from the Hamiltonian model of the system transformed into a second interaction picture, and under special detuning conditions and the rotating-wave approximation, we derived the interaction Hamiltonian of the system.

Substituting the Hamiltonian into the time-dependent Schrödinger equation, we solved the model so that the obtained solution contained all the possible initial conditions. Consequently, we could retain all the elements of the following density matrix of the system. Thus we had the possibility to derive a master equation for the reduced field-density operator and the laser photon statistics. Besides the atomic coherence resulting from the applied external microwave field, this equation contained all the atomic coherences between the different atomic levels. The master equation is the same as that of the two-level laser with injected atomic coherence,[19] if we substitute ρ_{bb} (or ρ_{cc}) into M, that is, we reobtain the usual master equation of an ordinary two-level laser if the injected coherence is set equal to zero ($R=0$) and $M=\rho_{bb}(\rho_{cc})$.

We presented a Fokker-Planck treatment of the system and obtained drift and diffusion coefficients. The laser operation is studied without and with injected atomic coherence. In the first case an incoherent noninversion laser operation is found depending on the phase of the external microwave field ϕ and the initial populations of the lower two levels. Diffusion coefficients are calculated and small noise reduction is shown compared to the ordinary laser case.

In the second case, when we apply injected atomic coherence the system is shown to be phase sensitive and the laser phase is locked to a particular value θ_0 in the steady state. θ_0 is determined by the phase of the injected atomic coherence φ_{ij} and the external signal ϕ. The diffusion coefficients are calculated at the steady state; they take the value of the locked phase angle θ_0. Both the intensity and the phase-diffusion coefficients are reduced compared to the case of no initial atomic coherence.

The system is studied in the special case, when the diffusion coefficients are minimal: the intensity diffusion vanishes, the phase diffusion has a small, positive value. A general formula for the steady-state photon number is derived, which is investigated for different arbitrary parameter choices of the microwave phase ϕ and the initial population of the lower levels. It is shown that with the appropriate choice of parameters the system can behave as (1) an incoherent noninversion laser:[9] setting $M=0$ ($\phi=\pi$, $\phi=0$ and $\rho_{bb}=0$, or $\rho_{cc}=0$, respectively), we obtained the steady-state photon number, diffusion and variance of Eqs. (54)–(57), or (2) a coherent noninversion laser if $M\neq0$. In this case we have the minimum diffusion coefficients if $M=\gamma/\alpha$,

$$D_{\rm II}(\theta_0,n_0)=0\ ,$$

$$D_{\theta\theta}(\theta_0,n_0)=\frac{\beta\gamma}{8\alpha}\ ,$$

and the steady-state photon number (71) depends on the values of the parameters of the phase of the microwave field ϕ and the ratio of the population of the lower levels $m=\rho_{bb}/\rho_{cc}$, (72) and (73), at the same $M=\gamma/\alpha$.

(a) At the "two-level"-like[19] parameter choice: when $M=\gamma/\alpha$ [$=\rho_{cc}$ (or ρ_{bb}), ρ_{bb} (or ρ_{cc})$=0$, $\phi=0$ (or π)],

$$n_0=\frac{\alpha}{\gamma}\frac{\alpha-\gamma}{\beta}\ ,$$

$$D_{\theta\theta}(\theta_0,n_0)=\frac{\alpha-\gamma}{8n_0}\ ,$$

operating if $1<\alpha/\gamma<2$.

(b) At the three-level case: when $M=\gamma/\alpha(=\rho_{bb}=\rho_{cc},\phi=0$, or π),

$$n_0=\frac{\alpha}{\gamma}\frac{\alpha-2\gamma}{\beta}\ ,$$

$$D_{\theta\theta}(\theta_0,n_0)=\frac{\alpha-2\gamma}{8n_0}\ ,$$

if $2<\alpha/\gamma<3$.

The photon number and phase variances are also discussed and it is found that the photon-number distribution is exactly Poissonian, while the phase distribution is (at the threshold) near Poissonian, which means that the laser field approaches a coherent state. This noninversion laser system becomes a quantum-noise-limited active device if, besides the external microwave-field-induced coherence, an injected atomic coherence is also applied.

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