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Quantum optics of dielectric media

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We discuss the quantum fluctuations of the fields associated with a broad class of optical scattering and transmission problems by developing the quantum electrodynamics of an idealized linear, but nonuniform, dielectric medium. We present and compare two quantization schemes for this purpose. The first is based on the expansion of the field in terms of a set of single-frequency solutions of the Maxwell equations. The second involves expanding the field in the set of plane-wave solutions of the Maxwell equations in the vacuum. The relation between the two quantization schemes is discussed in the framework of the scattering theory that connects them. The methods presented are used to show that various field components within a dielectric medium may be either superfluctuant or subfluctuant relative to their fundamental uncertainties in the vacuum. These alterations of the fluctuation properties of the fields are shown to lead to changes in the spontaneous emission rates for both electric and magnetic dipole transitions of excited atoms within or near dielectric media. We also analyze the quantum properties of the transition radiation emitted by a fast charged particle in passing from one dielectric medium to another.

I. INTRODUCTION

Some of the most familiar problems of electromagnetic theory concern the determination of radiation fields in the presence of polarizable matter. These problems may be as simple as those of light transmission in uniform media, or as complex as light scattering by bodies of arbitrary size and shape. The appropriate calculations have been addressed in general by the well-established methods of classical electromagnetic theory.[1,2] Although these calculations have long been known to give correct predictions of average field intensities in the quantum theory as well, there are a number of quantum statistical problems of interest that cannot be approached by those classical means. The presence of polarizable media alters considerably the quantum fluctuation properties of the fields, and certain of these properties may be subjected directly to measurement.

In order to deal explicitly with the quantum properties of fields in the presence of polarizable media, we have developed a generalization of the familiar technique of canonical field quantization. It is a generalization that admits treatment of a broad range of inhomogeneous media with linear susceptibilities.[3] We should emphasize that these inhomogeneities include boundary discontinuities for media of finite extent, so that the formally soluble field theory we construct is applicable to many familiar photon transmission and scattering problems. To illustrate the quantization technique we have applied it in detail to the case of an inhomogeneous dielectric medium with a scalar dielectric constant $\epsilon(\mathbf{r})$ that depends explicitly on position. The procedure permits directly the introduction of more general tensor electric and magnetic susceptibilities.

The vacuum state of the electromagnetic field and indeed all coherent states[4] have the property that the variances of the fluctuating electric and magnetic fields are equal. The same equality of variances holds for any pair of linear combinations of those fields that oscillate 90° out of phase with each other. Considerable attention has been devoted recently to the possibility of altering this balance.[5] It is possible to reduce the value of one of these variances at the expense of increasing the other, and the quantum states in which this occurs have been referred to as "squeezed." Nonlinear interactions often leave the field in such "squeezed states." All of the schemes proposed for the generation of such states have, for this reason, been based on the techniques of nonlinear optics.[6,7] It is worth emphasizing, therefore, that linear interactions of the sort associated with simple polarizable media can likewise break the vacuum symmetry of the electric and magnetic fields, and in this way can lead much more simply to certain of the effects of squeezing.

The quantum fluctuation problems we shall study are those characteristic of quantum field theories in three dimensions. That is to say, they deal with the propagation of fields that have an infinite number of degrees of freedom.[8] It may be helpful, however, to begin by discussing the properties of a single field degree of freedom, let us say, for example, the amplitude of one particular mode of oscillation of the field at the frequency ω within a closed box that has perfectly conducting walls.[9] The familiar quadratic field Hamiltonian then reduces to that for a single harmonic oscillator of unit mass. If we introduce the annihilation and creation operators a and $a^\dagger$, which fulfill the commutation relation

$$[a,a^\dagger]=1 , \tag{1.1}$$

we can write the Hamiltonian for the mode as

$$H=\hbar\omega(a^\dagger a+\tfrac{1}{2}) . \tag{1.2}$$

The oscillator coordinates that we would like to measure are usually Hermitian combinations of a and $a^\dagger$ that we can write in the form

$$X(\theta)=\frac{1}{\sqrt{2}}(ae^{-i\theta}+a^{\dagger}e^{i\theta}) , \tag{1.3}$$

for suitably chosen values of the parameter θ. The operator $X(0)$, for example, is usually taken to be proportional to the amplitude of the vector potential for the mode or its magnetic-field amplitude. Then $X(\pi/2)$ is proportional to the rate of change of the vector potential of the mode or the amplitude of its electric field.

An operator canonically conjugate to $X(\theta)$ can be defined more generally by the expression

$$Y(\theta)=-\frac{i}{\sqrt{2}}(ae^{-i\theta}-a^{\dagger}e^{i\theta})=X(\theta+\pi/2) , \tag{1.4}$$

and, because of the phase relation indicated, the two variables $X(\theta)$ and $Y(\theta)$ are said to be in quadrature. Their variances, in any given quantum state, are defined as

$$(\Delta X)^2=\langle (X-\langle X\rangle)^2\rangle , \tag{1.5a}$$

$$(\Delta Y)^2=\langle (Y-\langle Y\rangle)^2\rangle . \tag{1.5b}$$

Since X and Y obey the commutation relation

$$[X(\theta),Y(\theta)]=i , \tag{1.6}$$

their variances must then satisfy the uncertainty inequality

$$[\Delta X(\theta)]^2[\Delta Y(\theta)]^2\geq\tfrac{1}{4} \tag{1.7}$$

for all values of the parameter θ.

It is clear from the definitions (1.3) and (1.4) that in all stationary states of the oscillator we have $[\Delta X(\theta)]^2=[\Delta Y(\theta)]^2$, but it is only for the ground state that this value corresponds to the lowest uncertainty bound in Eq. (1.7). A much broader class of states that satisfy the lowest uncertainty bound is the nonstationary set of coherent states.[4] These can be regarded as ground states that have been given an arbitrary displacement in coordinate and momentum. Since adding complex conjugate constants to a and $a^{\dagger}$ does not affect their variances, it is clear that in these states $[\Delta X(\theta)]^2=[\Delta Y(\theta)]^2=\frac{1}{2}$, just as in the ground state.

The most general class of minimum uncertainty states, satisfying

$$[\Delta X(\theta)]^2[\Delta Y(\theta)]^2=\tfrac{1}{4} , \tag{1.8}$$

corresponds to the Kennard wave packets.[10] For them the variances $[\Delta X(\theta)]^2$ and $[\Delta Y(\theta)]^2$ will depend on θ in general. One will be smaller than $\frac{1}{2}$ and the other consequently larger. Although states with this property have been called "squeezed" in the recent literature,[5-7] it is much more meaningful, as we shall indicate later, to think of a particular variable as squeezed in a given state rather than the quantum state as a whole. Since the effect of "stretching," opposite to "squeezing," is an equally important one, and neither term is an accurately descriptive one, we would prefer to alter the terminology by referring to a variable $X(\theta)$ as *subfluctuant* in a given state if

$$[\Delta X(\theta)]^2<\tfrac{1}{2} \tag{1.9}$$

and *superfluctuant* if

$$[\Delta X(\theta)]^2>\tfrac{1}{2} . \tag{1.10}$$

To begin our discussion of polarizable media let us now assume that our closed and resonant box has been filled with a uniform substance of dielectric constant ϵ. The field Lagrangian in that case takes the form

$$L=\tfrac{1}{2}\int(\epsilon E^2-B^2)d\mathbf{r} . \tag{1.11}$$

If we let the time-dependent amplitude of the vector potential for the mode we are discussing be $A(t)$, then, since the presence of the dielectric does not change the wave number of the mode from its vacuum value ω/c, the magnitude of its magnetic field can be taken to be $B(t)=(\omega/c)A(t)$. We can thus take the coordinate of the corresponding oscillator to be

$$q=A/c=B/\omega . \tag{1.12}$$

Then $\dot{q}$ is proportional to the electric-field strength

$$\dot{q}=\frac{1}{c}\dot{A}=-E , \tag{1.13}$$

and the Lagrangian for the field mode becomes

$$L=\tfrac{1}{2}(\epsilon\dot{q}^{\,2}-\omega^2q^2) . \tag{1.14}$$

We note that the dielectric constant in this expression plays the role of a mass for a mechanical oscillator. The canonical momentum

$$p=\frac{\partial L}{\partial\dot{q}}=\epsilon\dot{q}=-D \tag{1.15}$$

is proportional to the electric displacement vector, and the corresponding Hamiltonian for the mode is

$$H=\tfrac{1}{2}\left[\frac{p^2}{\epsilon}+\omega^2q^2\right] . \tag{1.16}$$

By carrying out the canonical scale transformation

$$p'=p/\sqrt{\epsilon} , \tag{1.17a}$$

$$q'=q\sqrt{\epsilon} , \tag{1.17b}$$

we can write the Hamiltonian as

$$H=\tfrac{1}{2}[(p')^2+\Omega^2(q')^2] , \tag{1.18}$$

where the new frequency is

$$\Omega=\omega/\sqrt{\epsilon} . \tag{1.19}$$

The stationary states of the field mode are evidently changed both in their frequency and in the fluctuations of their field strengths by the presence of the dielectric. The annihilation operator appropriate to the Hamiltonian (1.18), and therefore the one that removes one quantum from the dielectric, can be written as

$$a=\frac{1}{\sqrt{2\hbar\Omega}}(\Omega q'+ip')=\frac{1}{\sqrt{2\hbar\Omega}}(\omega q+ip/\sqrt{\epsilon}) . \tag{1.20}$$

This is not the same annihilation operator as we had previously for photons in the empty cavity. If we write that operator as

$$b=\frac{1}{\sqrt{2\hbar\omega}}(\omega q+ip)\ , \tag{1.21}$$

then we find that the transformation from the operators for the empty cavity to those for the dielectric-filled cavity is

$$a=\frac{\omega+\Omega}{2\sqrt{\omega\Omega}}b+\frac{\omega-\Omega}{2\sqrt{\omega\Omega}}b^\dagger\ , \tag{1.22a}$$

$$a^\dagger=\frac{\omega+\Omega}{2\sqrt{\omega\Omega}}b^\dagger+\frac{\omega-\Omega}{2\sqrt{\omega\Omega}}b\ . \tag{1.22b}$$

The ground state of the field mode in the presence of the dielectric is determined by the condition

$$a|0\rangle_{\text{diel}}=0\ . \tag{1.23}$$

Because that state is the ground state of an oscillator of frequency Ω, the variances of p' and q' are given by the familiar values

$$(\Delta p')^2=\Omega^2(\Delta q')^2=\tfrac{1}{2}\hbar\Omega\ . \tag{1.24}$$

The amplitudes of the magnetic field and the electric displacement are given by Eqs. (1.12) and (1.15). If we introduce the operators

$$X_b(0)=\frac{1}{\sqrt{2}}(b+b^\dagger)\ , \tag{1.25a}$$

$$Y_b(0)=-\frac{i}{\sqrt{2}}(b-b^\dagger)\ , \tag{1.25b}$$

analogous to those of Eqs. (1.3) and (1.4), we can write the field amplitudes as

$$B=\omega q=\sqrt{\hbar\omega}X_b(0)\ , \tag{1.26a}$$

$$D=-p=-\sqrt{\hbar\omega}Y_b(0)\ . \tag{1.26b}$$

An advantage of writing the field amplitudes in this way is that these expressions are medium independent; they take the same form in the dielectric as in the vacuum.

To find the variances of B and D in the ground state of the dielectric we make use of the scaled variables defined by Eq. (1.17) and their variances given by Eq. (1.24) to write

$$(\Delta B)^2_{\text{diel}}=\omega^2(\Delta q)^2=\omega^2(\Delta q')^2\epsilon=\tfrac{1}{2}\hbar\Omega=\tfrac{1}{2}\hbar\omega/\sqrt{\epsilon}\ , \tag{1.27a}$$

$$(\Delta D)^2_{\text{diel}}=(\Delta p)^2=\epsilon(\Delta p')^2=\tfrac{1}{2}\epsilon\hbar\Omega=\tfrac{1}{2}\sqrt{\epsilon}\hbar\omega\ . \tag{1.27b}$$

It is worth noting that the product of the variances of D and B

$$(\Delta D)^2_{\text{diel}}(\Delta B)^2_{\text{diel}}=\tfrac{1}{4}(\hbar\omega)^2 \tag{1.28}$$

is independent of the dielectric constant. That is true because D and B have a medium-independent commutator

$$[D,B]=[-p,\omega q]=i\hbar\omega\ . \tag{1.29}$$

That property of the fields **D** and **B** holds much more generally[3(b)] and will later be of considerable use to us.

The variances we have found for B and D show, according to Eqs. (1.26), that the variances of $X_b(0)$ and $Y_b(0)$ are given in the ground state of the dielectric by

$$[\Delta X_b(0)]^2=\frac{1}{2\sqrt{\epsilon}}\ , \tag{1.30a}$$

$$[\Delta Y_b(0)]^2=\tfrac{1}{2}\sqrt{\epsilon}\ . \tag{1.30b}$$

In this sense it follows then that the ground state of the dielectric-filled cavity is what has been called a squeezed state. We have used the operators X_b and Y_b to demonstrate this in order to exploit the medium independence of the expressions in the Eqs. (1.26). A somewhat more physical way of stating the same results is to compare the field fluctuations in the dielectric ground state with those in the vacuum ground state (obtained by setting $\epsilon=1$). We then find the ratios of the variances

$$\frac{(\Delta B)^2_{\text{diel}}}{(\Delta B)^2_{\text{vac}}}=\frac{1}{\sqrt{\epsilon}}\ , \tag{1.31a}$$

$$\frac{(\Delta D)^2_{\text{diel}}}{(\Delta D)^2_{\text{vac}}}=\sqrt{\epsilon}\ . \tag{1.31b}$$

Here we have a concrete sense in which the magnetic-field amplitude B must be regarded as subfluctuant in the dielectric ground state (for $\epsilon>1$), while the displacement amplitude D is superfluctuant. The electric-field amplitude $E=D/\epsilon$, on the other hand, has the variance ratio

$$\frac{(\Delta E)^2_{\text{diel}}}{(\Delta E)^2_{\text{vac}}}=\frac{1}{\epsilon^{3/2}}\ , \tag{1.32}$$

so that it is subfluctuant in the dielectric.

It may be helpful at this point to contrast the variances indicated by Eqs. (1.12)–(1.32) with the variances shown by some other states. For a coherent state in free space, as we have noted earlier, the variances are equal and are constant in time. They are schematically represented in Fig. 1, for example, by the small circles that represent the uncertainty of the locus of the field vector at three different times. In Fig. 2, which corresponds to what has usually been called a "squeezed state" in free space, the domains of uncertainty are elliptical in shape. Those ellipses, furthermore, rotate with the field vector itself. They correspond, in the example shown, to suppressed amplitude modulation. For a coherent state in a dielectric medium, on the other hand, the field vector, as shown in Fig. 3, rotates on an elliptical trajectory. The domains of uncertainty are likewise elliptical, but the ellipses do not rotate with time. The uncertainties in modulus and phase of the field vector therefore vary periodically with time.

The results we have presented indicate that there is, in fact, a substantial ambiguity involved in speaking of the quantum states rather than particular observables as being "squeezed." The ground state of the dielectric-filled cavity, as we have seen, is a "squeezed state" when analyzed in terms of the fluctuations of the variable pair p and q or the vacuum operator b. On the other hand, it is *not* a "squeezed state" when analyzed in terms of the scaled variables p' and q', or the annihilation operator a. This is simply to say that in a given quantum state we can

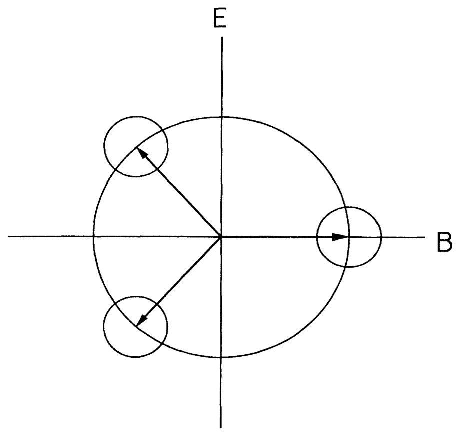

FIG. 1. Schematic representation of the time dependence of the field amplitude and its variance for a coherent state in free space. The expectation value of the field vector rotates clockwise about the larger circle. Its instantaneous value is uncertain, however, and tends to lie within the smaller circles that indicate its uncertainty or variance. The mean field and its variance, which is constant, are shown at three different times.

define the variables in alternative ways some of which exhibit "squeezing" and others of which do not.

Some further indications of the ambiguity implicit in speaking of the quantum states as "squeezed" may be seen, if any more is needed, by considering a simple hypothetical example. Let us suppose that the empty cavity can be quite suddenly filled with a uniform dielectric; that is, its dielectric constant can be changed instantaneously[11] from 1 to ϵ. The quantum state of the field undergoes no change during the sudden change of the dielectric constant. The vacuum ground state, which it continues to be immediately after the discontinuity of the dielectric constant, may be shown, however, to be a squeezed state when analyzed in terms of the annihilation operator a of

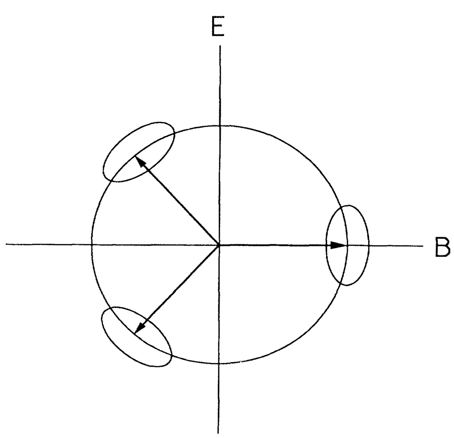

FIG. 2. Time dependence of the field amplitude and its variance for a "squeezed state" in the vacuum. The domains of uncertainty of the field amplitude are ellipses which rotate rigidly with the amplitude. In the example shown they represent the suppression of intensity fluctuation.

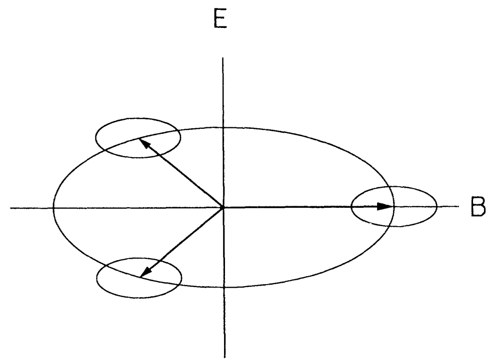

FIG. 3. Time dependence of the field amplitude and its variance for a coherent state in a dielectric. The expectation value of the field amplitude rotates about the large ellipse, and the domains of uncertainty of the field amplitudes are likewise elliptical. The latter ellipses do not rotate, however, and the electric- and magnetic-field variances remain constant in time.

Eq. (1.20) appropriate to the dielectric. If one insists on regarding quantum states as squeezed rather than a particular choice of observables, then one is confronted by an example of a state that suddenly becomes squeezed while remaining unchanged.

The photons counted in ideal laser beams characteristically exhibit Poisson statistics.[4] Certain laser-generated fields, however, for which the field variances exhibit squeezing, have been shown to have photon statistics that deviate appreciably from the Poissonian form.[12] Since squeezed fields are indeed present in dielectric media, it is interesting to discuss the problem of counting photons within them. We might imagine, for example, that a beam of photons is directed from the vacuum into a dielectric medium and ask whether any change in the photon statistics would be observed within the medium. Our single-mode model of the field omits many of the features of the problem that are necessary, we shall presently show, for a more realistic treatment. It does, however, pose an interesting puzzle that is indeed a part of the problem.

There is a substantial physical difference between the photons defined in the vacuum, i.e., the empty cavity in the single-mode model, and those defined in the dielectric-filled cavity. Because the transformation (1.22) mixes the operators b and $b^\dagger$ it does not maintain any one-to-one correspondence between the numbers of "vacuum" and "dielectric" photons. The ground state of the dielectric, for example, can easily be shown to represent a distribution of $n=0,2,4,6,\ldots$, i.e., any even number of vacuum photons. Those vacuum photons might seem to be of precisely the same sort as the ones detected in most photon-counting experiments. But could we insert a photon counter into the dielectric and count them? Obviously not, because we cannot draw energy from the dielectric if it is already in its ground state. Those pairs of vacuum quanta are not really present; they are only virtual.

We shall show in the later sections of this paper that although transformations like Eq. (1.22) fail to conserve photon number, they do not lead to any qualitative changes in the nature of the photon-counting distributions that we can actually observe. In order to demonstrate that, however, we shall have to pay a certain amount of attention to the theory underlying such photon-counting experiments.

A photodetector consists of atoms free to undergo photoabsorption processes that can somehow be counted in number. The photosensitive atoms can be regarded as present within a small cavity in the dielectric medium, inside which they are subject to an oscillating local field $E_{\text{loc}}(\mathbf{r},t)$. The photocount distribution, it has been shown,[4] can be constructed from a knowledge of the expectation values of the correlation products

$$E_{\text{loc}}^{(-)}(\mathbf{r}_1,t_1)\cdots E_{\text{loc}}^{(-)}(\mathbf{r}_n,t_n)\times E_{\text{loc}}^{(+)}(\mathbf{r}_{n+1},t_{n+1})\cdots E_{\text{loc}}^{(+)}(\mathbf{r}_{2n},t_{2n}) , \tag{1.33}$$

in which $E_{\text{loc}}^{(+)}$ and $E_{\text{loc}}^{(-)}$ are the positive- and negative-frequency components of the local field, respectively. The particular ordering required in these products, positive-frequency operators to the right of negative-frequency operators, responds to an elementary property of the photoabsorption process: when a photon is recorded, the energy in the field must decrease.

In vacuum quantum electrodynamics the photon annihilation operators all have positive frequencies and the creation operators negative frequencies. The ordering indicated by Eq. (1.33) is then what has usually been referred to as "normal ordering," one that places all annihilation operators to the right of the creation operators. That definition of normal ordering indeed suffices for a discussion of all photon-counting experiments carried out in vacuum. The more fundamental definition, however, for the present purposes is the ordering according to the sign of the frequency indicated by Eq. (1.33).

To underscore the importance of this distinction we should note that it is often more convenient to evaluate field operators $E_{\text{loc}}^{(+)}$ in terms of the annihilation and creation operators such as b and $b^\dagger$ for vacuum photons, rather than those defined in the presence of the dielectric. The operators b and $b^\dagger$ are given by the relations inverse to Eq. (1.22)

$$b=\frac{\omega+\Omega}{2\sqrt{\omega\Omega}}a-\frac{\omega-\Omega}{2\sqrt{\omega\Omega}}a^\dagger , \tag{1.34a}$$

$$b^\dagger=\frac{\omega+\Omega}{2\sqrt{\omega\Omega}}a^\dagger-\frac{\omega-\Omega}{2\sqrt{\omega\Omega}}a . \tag{1.34b}$$

Since the operators a and $a^\dagger$ are defined to oscillate with positive and negative frequencies, respectively, it is clear that the operators b and $b^\dagger$ oscillate in general with both signs of the frequency. They do not generate energy eigenstates; absorption of a physical photon may be accomplished either by absorbing or emitting a "vacuum photon." The expectation values of products like that of Eq. (1.33) can nonetheless be evaluated in terms of the vacuum operators b and $b^\dagger$. It is only necessary, for that purpose, to observe that the ordering called for is defined according to the sign of the frequency, and is no longer normal ordering as it is usually construed.

All of the points we have noted in discussing a single mode of the field in a dielectric are, of course, equally characteristic of the multimode field theory more generally. We shall devote the remainder of this paper to discussing them and illustrating them in the much broader context of electromagnetic scattering and transmission problems in three dimensions. In Sec. II we present a detailed description of a quantization procedure that can be used in the presence of nonuniform dielectric media. It is based on a global expansion of the field in terms of single-frequency solutions of the Maxwell equations. In Sec. III we discuss a quantization procedure based on an expansion of the fields in terms of the plane-wave solutions of the Maxwell equations in the vacuum. The relation between the two expansion schemes is then discussed in Sec. IV. We show there that the creation and annihilation operators associated with the plane-wave expansion are related to those of physical photons by means of a linear transformation. This linear transformation, we show, is generated by the Møller operators of the corresponding classical scattering problem.[2]

The quantization schemes discussed in the earlier sections are used in Sec. V to determine the fluctuation properties of various field components, or more precisely, the variances of their averages, taken over arbitrary volumes. Although the foregoing results demonstrate that dielectric media introduce no qualitative changes in the photon-counting distributions that can be measured by photoabsorption processes, they do indeed introduce substantial changes that can be observed in spontaneous emission processes. We show in Sec. VI that a dielectric medium, by altering the strength of the zero-point fluctuations of the electric field, changes the rate of spontaneous electric dipole emission by atoms located within or close to the medium. More interestingly perhaps, any such change in the electric-field fluctuations implies quantum mechanically a complementary change of the magnetic-field fluctuations. The rate of spontaneous magnetic dipole emission must therefore also be altered when an excited atom is in or near a dielectric medium.

As a final illustration of our formalism we present in Sec. VII a quantum-mechanical treatment of transition radiation. This is the radiation given off when a fast charged particle passes from a medium of one dielectric constant into another. The mean number of photons emitted is shown to follow the classically derived formula of Frank and Ginsburg.[13] The fluctuations in the photon numbers are shown to be those characteristic of Poisson distributions.

II. QUANTIZATION IN THE PRESENCE OF DIELECTRIC MEDIA— NORMAL-MODE EXPANSION

We shall present in this section a detailed description of the quantization of the electromagnetic field in the presence of an idealized linear, but nonuniform, dielectric medium with frequency-independent polarizability. Although special cases of this problem have received some discussion,[3,14–16] a number of its more interesting aspects have evidently been left untouched. We shall try, therefore, to formulate the problem in fairly general terms. Furthermore, we shall pay special attention to the quantum-statistical properties of the electromagnetic field in and near dielectric media.

The source-free Maxwell equations in matter take the form[1]

$$\begin{aligned} &\nabla\cdot\mathbf{D}=0\ ,\\ &\nabla\cdot\mathbf{B}=0\ ,\\ &\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}=\nabla\times\mathbf{H}\ ,\\ &\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}=-\nabla\times\mathbf{E}\ . \end{aligned} \tag{2.1}$$

In the following we shall limit our consideration to the case of linear, isotropic dielectric media, for which the electric displacement vector and magnetic induction are given simply by

$$\begin{aligned} &\mathbf{D}=\epsilon(\mathbf{r})\mathbf{E}\ ,\\ &\mathbf{B}=\mathbf{H}\ , \end{aligned} \tag{2.2}$$

where $\epsilon(\mathbf{r})$ is a position-dependent dielectric constant. Generalizations to the anisotropic case or the case of magnetically susceptible media are straightforward. We shall quantize the field by generalizing appropriately the familiar procedure which consists of (a) the introduction of potentials, (b) fixing the gauge, and (c) replacing Poisson bracket expressions by canonical commutation relations for the vector potential and its canonically conjugate momentum.

From Eq. (2.1) it is clear that we can introduce the vector potential $\mathbf{A}$ and the scalar potential Φ via the familiar relations

$$\begin{aligned} &\mathbf{B}=\nabla\times\mathbf{A}\ ,\\ &\mathbf{E}=-\nabla\Phi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\ . \end{aligned} \tag{2.3}$$

The gauge that is most commonly used in nonrelativistic QED is the Coulomb or "radiation" gauge. It corresponds, in the absence of charges, to the choice

$$\Phi=0\ . \tag{2.4}$$

With this choice, the transversality condition on $\mathbf{D}$ becomes

$$\nabla\cdot[\epsilon(\mathbf{r})\dot{\mathbf{A}}]=0\ . \tag{2.5a}$$

We may now fix the gauge by imposing the requirement

$$\nabla\cdot[\epsilon(\mathbf{r})\mathbf{A}]=0\ , \tag{2.5b}$$

which automatically fulfills condition (2.5a). The gauge condition (2.5b) is a generalization, appropriate to the presence of a dielectric, of the Coulomb gauge condition ($\nabla\cdot\mathbf{A}=0$).

The equation of motion for the vector potential $\mathbf{A}$ is

$$\frac{\epsilon(\mathbf{r})}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}+\nabla\times(\nabla\times\mathbf{A})=0\ , \tag{2.6}$$

and is obviously compatible with the gauge condition (2.5b).

It is not difficult to find a Lagrangian function for which Eq. (2.6) is an equation of motion. An elementary calculation shows that Eq. (2.6) follows from Hamilton's principle for the Lagrangian

$$\begin{aligned} \mathcal{L}&=\tfrac{1}{2}\int d\mathbf{r}[\epsilon(\mathbf{r})\mathbf{E}(\mathbf{r})^2-\mathbf{B}(\mathbf{r})^2]\\ &=\tfrac{1}{2}\int d\mathbf{r}\left[\frac{\epsilon(\mathbf{r})}{c^2}\dot{\mathbf{A}}(\mathbf{r})^2-[\nabla\times\mathbf{A}(\mathbf{r})]^2\right]\ . \end{aligned} \tag{2.7}$$

The Hamiltonian description of the motion is obtained by introducing the canonical momentum

$$\Pi_\alpha(\mathbf{r})=\frac{\delta\mathcal{L}}{\delta\dot{A}_\alpha(\mathbf{r})}=\frac{\epsilon(\mathbf{r})\dot{A}_\alpha(\mathbf{r})}{c^2}\ , \tag{2.8}$$

and performing the Legendre transformation to define the Hamiltonian function

$$\mathcal{H}[\mathbf{A},\mathbf{\Pi}]=\int d\mathbf{r}\,\mathbf{\Pi}(\mathbf{r},t)\,\dot{\mathbf{A}}(\mathbf{r},t)-\mathcal{L}\ . \tag{2.9}$$

We may note that the canonical momentum field $\mathbf{\Pi}$ is in fact proportional to the electric displacement vector

$$\mathbf{\Pi}(\mathbf{r},t)=-\frac{1}{c}\mathbf{D}(\mathbf{r},t)\ . \tag{2.10}$$

This fact, first recognized by Born and Infeld,[3(b)] leads to medium-independent commutation relations for the fields $\mathbf{D}$ and $\mathbf{B}$ (rather than $\mathbf{E}$ and $\mathbf{B}$).

The Hamiltonian (2.9) takes the form

$$\mathcal{H}=\tfrac{1}{2}\int d\mathbf{r}\left[\frac{c^2\mathbf{\Pi}(\mathbf{r},t)^2}{\epsilon(\mathbf{r})}+[\nabla\times\mathbf{A}(\mathbf{r},t)]^2\right]\ . \tag{2.11}$$

The equation of motion (2.6) can, in principle, be derived from Eq. (2.11) by means of a suitable commutation relation (Poisson bracket) between $\mathbf{A}(\mathbf{r},t)$ and $\mathbf{\Pi}(\mathbf{r},t)$. An equivalent and more elementary approach is to expand the fields in an appropriate set of mode functions and to find equations of motion for the expansion coefficients. We shall follow the latter method here. By analogy with the standard free-space quantization procedure, we shall expand the vector potential $\mathbf{A}$ in a set of vector functions $\mathbf{f}_k(\mathbf{r})$ that obey the eigenmode equations

$$\frac{\epsilon(\mathbf{r})\omega_k^2}{c^2}\mathbf{f}_k(\mathbf{r})-\nabla\times[\nabla\times\mathbf{f}_k(\mathbf{r})]=0\ , \tag{2.12a}$$

and the transversality condition

$$\nabla\cdot[\epsilon(\mathbf{r})\mathbf{f}_k(\mathbf{r})]=0\ , \tag{2.12b}$$

together with an appropriate set of boundary or asymptotic conditions. The asymptotic conditions, for example, could correspond to the formulation of a scattering problem with plane waves in an initial or a final state, or perhaps to the presence of standing waves. The parameter ω_k in Eq. (2.12a) is to be regarded as an eigenvalue, while the subscript k labels the available solutions. It may run through discrete values, as it does in ideal cavities, or through a continuum of values, as it does in unbounded space. In free space, for example, [$\epsilon(\mathbf{r})=1$], the index k may be taken to correspond to a pair of indices $(\mathbf{k},\mu)$, where $\mathbf{k}$ denotes the propagation vector of a plane wave, while μ labels its polarization. In the following we shall assume, to simplify the notation, that k runs through a discrete set of values; dealing with continuous spectra is then a matter of adjusting the notation.

The vector potential $\mathbf{A}$ is assumed to have an expansion

$$\mathbf{A}(\mathbf{r},t)=c\sum_k Q_k(t)\mathbf{f}_k(\mathbf{r}) , \tag{2.13}$$

in which the $Q_k(t)$ are regarded as a set of time-dependent coordinate operators. The full set of the solutions of Eqs. (2.12) can be chosen to fulfill an orthonormality condition. That property follows from the observation that under the substitution

$$\mathbf{f}_k(\mathbf{r})=\frac{1}{\sqrt{\epsilon(\mathbf{r})}}\mathbf{g}_k(\mathbf{r}) ,$$

Eq. (2.12) becomes

$$\frac{\omega_k^2}{c^2}\mathbf{g}_k(\mathbf{r})-\frac{1}{\sqrt{\epsilon(\mathbf{r})}}\nabla\times\left[\nabla\times\frac{\mathbf{g}_k(\mathbf{r})}{\sqrt{\epsilon(\mathbf{r})}}\right]=0 ,$$

so that the functions $\mathbf{g}_k$ are eigenvectors of a Hermitian differential operator. By choosing an orthonormal set of g's, we are led to the orthonormality condition

$$\int d\mathbf{r}\,\epsilon(\mathbf{r})\mathbf{f}_k(\mathbf{r})\cdot\mathbf{f}_{k'}^*(\mathbf{r})=\delta_{kk'} \tag{2.14a}$$

on the functions $\mathbf{f}_k(\mathbf{r})$.

It is a little more difficult to describe the corresponding completeness relation. The functions $\mathbf{g}_k$ obviously provide a complete set in the subspace of L^2 functions, that is defined by the gauge condition,

$$\nabla\cdot[\sqrt{\epsilon(\mathbf{r})}\mathbf{g}_k(\mathbf{r})]=0 .$$

The distribution $\sum_k \mathbf{g}_k(\mathbf{r})\cdot\mathbf{g}_k^*(\mathbf{r}')$ is therefore an identity on this subspace of functions.

We find it useful then to define the analogous distribution

$$\delta_{\alpha\beta}^{\epsilon}(\mathbf{r},\mathbf{r}')=\sum_k f_{k\alpha}(\mathbf{r})f_{k\beta}^*(\mathbf{r}) . \tag{2.14b}$$

In free space [i.e., when $\epsilon(\mathbf{r})=1$] the distribution (2.14b) reduces to a standard transverse δ function[17] defined as

$$\delta_{\alpha\beta}^T(\mathbf{r}-\mathbf{r}')=\frac{1}{(2\pi)^3}\int d\mathbf{k}\left[\delta_{\alpha\beta}-\frac{k_\alpha k_\beta}{k^2}\right]e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} . \tag{2.15}$$

The action of the transverse δ^T function can be explained as follows. Let $\mathbf{X}^T$ denote an arbitrary transverse vector field

$$\nabla\cdot\mathbf{X}^T=0 ,$$

while $\mathbf{X}^L$ denotes a longitudinal vector field

$$\nabla\times\mathbf{X}^L=0 .$$

We then have

$$\int d\mathbf{r}'\delta_{\alpha\beta}^T(\mathbf{r}-\mathbf{r}')\mathbf{X}_\beta^T(\mathbf{r}')=\mathbf{X}_\alpha^T(\mathbf{r}) ,$$

$$\int d\mathbf{r}'\delta_{\alpha\beta}^T(\mathbf{r}-\mathbf{r}')X_\beta^L(\mathbf{r}')=0 .$$

In the above expressions the summation convention with respect to the repeated vector indices (denoted by Greek letters $\alpha,\beta,\ldots$) has been used. The distribution δ^T transforms transverse vector fields into themselves without changing them.

The distribution defined by Eq. (2.14b) is a simple generalization of the transverse δ function (2.15). For any transverse vector field we have

$$\int d\mathbf{r}'\epsilon(\mathbf{r})\delta_{\alpha\beta}^{\epsilon}(\mathbf{r},\mathbf{r}')\mathbf{X}_\beta^T(\mathbf{r}')=\mathbf{X}_\alpha^T(\mathbf{r}) , \tag{2.16a}$$

while for a longitudinal vector field $\mathbf{X}^L$,

$$\int d\mathbf{r}'\delta_{\alpha\beta}^{\epsilon}(\mathbf{r},\mathbf{r}')\epsilon(\mathbf{r}')\mathbf{X}_\beta^L(\mathbf{r}')=0 . \tag{2.16b}$$

The vector potential $\mathbf{A}$ has to fulfill the reality condition $\mathbf{A}=\mathbf{A}^\dagger$, which implies

$$\sum_k Q_k\mathbf{f}_k(\mathbf{r})=\sum_k Q_k^\dagger f_k^*(\mathbf{r}) . \tag{2.17a}$$

By using the orthonormality condition (2.14a) we may easily derive from this identity the requirement

$$Q_k=\sum_{k'} Q_{k'}^\dagger U_{k'k}^* , \tag{2.17b}$$

in which the matrix U^* is defined as

$$U_{k'k}^*=\int d\mathbf{r}\,\epsilon(\mathbf{r})\mathbf{f}_{k'}^*(\mathbf{r})\cdot\mathbf{f}_k^*(\mathbf{r}) . \tag{2.18a}$$

It is clear from Eq. (2.14a) that the integrals $U_{kk'}^*$ are the expansion coefficients of the functions $\mathbf{f}_k^*(\mathbf{r})$ in terms of the $\mathbf{f}_k(\mathbf{r})$, i.e.,

$$\mathbf{f}_k^*(\mathbf{r})=\sum_{k'} U_{kk'}^*\mathbf{f}_{k'}(\mathbf{r}) . \tag{2.18b}$$

We shall presently see that when the functions $\mathbf{f}_k$ are chosen to fulfill appropriate asymptotic boundary conditions, the matrix U becomes related to the scattering matrix $\hat{S}$ for the classical scattering problem defined by Eq. (2.12).

The matrix U^* has three important properties.

(i) It is symmetric:

$$U_{kk'}^*=U_{k'k}^* . \tag{2.19a}$$

(ii) It is unitary:

$$\sum_{k'} U_{kk'}U_{k''k'}^*=\delta_{kk''} . \tag{2.19b}$$

These properties are derived from Eqs. (2.18) and (2.14a) together with the definition (2.14b) and its properties.

(iii) Finally, it vanishes everywhere off the "energy shell":

$$U_{kk'}\sim\delta(\omega_k-\omega_{k'}) . \tag{2.19c}$$

This latter property follows from the fact that the matrix elements $U_{kk'}$, defined as scalar products of the eigenmode solutions of Eq. (2.12), must vanish when the two solutions correspond to different eigenvalues.

We may construct analogously an expansion of the $\mathbf{\Pi}(\mathbf{r},t)$ field, which is canonically conjugate to $\mathbf{A}(\mathbf{r},t)$, by writing

$$\mathbf{\Pi}(\mathbf{r},t)=\frac{1}{c}\sum_k P_k(t)\epsilon(\mathbf{r})\mathbf{f}_k^*(\mathbf{r}) \ . \tag{2.20}$$

The need for a factor of $\epsilon(\mathbf{r})$ in this expression arises from the transversality condition $\nabla\cdot\mathbf{\Pi}(\mathbf{r},t)=0$. The reality condition $\mathbf{\Pi}=\mathbf{\Pi}^\dagger$ takes a form analogous to that of Eq. (2.18a),

$$\sum_k P_k \mathbf{f}_k^*(\mathbf{r})=\sum_k P_k^\dagger \mathbf{f}_k(\mathbf{r}) \ , \tag{2.21a}$$

and requires that

$$P_k^\dagger=\sum_{k'} P_{k'}U_{k'k}^* \ . \tag{2.21b}$$

When the expressions (2.13) and (2.20) are used to evaluate the Hamiltonian (2.11) in terms of the variables Q_k, $Q_k^\dagger$, P_k, and $P_k^\dagger$, we find it to reduce to the diagonal quadratic form

$$\mathcal{H}=\tfrac{1}{2}\sum_k (P_k^\dagger P_k+\omega_k^2 Q_k^\dagger Q_k) \ . \tag{2.22}$$

It is easily verified that the Maxwell equations follow from the Heisenberg equations of motion for the Q and P variables under the assumption of the following equal-time commutation relations (or Poisson brackets in the classical case):

$$[Q_k,Q_{k'}]=[Q_k^\dagger,Q_{k'}^\dagger]=[Q_k,Q_{k'}^\dagger]=0 \ , \tag{2.23a}$$

$$[P_k,P_{k'}]=[P_k^\dagger,P_{k'}^\dagger]=[P_k,P_{k'}^\dagger]=0 \ , \tag{2.23b}$$

$$[Q_k,P_{k'}]=i\hbar\delta_{kk'} \ . \tag{2.23c}$$

From the reality condition (2.21b) we can then derive the remaining commutation relation

$$[Q_k,P_{k'}^\dagger]=i\hbar U_{kk'}^* \ . \tag{2.23d}$$

The commutation relations (2.23) are equivalent to the canonical commutation relation for $\mathbf{A}(\mathbf{r})$ and $\mathbf{\Pi}(\mathbf{r}')$:

$$\begin{aligned}[A_\alpha(\mathbf{r}),\Pi_\beta(\mathbf{r}')]&=i\hbar\sum_k f_{k\alpha}(\mathbf{r})f_{k\beta}^*(\mathbf{r}')\epsilon(\mathbf{r})\\ &=i\hbar\delta_{\alpha\beta}^{\epsilon}(\mathbf{r},\mathbf{r}')\epsilon(\mathbf{r}') \ .\end{aligned} \tag{2.24}$$

The distribution δ^ϵ appearing in this commutator is precisely the generalization of the transverse δ function defined by the formula (2.14b).

The final step of our quantization procedure requires that we represent the commutation relations (2.23) in terms of photon creation and annihilation operators. In order to do that we assume that both Q_k and P_k are linear combinations of Hermitian conjugate creation and annihilation operators a_k and $a_k^\dagger$, which fulfill the canonical commutation relations

$$[a_k,a_{k'}]=0 \ , \tag{2.25a}$$

$$[a_k,a_{k'}^\dagger]=\delta_{kk'} \ . \tag{2.25b}$$

Both Q_k and P_k must fulfill the reality conditions stated by Eqs. (2.17b) and (2.21b). It is easy to check that the following representation of Q_k, $Q_k^\dagger$, P_k, and $P_k^\dagger$ fulfills the reality requirements:

$$Q_k=\left[\frac{\hbar}{2\omega_k}\right]^{1/2}\left[a_k+\sum_{k'}U_{kk'}^\dagger a_{k'}^\dagger\right] \ , \tag{2.26a}$$

$$P_k=i\left[\frac{\hbar\omega_k}{2}\right]^{1/2}\left[a_k^\dagger-\sum_{k'}U_{kk'}a_{k'}\right] \ . \tag{2.26b}$$

The expressions (2.25) and (2.26) realize, at the same time, a representation of the commutation relations (2.23). The proof that Eqs. (2.25) and (2.26) indeed assure the relations (2.23) requires use of the symmetry and unitarity properties of the matrix U.

When the expression (2.26) for Q_k and P_k are substituted in the Hamiltonian (2.22) and further use is made of the properties of the matrix U, we reach the familiar expression

$$\mathcal{H}=\tfrac{1}{2}\sum_k \hbar\omega_k(a_k^\dagger a_k+a_k a_k^\dagger)=\sum_k \hbar\omega_k a_k^\dagger a_k+C[\epsilon] \ . \tag{2.27}$$

The constant $C[\epsilon]$ depends functionally on the dielectric susceptibility $\epsilon(\mathbf{r})$ and is formally infinite. It does, however, contain important physical information; its derivatives with respect to geometrical parameters are related to the so-called Casimir forces,[18] the forces produced by the cumulative effects of zero-point oscillations.

The operators a_k and $a_k^\dagger$ obey the elementary equations of motion that give them the time dependences

$$a_k(t)=e^{-i\omega_k(t-t_0)}a_k(t_0) \ , \tag{2.28a}$$

$$a_k^\dagger(t)=e^{i\omega_k(t-t_0)}a_k^\dagger(t_0) \ . \tag{2.28b}$$

Expansion of any of the physical fields in terms of a_k and $a_k^\dagger$ then defines simultaneously the positive- and negative-frequency parts of the field. For example, by inserting the expressions (2.26a) and (2.26b) into the expressions

$$\mathbf{A}(\mathbf{r},t)=c\sum_k Q_k\mathbf{f}_k(\mathbf{r}) \ ,$$

$$\mathbf{E}(\mathbf{r},t)=-\sum_k P_k^\dagger\mathbf{f}_k(\mathbf{r}) \ ,$$

we obtain

$$\mathbf{A}(\mathbf{r},t)=c\sum_k\left[\frac{\hbar}{2\omega_k}\right]^{1/2}[a_k\mathbf{f}_k(\mathbf{r})+a_k^\dagger\mathbf{f}_k^*(\mathbf{r})] \ , \tag{2.29a}$$

$$\mathbf{E}(\mathbf{r},t)=i\sum_k\left[\frac{\hbar\omega_k}{2}\right]^{1/2}[a_k\mathbf{f}_k(\mathbf{r})-a_k^\dagger\mathbf{f}_k^*(\mathbf{r})] \ , \tag{2.29b}$$

and the positive- (negative-) frequency parts are simply equal to the terms in this sum proportional to a_k ($a_k^\dagger$):

$$\mathbf{A}^{(+)}(\mathbf{r},t)=+c\sum_k\left[\frac{\hbar}{2\omega_k}\right]^{1/2}a_k\mathbf{f}_k(\mathbf{r}) \ , \tag{2.30a}$$

$$\mathbf{A}^{(-)}(\mathbf{r},t)=+c\sum_k\left[\frac{\hbar}{2\omega_k}\right]^{1/2}a_k^\dagger\mathbf{f}_k^*(\mathbf{r}) \ , \tag{2.30b}$$

$$\mathbf{E}^{(+)}(\mathbf{r},t)=+i\sum_k\left[\frac{\hbar\omega_k}{2}\right]^{1/2}a_k\mathbf{f}_k(\mathbf{r}) \ , \tag{2.30c}$$

$$\mathbf{E}^{(-)}(\mathbf{r},t)=-i\sum_k\left[\frac{\hbar\omega_k}{2}\right]^{1/2}a_k^\dagger\mathbf{f}_k^*(\mathbf{r})\ . \tag{2.30d}$$

The expansion of the magnetic field analogous to Eq. (2.29) is

$$\mathbf{B}(\mathbf{r},t)=c\nabla\times\left[\sum_k\left[\frac{\hbar}{2\omega_k}\right]^{1/2}[a_k\mathbf{f}_k(\mathbf{r})+a_k^\dagger\mathbf{f}_k^*(\mathbf{r})]\right]\ . \tag{2.31}$$

As we can see, the quantization of the electromagnetic field in polarizable media in terms of the eigenmode solutions of Eq. (2.6) has a number of appealing properties. One of the most important of these is that the normal ordering of observables with respect to the eigenmode operators a_k and $a_k^\dagger$ corresponds to ordering with respect to positive- (negative-) frequency parts. The latter form of ordering has an important physical significance since it corresponds to the ordering of all the operator products measured in photon-counting experiments.[4]

It can also be useful, however, to carry out the quantization in ways that do not rely on the eigenmode expansion (2.13), but are based instead on expansion in some other set of functions. For example, if we consider a problem in which a plane electromagnetic wave is incident upon a dielectric medium, it seems natural to work with the plane-wave expansion of the field. The quantization procedure, based on such an expansion, will be described in Sec. III.

III. QUANTIZATION IN THE PRESENCE OF DIELECTRIC MEDIA —PLANE-WAVE EXPANSION

There are certain problems for which it is interesting to carry out the quantization procedure in an orthogonal basis quite different from that defined by Eq. (2.12). An example is the frequently occurring one in which the dielectric medium is finite in size and surrounded by free space. In that case it is often natural to use plane waves to describe both the incident and scattered quanta. In this section we shall discuss the quantization of the electromagnetic field, on the basis of the plane-wave expansion.

Our starting point will be the expansion of the vector potential $\mathbf{A}$ in terms of normalized plane-wave modes that obey periodic boundary conditions in a box of volume V. An expansion that is often used in free space takes the form

$$\mathbf{A}(\mathbf{r},t)=c\sum_k Q_k(t)\hat{\mathbf{e}}_k\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{V^{1/2}}\ . \tag{3.1}$$

In this expression the index k should be thought of as corresponding to a pair of indices $\mathbf{k}$ and μ, where $\mathbf{k}$ is a plane-wave propagation vector and μ is a polarization index. The $Q_k(t)$ are time-dependent operators that represent the complex mode amplitudes. The polarization vectors $\hat{\mathbf{e}}_k$ are transverse to the propagation vector, $\mathbf{k}\cdot\hat{\mathbf{e}}_k=0$. It is convenient to choose them in such a way that

$$\hat{\mathbf{e}}^*_{\mathbf{k},\mu}=\hat{\mathbf{e}}_{-\mathbf{k},\mu}\ . \tag{3.2}$$

It is exactly because of the transverse character of the plane waves, however, that the expansion (3.1) is inconvenient to use without alteration. That is because in inhomogeneous media it fails to fulfill the generalized Coulomb gauge condition (2.5b). That failure implies that the scalar potential Φ is necessarily different from zero in this case, since otherwise the Maxwell equation $\nabla\cdot\mathbf{D}=0$ would not be fulfilled. It is reasonable to assume, however, that by applying a gauge transformation to the vector potential defined by Eq. (3.1), we can satisfy the two conditions

$$\nabla\cdot[\epsilon(\mathbf{r})\mathbf{A}(\mathbf{r})]=0 \tag{3.3a}$$

and

$$\Phi(\mathbf{r},t)=0\ . \tag{3.3b}$$

In general, in classical electrodynamics a gauge transformation takes the form

$$\mathbf{A}\to\mathbf{A}+\nabla\psi\ ,$$

$$\Phi\to\Phi-\frac{1}{c}\dot{\psi}\ ,$$

where ψ is an arbitrary function of $\mathbf{r}$ and t. In the present quantum-mechanical case we must further require that ψ be an operator expressible as a linear combination of the independent coordinate operators Q_k, so that

$$\nabla\psi(\mathbf{r},t)=c\sum_k Q_k(t)\nabla g_k(\mathbf{r}) \tag{3.4}$$

for some suitable set of time-independent functions $g_k(\mathbf{r})$. After such a transformation the vector potential becomes

$$\mathbf{A}(\mathbf{r},t)=c\sum_k Q_k(t)\left[\hat{\mathbf{e}}_k\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{V^{1/2}}+\nabla g_k(\mathbf{r})\right]\ . \tag{3.5}$$

This gauge transformation is intended to turn the transverse vector field $\mathbf{A}(\mathbf{r},t)$ into one that satisfies the gauge condition (3.3) (without, of course, altering the fields $\mathbf{D}$, $\mathbf{E}$, or $\mathbf{B}$). The condition that the functions g_k must satisfy for this to be so is

$$\nabla\cdot\left[\epsilon(\mathbf{r})\left[\hat{\mathbf{e}}_k\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{V^{1/2}}+\nabla g_k(\mathbf{r})\right]\right]=0\ . \tag{3.6}$$

The reality property of $\mathbf{A}(\mathbf{r},t)$ requires, furthermore, that we choose the g_k to satisfy the relation

$$g^*_{k,\mu}=g_{-\mathbf{k},\mu}\ . \tag{3.7}$$

As an existence proof for this gauge transformation, we shall show presently how the functions g_k can be constructed.

Two convenient properties of the gauge transformation (3.5) are the following.

(i) The reality condition for $\mathbf{A}(\mathbf{r},t)$ may be reduced, by using Eqs. (3.2) and (3.7), to the simple condition

$$Q^\dagger_{\mathbf{k},\mu}(t)=Q_{-\mathbf{k},\mu}(t)\ . \tag{3.8a}$$

By introducing the abbreviation $k=(\mathbf{k},\mu)$ and

$-k=(-\mathbf{k},\mu)$ for pairs of indices, the above condition can be rewritten in the simpler form

$$Q_k^\dagger(t)=Q_{-k}(t)\ . \tag{3.8b}$$

(ii) The magnetic field $\mathbf{B}(\mathbf{r},t)$ is easily found to have the simple expansion

$$\mathbf{B}(\mathbf{r},t)=ic\sum_k Q_k(t)(\mathbf{k}\times\hat{\mathbf{e}}_k)\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{V^{1/2}}\ . \tag{3.8c}$$

The field $\mathbf{\Pi}(\mathbf{r},t)=-(1/c)\mathbf{D}(\mathbf{r},t)$ which is canonically conjugate to $\mathbf{A}(\mathbf{r},t)$, is also transverse, and can therefore be expanded as

$$\mathbf{\Pi}(\mathbf{r},t)=\frac{1}{c}\sum_k P_k(t)\hat{\mathbf{e}}_k^*\frac{e^{-i\mathbf{k}\cdot\mathbf{r}}}{V^{1/2}}\ . \tag{3.9}$$

By substituting this expansion and Eq. (3.5) for $\mathbf{A}(\mathbf{r},t)$ into the Hamiltonian (2.11), we can easily express the latter in terms of the variables Q_k and P_k and their adjoints. The result can be written in the form

$$\mathcal{H}=\tfrac{1}{2}\sum_{k,k'}P_k^\dagger M_{kk'}P_{k'}+\tfrac{1}{2}\sum_k \omega_k^2 Q_k^\dagger Q_k\ , \tag{3.10}$$

where the matrix $M_{kk'}$ is defined as

$$M_{kk'}=\frac{1}{V}\int d\mathbf{r}\,\hat{\mathbf{e}}_k\cdot\hat{\mathbf{e}}_{k'}^*\frac{e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}}{\epsilon(\mathbf{r})}\ . \tag{3.11}$$

The matrix $M_{kk'}$ can be used, furthermore, to construct the gauge transformation (3.5). It is important to note at this point that the matrix $M_{kk'}$ is invertible. That property is easily seen for $\epsilon(\mathbf{r})=1$, since the matrix M in that case is just the unit matrix

$$M_{kk'}=\delta_{kk'}\ .$$

It is then natural to expect that, for bounded functions $1/\epsilon(\mathbf{r})$, this property of M is preserved.

To prove the existence of the gauge transformation (3.5), let us assume that the functions g_k have the property

$$\nabla g_k(\mathbf{r})=\sum_{k'}M_{kk'}^{-1}\hat{\mathbf{e}}_{k'}\frac{1}{V^{1/2}}\frac{e^{i\mathbf{k}'\cdot\mathbf{r}}}{\epsilon(\mathbf{r})}-\hat{\mathbf{e}}_k\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{V^{1/2}}\ . \tag{3.12}$$

It is evident, in that case, that the functions g_k obey the Coulomb gauge constraint (3.6). Equation (3.12), however, admits solutions for g_k if and only if the curl of the right-hand side of Eq. (3.12) vanishes identically for all k:

$$\sum_{k'}M_{kk'}^{-1}\nabla\times\left[\hat{\mathbf{e}}_{k'}\frac{1}{V^{1/2}}\frac{e^{i\mathbf{k}'\cdot\mathbf{r}}}{\epsilon(\mathbf{r})}\right]-\nabla\times\left[\hat{\mathbf{e}}_k\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{V^{1/2}}\right]=0\ . \tag{3.13}$$

After matrix multiplication by M this relation becomes

$$\nabla\times\left[\hat{\mathbf{e}}_k\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\epsilon(\mathbf{r})}\right]=\nabla\times\left[\sum_{k'}M_{kk'}\hat{\mathbf{e}}_{k'}e^{i\mathbf{k}'\cdot\mathbf{r}}\right]\ . \tag{3.14}$$

According to the definition (3.11) of the matrix $M_{kk'}$, however, we can express the sum on the right in terms of the transverse δ function (2.15). If we use the summation convention for spatial indices, we can write

$$\begin{aligned}\left[\nabla\times\left[\sum_{k'}M_{kk'}\hat{\mathbf{e}}_{k'}e^{i\mathbf{k}'\cdot\mathbf{r}}\right]\right]_\alpha &=\epsilon_{\alpha\beta\gamma}\frac{\partial}{\partial x_\beta}\int d\mathbf{r}'\delta^T_{\gamma\rho}(\mathbf{r}-\mathbf{r}')\hat{\mathbf{e}}_{k\rho}\frac{e^{i\mathbf{k}\cdot\mathbf{r}'}}{\epsilon(\mathbf{r}')}\\ &=\left[\nabla\times\left[\hat{\mathbf{e}}_k\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\epsilon(\mathbf{r})}\right]\right]_\alpha\ .\end{aligned} \tag{3.15}$$

The latter equality, which demonstrates the desired relation (3.13), follows from the fact that curl of the transverse δ function is equal to the curl of the normal δ function. From Eq. (3.15) we conclude that the condition (3.12) does indeed admit solutions for g_k. We note that this solution is also consistent with the reality condition (3.7). We have thus proved the existence of the gauge transformation (3.6).

Having discussed the required gauge transformation, we may postulate a set of commutation relations analogous to those of Eq. (2.23), that is

$$[Q_k,Q_{k'}]=[Q_k,Q_{k'}^\dagger]=0\ , \tag{3.16a}$$

$$[P_k,P_{k'}]=[P_k,P_{k'}^\dagger]=0\ , \tag{3.16b}$$

$$[Q_k,P_{k'}]=i\hbar\delta_{kk'}\ . \tag{3.16c}$$

The above commutation relations have a standard representation in terms of creation and annihilation operators $b_k^\dagger$ and b_k,

$$Q_k=\left[\frac{\hbar}{2\omega_k}\right]^{1/2}(b_k+b_{-k}^\dagger)\ , \tag{3.17a}$$

$$P_k=i\left[\frac{\hbar\omega_k}{2}\right]^{1/2}(b_k^\dagger-b_{-k})\ , \tag{3.17b}$$

and these expressions lead to a standard plane-wave expansion for the fields

$$\mathbf{\Pi}(\mathbf{r},t)=\frac{i}{c}\sum_k\left[\frac{\hbar\omega_k}{2V}\right]^{1/2}(b_k^\dagger\hat{\mathbf{e}}_k^*e^{-i\mathbf{k}\cdot\mathbf{r}}-b_k\hat{\mathbf{e}}_k e^{i\mathbf{k}\cdot\mathbf{r}})\ , \tag{3.18a}$$

$$\mathbf{B}(\mathbf{r},t)=-ic\sum_k\left[\frac{\hbar}{2\omega_k V}\right]^{1/2}\mathbf{k}(b_k^\dagger\hat{\mathbf{e}}_k^*e^{-i\mathbf{k}\cdot\mathbf{r}}-b_k\hat{\mathbf{e}}_k e^{i\mathbf{k}\cdot\mathbf{r}})\ . \tag{3.18b}$$

The Hamiltonian, when expressed in terms of the operators b_k and $b_k^\dagger$, takes the nondiagonal form

$$\begin{aligned}\mathcal{H}=&\sum_k\hbar\omega_k b_k^\dagger b_k\\ &+\frac{\hbar}{4}\left[\sum_{k,k'}b_k^\dagger\sqrt{\omega_k\omega_{k'}}\hat{\mathbf{e}}_k^*\cdot\hat{\mathbf{e}}_{k'}^*v^*(\mathbf{k}+\mathbf{k}')b_{k'}^\dagger\right.\\ &\qquad\left.-\sum_{k,k'}b_k^\dagger\sqrt{\omega_k\omega_{k'}}\hat{\mathbf{e}}_k^*\cdot\hat{\mathbf{e}}_{k'}v(\mathbf{k}'-\mathbf{k})b_{k'}+\text{H.c.}\right]\\ &+\tilde{C}[\epsilon]\ ,\end{aligned} \tag{3.19}$$

in which the function v is defined by

$$v(\mathbf{k})=\frac{1}{V}\int d\mathbf{r}\frac{\epsilon(\mathbf{r})-1}{\epsilon(\mathbf{r})}e^{i\mathbf{k}\cdot\mathbf{r}}\ , \tag{3.20}$$

and $\tilde{C}[\epsilon]$ represents a c-number functional of the dielectric susceptibility $\epsilon(\mathbf{r})$.

The expression (3.19) for the Hamiltonian indicates an important feature of the plane-wave expansion. It shows that the equations of motion for the operators b_k and $b_k^\dagger$ must contain a coupling of b_k's to the $b_k^\dagger$'s, whenever $\epsilon(\mathbf{r})$ deviates from unity. This fact implies that the plane-wave amplitudes do not simply oscillate, as they do in free space, with definite signs of the frequency. In fact, both b_k and $b_k^\dagger$ contain, in general, Fourier components of both negative and positive frequencies. This mixing of the signs of plane-wave frequencies is a general feature of scattering phenomena, described by wave equations that are of second order in the time derivative. Normal ordering of the operators b_k and $b_k^\dagger$ therefore, no longer has any simple relation to the ordering with respect to positive and negative frequencies, which is required for the description of photon-counting measurements carried out in dielectric media.

The n-quantum states of the system defined in terms of physical photons correspond to those generated by the eigenmode operators a_k and $a_k^\dagger$. Because of the presence of the $b_k^\dagger b_{k'}^\dagger$ terms and their conjugates in the Hamiltonian (3.19), those states may contain unlimited numbers of pairs of virtual photons of the kind generated by the plane-wave operators b_k and $b_k^\dagger$. The latter photons, however, because of their partially virtual nature, are not in general the ones registered by any photon counter. Absorbing or annihilating a physical photon, for example, may mean creating one of the plane-wave photons. The plane-wave representation, nevertheless, does have a number of useful formal properties. We shall examine some of them in Sec. IV.

IV. INTERPRETATION OF PLANE-WAVE QUANTIZATION PROCEDURE IN SCATTERING THEORY

In this section we shall show how the plane-wave photons, which are generated by the operators $b_k^\dagger$ and b_k of Sec. III, and which contain virtual as well as real excitations of the electromagnetic field, can be related within the framework of scattering theory to the physical photons that correspond to the operators $a_k^\dagger$ and a_k of Sec. II. Since the problem we are considering is a linear one, it is clear that there must exist a linear transformation connecting the physical and the plane-wave photons. Let us assume that it takes the general form

$$b_k=\sum_{k'}[A(k,k')a_{k'}+B(k,k')a_{k'}^\dagger]\ . \tag{4.1}$$

We shall show that the inverse relation, which expresses the physical operators in terms of the plane-wave operators, takes a closely analogous form involving the same coefficient matrices. The transformation must preserve the commutation relations, and corresponds thus to a unitary transformation of the quantum-mechanical state vectors

$$|\Psi\rangle'=U|\Psi\rangle$$

and of the related operators

$$O'=UOU^\dagger\ .$$

These conditions imply the existence of orthogonality and completeness relations for the coefficients $A(k,k')$ and $B(k,k')$. From the vanishing of the commutator $[b_k,b_{k''}]$ and the canonical commutation relations for the a_k and $a_{k'}^\dagger$, we find the relation

$$\sum_{k'}[A(k,k')B(k'',k')-B(k,k')A(k'',k')]=0\ . \tag{4.2a}$$

From the commutation relation of b_k and $b_{k''}^\dagger$ we likewise find

$$\sum_{k'}[A(k,k')A^*(k'',k')-B(k,k')B^*(k'',k')]=\delta_{kk''}\ . \tag{4.2b}$$

It is convenient at this point to introduce the matrix notation $\underline{A}$ for $A(k,k')$ and $\underline{B}$ for $B(k,k')$. The transformation of the a_k and $a_k^\dagger$ operators into the b_k and $b_k^\dagger$ can then be regarded as the vector transformation

$$\begin{bmatrix} b \\ b^\dagger \end{bmatrix}=\underline{M}\begin{bmatrix} a \\ a^\dagger \end{bmatrix}\ , \tag{4.3}$$

where $\underline{M}$ is the supermatrix

$$\underline{M}=\begin{bmatrix} \underline{A} & \underline{B} \\ \underline{B}^* & \underline{A}^* \end{bmatrix}\ . \tag{4.4}$$

The relations (4.2a) and (4.2b) can now be abbreviated as the matrix identities

$$\underline{A}\,\underline{B}^T-\underline{B}\,\underline{A}^T=0\ , \tag{4.5a}$$

$$\underline{A}\,\underline{A}^\dagger-\underline{B}\,\underline{B}^\dagger=\underline{\mathbb{1}}\ , \tag{4.5b}$$

where the index T stands for the transposed form of the matrix. If we now introduce the supermatrix

$$\underline{G}=\begin{bmatrix} \underline{\mathbb{1}} & 0 \\ 0 & -\underline{\mathbb{1}} \end{bmatrix}\ , \tag{4.6}$$

and the Hermitian adjoint of M,

$$\underline{M}^\dagger=\begin{bmatrix} \underline{A}^\dagger & \underline{B}^T \\ \underline{B}^\dagger & \underline{A}^T \end{bmatrix}\ , \tag{4.7}$$

we find that the relations (4.5) imply the identity

$$\underline{M}\,\underline{G}\,\underline{M}^\dagger\underline{G}=\underline{\mathbb{1}}\ . \tag{4.8}$$

It follows from this relation that the determinant of $\underline{M}$ cannot vanish. In fact, we have

$$|\det\underline{M}|^2=1\ . \tag{4.9}$$

The supermatrix $\underline{M}$ therefore has an inverse, and Eq. (4.8) shows that it can only be

$$\underline{M}^{-1}=\underline{G}\,\underline{M}^\dagger\underline{G}=\begin{bmatrix} \underline{A}^\dagger & -\underline{B}^T \\ -\underline{B}^\dagger & \underline{A}^T \end{bmatrix}\ . \tag{4.10}$$

The matrices $\underline{A}$ and $\underline{B}$, in other words, must obey the identities implicit in the relation[19]

$$\underline{G}\,\underline{M}^{\dagger}\underline{G}\,\underline{M}=\mathbb{1}\ , \tag{4.11a}$$

that is to say

$$\underline{A}^{\dagger}\underline{B}-\underline{B}^{T}\underline{A}^{*}=0\ , \tag{4.11b}$$

$$\underline{A}^{\dagger}\underline{A}-\underline{B}^{T}\underline{B}^{*}=\mathbb{1}\ , \tag{4.11c}$$

or more explicitly

$$\sum_{k'}[A^{*}(k',k)B(k',k'')-B(k',k)A^{*}(k',k'')]=0\ , \tag{4.12a}$$

$$\sum_{k'}[A^{*}(k',k)A(k',k'')-B(k',k)B^{*}(k',k'')]=\delta_{kk''}\ . \tag{4.12b}$$

Furthermore, the relation inverse to Eqs. (4.1) and (4.3) is given, according to Eq. (4.10), by

$$a_k=\sum_{k'}[A^{*}(k',k)b_{k'}-B(k',k)b_{k'}^{\dagger}]\ . \tag{4.13}$$

Thus the expression for the physical operators in terms of the plane-wave operators is no more complicated in form than its inverse.

The transformation (4.1) has the further property that it relates the eigenmode expansion of the electromagnetic field (2.29) and (2.31) to the plane-wave expansion (3.18). Let us consider the fields $\mathbf{\Pi}(\mathbf{r},t)$ of Eq. (3.18a), and $\mathbf{B}(\mathbf{r},t)$ of Eq. (3.18b), which are given by

$$\mathbf{\Pi}(\mathbf{r},t)=\frac{i}{c}\sum_{k'}\left[\frac{\hbar\omega_{k'}}{2V}\right]^{1/2}(b_{k'}^{\dagger}\hat{\mathbf{e}}_{k'}^{*}e^{-i\mathbf{k}'\cdot\mathbf{r}}-b_{k'}\hat{\mathbf{e}}_{k'}e^{+i\mathbf{k}'\cdot\mathbf{r}})\ , \tag{4.14a}$$

$$\mathbf{B}(\mathbf{r},t)=-ic\sum_{k'}\left[\frac{\hbar}{2\omega_{k'}V}\right]^{1/2}[b_{k'}^{\dagger}(\mathbf{k}'\times\hat{\mathbf{e}}_{k'}^{*})e^{-i\mathbf{k}'\cdot\mathbf{r}} -b_{k'}(\mathbf{k}'\times\hat{\mathbf{e}}_{k'})e^{+i\mathbf{k}'\cdot\mathbf{r}}]\ . \tag{4.14b}$$

When we insert the relation (4.1) these expressions become

$$\mathbf{\Pi}(\mathbf{r},t)=\frac{i}{c}\left[\sum_{k,k'}\left[\frac{\hbar\omega_{k'}}{2V}\right]^{1/2}[\hat{\mathbf{e}}_{k'}^{*}e^{-i\mathbf{k}'\cdot\mathbf{r}}B^{*}(k',k) -\hat{\mathbf{e}}_{k'}e^{+i\mathbf{k}'\cdot\mathbf{r}}A(k',k)]a_k -\mathrm{H.c.}\right]\ , \tag{4.14c}$$

$$\mathbf{B}(\mathbf{r},t)=-ic\left[\sum_{k,k'}\left[\frac{\hbar}{2\omega_{k'}V}\right]^{1/2}\times[(\mathbf{k}'\times\hat{\mathbf{e}}_{k'}^{*})e^{-i\mathbf{k}'\cdot\mathbf{r}}B^{*}(k',k) -(\mathbf{k}'\times\hat{\mathbf{e}}_{k'})e^{+i\mathbf{k}'\cdot\mathbf{r}}A(k',k)]a_k -\mathrm{H.c.}\right]\ . \tag{4.14d}$$

On the other hand, Eqs. (2.10), (2.29), and (2.31) state that

$$\mathbf{\Pi}(\mathbf{r},t)=-\frac{i}{c}\epsilon(\mathbf{r})\sum_{k}\left[\frac{\hbar\omega_k}{2}\right]^{1/2}[a_k\mathbf{f}_k(\mathbf{r})-a_k^{\dagger}\mathbf{f}_k^{*}(\mathbf{r})]\ , \tag{4.15a}$$

$$\mathbf{B}(\mathbf{r},t)=c\sum_{k}\left[\frac{\hbar}{2\omega_k}\right]^{1/2}\{a_k[\nabla\times\mathbf{f}_k(\mathbf{r})]+a_k^{\dagger}[\nabla\times\mathbf{f}_k^{*}(\mathbf{r})]\}\ . \tag{4.15b}$$

The coefficients multiplying a_k on the right-hand side of Eqs. (4.14c) and (4.14d) must be equal, therefore, to the ones on the right-hand side of Eqs. (4.15); that is, we must have

$$\sum_{k'}\left[\frac{\hbar\omega_{k'}}{2V}\right]^{1/2}\hat{\mathbf{e}}_{k'}^{*}e^{-i\mathbf{k}'\cdot\mathbf{r}}[B^{*}(k',k)-A(-k',k)] =-\left[\frac{\hbar\omega_k}{2}\right]^{1/2}\epsilon(\mathbf{r})\mathbf{f}_k(\mathbf{r})\ , \tag{4.16a}$$

$$i\sum_{k'}\left[\frac{\hbar}{2\omega_{k'}V}\right]^{1/2}(\mathbf{k}'\times\hat{\mathbf{e}}_{k'}^{*})e^{-i\mathbf{k}'\cdot\mathbf{r}}\times[B^{*}(k',k)+A(-k',k)] =-\left[\frac{\hbar}{2\omega_k}\right]^{1/2}[\nabla\times\mathbf{f}_k(\mathbf{r})]\ . \tag{4.16b}$$

By using the orthogonality of plane-wave modes we then obtain from Eq. (4.16)

$$A(-k',k)-B^{*}(k',k)=\left[\frac{\omega_k}{\omega_{k'}}\right]^{1/2}\hat{v}_k(k')\ , \tag{4.17a}$$

$$A(-k',k)+B^{*}(k',k)=\left[\frac{\omega_{k'}}{\omega_k}\right]^{1/2}v_k(k')\ . \tag{4.17b}$$

In the above expressions we have introduced the transverse Fourier transforms of the eigenmode functions $\mathbf{f}_k$ and $\epsilon(\mathbf{r})\mathbf{f}_k$,

$$v_k(k')=\int d\mathbf{r}\,\mathbf{f}_k(\mathbf{r})\cdot\hat{\mathbf{e}}_{k'}\frac{e^{i\mathbf{k}'\cdot\mathbf{r}}}{V^{1/2}}\ , \tag{4.18a}$$

$$\hat{v}_k(k')=\int d\mathbf{r}\,\epsilon(\mathbf{r})\mathbf{f}_k(\mathbf{r})\cdot\hat{\mathbf{e}}_{k'}\frac{e^{i\mathbf{k}'\cdot\mathbf{r}}}{V^{1/2}}\ . \tag{4.18b}$$

We may now construct a unique solution for the coefficient functions $A(k',k)$ and $B(k',k)$ which obey the conditions (4.2), (4.12), and (4.17),

$$A(k',k)=\frac{1}{2}\left[\left[\frac{\omega_{k'}}{\omega_k}\right]^{1/2}v_k(-k')+\left[\frac{\omega_k}{\omega_{k'}}\right]^{1/2}\hat{v}_k(-k')\right], \tag{4.19a}$$

$$B^*(k',k)=\frac{1}{2}\left[\left[\frac{\omega_{k'}}{\omega_k}\right]^{1/2}v_k(k')-\left[\frac{\omega_k}{\omega_{k'}}\right]^{1/2}\hat{v}_k(k')\right]. \tag{4.19b}$$

It is straightforward to verify that these coefficients obey the conditions (4.17) and correctly convert the plane-wave expansion of the fields into the eigenmode expansion. It requires a little more calculation in order to show that conditions (4.2) and (4.12) hold. Indeed we have, for example,

$$\sum_{k'}[A(k',k)A^*(k',k'')-B(k',k'')B^*(k',k)]$$
$$=\frac{1}{2}\sum_{k'}\left[\left[\frac{\omega_k}{\omega_{k''}}\right]^{1/2}\hat{v}_k(k')v^*_{k''}(k')+\left[\frac{\omega_{k''}}{\omega_k}\right]^{1/2}v_k(k')\hat{v}^*_{k''}(k')\right]. \tag{4.20}$$

If we next note that

$$\sum_{k'}\hat{v}_k(k')v^*_{k''}(k')=\int\int d\mathbf{r}\,d\mathbf{r}'\epsilon(\mathbf{r})f_{k\alpha}(\mathbf{r})\delta^T_{\alpha\beta}(\mathbf{r}-\mathbf{r}')\times f^*_{k''\beta}(\mathbf{r}')=\delta_{kk''}, \tag{4.21}$$

we reduce Eq. (4.20) to the form of Eq. (4.12b). To evaluate the integrals in Eq. (4.21) we have used the gauge condition $\nabla\cdot[\epsilon(\mathbf{r})\mathbf{f}_k(\mathbf{r})]=0$, and the fact that the transverse δ^T function applied to a transverse vector field $\epsilon(\mathbf{r})\mathbf{f}_k(\mathbf{r})$ acts as an identity.

As we see, for any choice of the eigenmode solutions (i.e., incoming waves, outgoing waves, etc.) the relations (4.19) establish a unique relation between $a_k,a_k^\dagger$ and $b_k,b_k^\dagger$. For some specific choices, however, this relation has an additional physical meaning.

Let us suppose that the dielectric is localized in space $[\epsilon(\mathbf{r})\to 1$ for $|\mathbf{r}|\to\infty]$ and that the interaction is turned on and off adiabatically. That means that in fact ϵ is a slowly varying function of time and obeys

$$\epsilon(\mathbf{r},t)\to 1$$

for $t\to\pm\infty$ with $\epsilon(\mathbf{r},0)=\epsilon(\mathbf{r})$.

Under such assumptions one can solve the Heisenberg equations of motion for the operators b_k and $b_k^\dagger$, which are governed by the Hamiltonian (3.19). Let us denote the corresponding operators in the interaction picture by $\tilde{b}_k$ and $\tilde{b}_k^\dagger$:

$$b_k(t)=e^{-i\omega_k t}\tilde{b}_k(t), \tag{4.22a}$$

$$b_k^\dagger(t)=e^{i\omega_k t}\tilde{b}^\dagger_k(t). \tag{4.22b}$$

Then the relation between $\tilde{b}_k(0)$ and $\tilde{b}_k(-\infty)\equiv b_k^{\rm in}$ must take a form similar to Eq. (4.1),

$$\tilde{b}_k(0)=\sum_{k}[A(k',k)b_{k'}^{\rm in}+B(k',k)(b_{k'}^{\rm in})^\dagger]. \tag{4.23}$$

The coefficients A and B must then fulfill relations analogous to those of Eqs. (4.2) and (4.12) because the time evolution of the operators preserves the commutators.

The displacement field $\mathbf{D}$ and the magnetic field $\mathbf{B}$ can be represented at $t=0$ as

$$\mathbf{D}(\mathbf{r},0)=-i\sum_{k'}\left[\frac{\hbar\omega_{k'}}{2V}\right]^{1/2}[\tilde{b}^\dagger_{k'}(0)\hat{\mathbf{e}}^*_{k'}e^{-i\mathbf{k}'\cdot\mathbf{r}}-\tilde{b}_{k'}(0)\hat{\mathbf{e}}_{k'}e^{+i\mathbf{k}'\cdot\mathbf{r}}], \tag{4.24a}$$

$$\mathbf{B}(\mathbf{r},0)=-ic\sum_{k'}\left[\frac{\hbar}{2\omega_{k'}V}\right]^{1/2}\times[\tilde{b}^\dagger_{k'}(0)(\mathbf{k}'\times\hat{\mathbf{e}}^*_{k'})e^{-i\mathbf{k}'\cdot\mathbf{r}}-\tilde{b}_{k'}(0)(\mathbf{k}'\times\hat{\mathbf{e}}_{k'})e^{+i\mathbf{k}'\cdot\mathbf{r}}]. \tag{4.24b}$$

For $t\to\infty$ these finds have an asymptotic expansion

$$\mathbf{D}(\mathbf{r},t)=-i\sum_{k'}\left[\frac{\hbar\omega_{k'}}{2V}\right]^{1/2}[(b_{k'}^{\rm in})^\dagger\hat{\mathbf{e}}^*_{k'}e^{-i\mathbf{k}'\cdot\mathbf{r}+i\omega_{k'}t}-b_{k'}^{\rm in}\hat{\mathbf{e}}_{k'}e^{+i\mathbf{k}'\cdot\mathbf{r}-i\omega_{k'}t}], \tag{4.25a}$$

$$\mathbf{B}(\mathbf{r},t)=-ic\sum_{k'}\left[\frac{\hbar}{2\omega_{k'}V}\right]^{1/2}\times[\tilde{b}^\dagger_{k'}(0)(\mathbf{k}'\times\hat{\mathbf{e}}^*_{k'})e^{-i\mathbf{k}'\cdot\mathbf{r}}-\tilde{b}_{k'}(0)(\mathbf{k}'\times\hat{\mathbf{e}}_{k'})e^{+i\mathbf{k}'\cdot\mathbf{r}}]. \tag{4.24b}$$

When we deal with a time-dependent dielectric susceptibility function $\epsilon(\mathbf{r},t)$, the mode functions we choose must, strictly speaking, obey an explicitly time-dependent form of the wave equation. In the adiabatic limit, however, that is, when we can neglect terms involving the time derivative of $\epsilon(\mathbf{r},t)$, we can write the mode functions in the form $e^{-i\omega_k t}\mathbf{f}_k(\mathbf{r},t)$, where $\mathbf{f}_k(\mathbf{r},t)$ obeys the equation

$$\frac{\epsilon(\mathbf{r},t)\omega_k^2}{c^2}\mathbf{f}_k(\mathbf{r},t)-\nabla\times[\nabla\times\mathbf{f}_k(\mathbf{r},t)]=0. \tag{4.26}$$

We note then that $\mathbf{f}(\mathbf{r},t)$ obeys a different wave equation analogous to Eq. (2.12a) at each instant of time and that its variation with time is adiabatically slow. At time $t\to-\infty$, since $\epsilon(\mathbf{r},t)\to 1$, the asymptotic solution can be taken to be time independent and to obey

$$\mathbf{f}_k(\mathbf{r},t)\to\hat{\mathbf{e}}_k\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{V^{1/2}}. \tag{4.27}$$

It is also necessary to specify spatial boundary conditions for Eq. (4.26) at all later times. A natural choice for those conditions is one that corresponds to a plane wave plus outgoing spherical waves. It is that choice that describes the evolution of a normal scattering state at time $t=0$ from an initial plane wave. An alternative choice that is also useful corresponds to a plane wave at $t\to+\infty$ plus the combination of a plane wave with incoming spherical waves at all earlier times. It is the latter choice that describes the state that must be present at time $t=0$ in order to have a plane wave present at

$t \to +\infty$.

Let us consider now the eigenmode functions, which obey the first of the above-mentioned boundary conditions, with the asymptotic behavior given by Eq. (4.27). Let us denote these functions by $\mathbf{f}_k^{\text{in}}(\mathbf{r},t)$. At any instant of time the fields **D** and **B** may be expanded uniquely in terms of the $\mathbf{f}_k^{\text{in}}(\mathbf{r},t)$, which form a specific example of the basis discussed in Sec. II,

$$\mathbf{D}(\mathbf{r},t)=i\epsilon(\mathbf{r},t)\sum_k \left[\frac{\hbar\omega_k}{2}\right]^{1/2}[a_k(t)\mathbf{f}_k^{\text{in}}(\mathbf{r},t)-a_k^\dagger(t)(\mathbf{f}_k^{\text{in}})^*(\mathbf{r},t)] , \tag{4.28a}$$

$$\mathbf{B}(\mathbf{r},t)=c\nabla\times\left[\sum_k \left[\frac{\hbar}{2\omega_k}\right]^{1/2}\times[a_k(t)\mathbf{f}_k^{\text{in}}(\mathbf{r},t)+a_k^\dagger(t)(\mathbf{f}_k^{\text{in}})^*(\mathbf{r},t)]\right] . \tag{4.28b}$$

Since the functions $\mathbf{f}_k^{\text{in}}(\mathbf{r},t)$ contain the effects of a slow variation of $\epsilon(\mathbf{r},t)$, the annihilation operators $a_k(t)$ and creation operators $a_k^\dagger(t)$ can be seen to have a very simple time dependence. Let us write them as

$$a_k(t)=e^{-i\omega_k t}\tilde{a}_k , \tag{4.29a}$$

$$a_k^\dagger(t)=e^{+i\omega_k t}\tilde{a}_k^\dagger , \tag{4.29b}$$

where $\tilde{a}_k$ is the interaction picture operator corresponding to $a_k(t)$, and is extremely slowly varying in the adiabatic limit. Then, by comparing Eq. (4.25) with Eqs. (4.28) and (4.29), we can immediately make the identifications

$$\tilde{a}_k=b_k^{\text{in}} , \tag{4.30a}$$

$$\tilde{a}_k^\dagger=(b_k^{\text{in}})^\dagger , \tag{4.30b}$$

so that the expression (4.28) takes at time $t=0$ the form

$$\mathbf{D}(\mathbf{r},0)=i\epsilon(\mathbf{r},t)\sum_k \left[\frac{\hbar\omega_k}{2}\right]^{1/2}\times[b_k^{\text{in}}\mathbf{f}_k^{\text{in}}(\mathbf{r},0)-(b_k^{\text{in}})^\dagger(\mathbf{f}_k^{\text{in}})^*(\mathbf{r},0)] , \tag{4.31a}$$

$$\mathbf{B}(\mathbf{r},0)=c\nabla\times\left[\sum_k \left[\frac{\hbar}{2\omega_k}\right]^{1/2}\times[b_k^{\text{in}}\mathbf{f}_k^{\text{in}}(\mathbf{r},0)+(b_k^{\text{in}})^\dagger(\mathbf{f}_k^{\text{in}})^*(\mathbf{r},0)]\right] . \tag{4.31b}$$

We may now use the same method as the one used for deriving the expressions (4.19) in order to calculate the coefficients A and B that appear in Eq. (4.23). We thus obtain

$$A(k',k)=\tfrac{1}{2}\left[\left[\frac{\omega_{k'}}{\omega_k}\right]^{1/2}\Omega_-(k,k')+\left[\frac{\omega_k}{\omega_{k'}}\right]^{1/2}\hat{\Omega}_-(k,k')\right] , \tag{4.32a}$$

$$B^*(k',k)=\tfrac{1}{2}\left[\left[\frac{\omega_{k'}}{\omega_k}\right]^{1/2}\Omega_-(k,k')-\left[\frac{\omega_k}{\omega_{k'}}\right]^{1/2}\hat{\Omega}_-(k,k')\right] . \tag{4.32b}$$

The coefficients that appear on right-hand side of Eqs. (4.32) can be identified with the elements of the Møller matrices[2] $\Omega_-(k,k')$ and $\hat{\Omega}_-(k,k')$. Namely,

$$\Omega_-(k,k')=\int d\mathbf{r}\,\mathbf{f}_k^{\text{in}}(\mathbf{r},0)\cdot\hat{\mathbf{e}}_{k'}\frac{e^{i\mathbf{k}'\cdot\mathbf{r}}}{V^{1/2}} , \tag{4.33a}$$

$$\hat{\Omega}_-(k,k')=\int d\mathbf{r}\,\epsilon(\mathbf{r})\mathbf{f}_k^{\text{in}}(\mathbf{r},0)\cdot\hat{\mathbf{e}}_{k'}\frac{e^{i\mathbf{k}'\cdot\mathbf{r}}}{V^{1/2}} , \tag{4.33b}$$

are the elements of matrices that express the solution of the full scattering problem [corresponding to the asymptotic condition (4.27) at $t\to\infty$] in terms of plane waves. Indeed, it is easy to check that the transverse wave functions obey

$$\epsilon(\mathbf{r})\mathbf{f}_k^{\text{in}}(\mathbf{r},0)=\sum_{k'}\hat{\Omega}_-(k,k')\hat{\mathbf{e}}_{k'}\frac{e^{i\mathbf{k}'\cdot\mathbf{r}}}{V^{1/2}} . \tag{4.33c}$$

The matrix $\hat{\Omega}_-$ evidently is not unitary, because of the role of the function $\epsilon(\mathbf{r})$ in the orthonormality condition (2.14a). The matrix adjoint to $\hat{\Omega}_-$ with respect to that scalar product has the elements

$$\Omega_-^*(k,k')=\int d\mathbf{r}[\mathbf{f}_k^{\text{in}}(\mathbf{r},0)]^*\cdot\hat{\mathbf{e}}_{k'}^*\frac{e^{-i\mathbf{k}'\cdot\mathbf{r}}}{V^{1/2}} . \tag{4.34a}$$

That it is an inverse of $\hat{\Omega}_-$ is evident from the relation

$$\sum_{k'}\Omega_-^*(k,k')\hat{\Omega}_-(k'',k')=\delta_{kk''} . \tag{4.34b}$$

Obviously Ω_-^* may be used to express the transverse plane waves in terms of the transverse functions $\epsilon(\mathbf{r})\mathbf{f}_k^{\text{in}}(\mathbf{r},0)$.

We may summarize our findings as follows.

(a) If the eigenmode functions $\mathbf{f}_k^{\text{in}}(\mathbf{r},t)$ are taken to correspond to plane waves plus outgoing spherical waves for all t, and to reduce to a single plane wave at $t=-\infty$, then the relation (4.1), which connects the plane-wave photons (defined through b_k and $b_k^\dagger$) to the physical photons (defined through a_k and $a_k^\dagger$), can be interpreted in terms of an action of the Møller matrices Ω_- and $\hat{\Omega}_-$ by making the identifications

$$\tilde{a}_k=\tilde{b}_k(-\infty)=b_k^{\text{in}} , \tag{4.35a}$$

$$b_k=\tilde{b}_k(0) , \tag{4.35b}$$

and using the expressions (4.32).

(b) Analogously, if the eigenmode functions $\mathbf{f}_k^{\text{out}}(\mathbf{r},t)$ are taken to correspond to plane waves plus incoming spherical waves for all t, and to reduce to a single plane wave at $t=+\infty$, then the relation (4.13), which connects the physical photons (defined through a_k and $a_k^\dagger$) to the plane wave photons (defined through b_k and $b_k^\dagger$), can be interpreted as an action of the Møller matrices Ω_+ and $\hat{\Omega}_+$ by making the identifications

$$\tilde{a}_k=\tilde{b}_k(+\infty)=b_k^{\text{out}} , \tag{4.36a}$$

$$b_k=\tilde{b}_k(0) , \tag{4.36b}$$

and replacing Ω_- and $\hat{\Omega}_-$ by Ω_+ and $\hat{\Omega}_+$ in the expressions (4.32). The matrices Ω_+ and $\hat{\Omega}_+$ are defined by the expressions

$$\Omega_+(k,k')=\int d\mathbf{r}\, \mathbf{f}_k^{\text{out}}(\mathbf{r},0)\cdot\hat{\mathbf{e}}_{k'}\frac{e^{i\mathbf{k}'\cdot\mathbf{r}}}{V^{1/2}} , \tag{4.37a}$$

$$\hat{\Omega}_+(k,k')=\int d\mathbf{r}\, \epsilon(\mathbf{r})\mathbf{f}_k^{\text{out}}(\mathbf{r},0)\cdot\hat{\mathbf{e}}_{k'}\frac{e^{i\mathbf{k}'\cdot\mathbf{r}}}{V^{1/2}} , \tag{4.37b}$$

and may be used to express the transverse wave functions $\epsilon(\mathbf{r})\mathbf{f}_k^{\text{out}}(\mathbf{r},0)$ in terms of the transverse plane waves, as in Eq. (4.33a).

The product of the two Møller matrices $(\Omega_+)^\dagger$ and Ω_-, according to the familiar formulation of quantum-mechanical scattering theory,[2] is the scattering matrix $\hat{S}$, which expresses the "in" states in terms of the "out" states,

$$\hat{S}(k,k')=\int d\mathbf{r}\, \epsilon(\mathbf{r})[\mathbf{f}_k^{\text{out}}(\mathbf{r})]^*\mathbf{f}_k^{\text{in}}(\mathbf{r}) \tag{4.38a}$$

and

$$\mathbf{f}_{k'}^{\text{in}}(\mathbf{r})=\sum_k \hat{S}(k,k')\mathbf{f}_k^{\text{out}}(\mathbf{r}) . \tag{4.38b}$$

In the present case we have two alternative representations of the $\hat{S}$ matrix,

$$\hat{S}(k,k'')=\sum_{k'}\Omega_+^*(k,k')\hat{\Omega}_-(k'',k') \tag{4.39a}$$

or

$$\hat{S}(k,k'')=\sum_{k'}\hat{\Omega}_+^*(k,k')\Omega_-(k'',k') . \tag{4.39b}$$

We can use Eqs. (4.1) and (4.13) to express the operators b_k^{out} in terms of the b_k^{in} by writing

$$b_k^{\text{out}}=\sum_{k',k''}\{[A^{\text{out}}(k',k)]^*A^{\text{in}}(k',k'')-B^{\text{out}}(k',k)[B^{\text{in}}(k',k'')]^*\}b_{k''}^{\text{in}} + \sum_{k',k''}\{-B^{\text{out}}(k',k)[A^{\text{in}}(k',k'')]^*+[A^{\text{out}}(k',k)]^*B^{\text{in}}(k',k'')\}(b_{k''}^{\text{in}})^\dagger . \tag{4.40}$$

Elementary calculation then shows

$$b_k^{\text{out}}=\sum_{k'}\hat{S}(k,k')b_{k'}^{\text{in}} , \tag{4.41}$$

where the coefficients $\hat{S}(k,k')$ are indeed given by Eq. (4.38a). Since the scattering matrix vanishes off the energy shell,

$$\hat{S}_{kk'}\sim\delta(\omega_k-\omega_{k'}) , \tag{4.42}$$

the expressions (4.41) and (4.42) assure us that the scattering is elastic (i.e., consists only of diffraction, reflection, and refraction). The photon number is conserved in the scattering process, as well as the photon energy. No mixing of creation and annihilation operators of the kind evident in Eqs. (4.1) and (4.13) is present in the asymptotic operator relations (4.41) (see also Ref. 15). If the system is prepared asymptotically in the coherent state $|\{\alpha_k^{\text{in}}\}\rangle$ for $t=-\infty$, then it will emerge in a coherent state $|\{\alpha_k^{\text{out}}\}\rangle$ at $t=+\infty$ as well, where the scattering matrix $\hat{S}$ effects an elementary linear transformation of the coherent state amplitudes,

$$\alpha_k^{\text{out}}=\sum_{k'}\hat{S}(k,k')\alpha_{k'}^{\text{in}} .$$

In fact, the conservation of the number of physical photons remains true for all intermediate times as well. The formula (4.1) suggests that the plane-wave creation and annihilation operators at $t=0$ are combinations of both creation and annihilation operators $(b_k^{\text{in}})^\dagger$ and b_k^{in}. The plane-wave photons, however, are not in general the ones detected in photon-counting experiments. The physical photons (i.e., the photons that are detected at $t=0$) correspond to a pair of creation and annihilation operators a_k^{ph} and $(a_k^{\text{ph}})^\dagger$ defined as in Eq. (4.13) with an appropriate choice of the functions $A(k',k)$ and $B(k',k)$. This choice corresponds uniquely to a set of basis functions $\mathbf{f}_k^{\text{ph}}(\mathbf{r})$ which describe specific properties of the detected photons. For example, detection may have a directional character; it may correspond to definite linear or circular polarizations. For a given photon frequency ω_k, it is therefore useful to choose the basis functions $\mathbf{f}_k^{\text{ph}}(\mathbf{r})$ in a way such that each of them represents a particular variety of photon that can be detected by the measuring device. Such physical photons are annihilated, therefore, by the operator

$$a_k^{\text{ph}}=\sum_{k',k''}\{[A(k',k)]^*A^{\text{in}}(k',k'') -B(k',k)[B^{\text{in}}(k',k'')]^*\}b_{k''}^{\text{in}} + \sum_{k',k''}\{-B(k',k)[A^{\text{in}}(k',k'')]^* +[A(k',k)]^*B^{\text{in}}(k',k'')\}(b_{k''}^{\text{in}})^\dagger . \tag{4.43}$$

A calculation analogous to that yielding Eq. (4.41) now gives us

$$a_k^{\text{ph}}=\sum_{k'}\hat{S}^{\text{ph}}(k,k')b_{k'}^{\text{in}} , \tag{4.44}$$

with

$$\hat{S}^{\text{ph}}(k,k')=\int d\mathbf{r}\, \epsilon(\mathbf{r})[\mathbf{f}_k^{\text{ph}}(\mathbf{r})]^*\cdot\mathbf{f}_{k'}^{\text{in}} . \tag{4.45}$$

Note that the formulas (4.44) and (4.45) imply that *physical photons are unchanged in number and frequency during their interactions with a static dielectric; only refraction, reflection, and diffraction, or in more general terms, scattering, can occur.* It is intuitively clear then that the presence of a lossless dielectric medium does not affect the nature of the quantum-statistical properties of the electromagnetic field at any instant. We shall examine the sense in which this statement holds in Sec. V.

V. ELECTROMAGNETIC FIELD FLUCTUATIONS

The results of the preceding sections may seem to present a certain contradiction. The creation and annihilation operators for plane-wave photons at $t=0$ are, according to the formula (4.1), linear combinations of both creation and annihilation operators for the plane-wave photons at $t=-\infty$. That means that a coherent state prepared at $t=-\infty$ and expressed in terms of those photons appears to be "squeezed" at time $t=0$. The plane-wave photons mix the signs of the frequencies at $t=0$, however, and are not uniquely the ones detected in photoabsorption experiments. The theorem formulated at the end of Sec. IV, which generalizes the results presented in the Introduction, shows, on the other hand, that the same state remains coherent at all times, when expressed in terms of the physically detectable photons. Those are the ones that correspond to definite signs of the frequency in the expansion of the electromagnetic field. Here again we see that "squeezing" is not a property of a quantum state, but of a particular choice of variables used to describe it.

The discussion of the uncertainty relations presented in the introductory section can easily now be generalized to the case of infinitely many modes, by using the result of Secs. II–IV. To do that we introduce, following Ref. 15, the spatially averaged field operators

$$X[\mathbf{u}]=\int d\mathbf{r}\,\mathbf{D}(\mathbf{r})\cdot\mathbf{u}(\mathbf{r}) , \tag{5.1a}$$

$$Y[\mathbf{v}]=\int d\mathbf{r}\,\mathbf{B}(\mathbf{r})\cdot\mathbf{v}(\mathbf{r}) , \tag{5.1b}$$

where $\mathbf{u}(\mathbf{r})$ and $\mathbf{v}(\mathbf{r})$ are real-valued vector functions. These are integrals of the same sort as typically occur in the evaluation of transition matrix elements. We may assume, without loss of the generality, that both functions $\mathbf{u}$ and $\mathbf{v}$ are transverse since longitudinal components of $\mathbf{u}$ and $\mathbf{v}$ do not contribute to the integrals.

Since the equal time commutator of the electromagnetic field operators is

$$[D_i(\mathbf{r}),B_j(\mathbf{r}')]=i\hbar\epsilon_{ijk}\partial_k\delta(\mathbf{r}-\mathbf{r}') , \tag{5.2}$$

it is easy to show that the commutator of the averaged operators is

$$[X[\mathbf{u}],Y[\mathbf{v}]]=-i\hbar\int d\mathbf{r}\,\mathbf{u}(\mathbf{r})\cdot[\nabla\times\mathbf{v}(\mathbf{r})] . \tag{5.3}$$

Therefore, the observables, defined by the expressions (5.1), fulfill the uncertainty relation

$$\Delta X[\mathbf{u}]\Delta Y[\mathbf{v}]\geq\frac{\hbar}{2}\left|\int d\mathbf{r}\,\mathbf{u}(\mathbf{r})\cdot[\nabla\times\mathbf{v}(\mathbf{r})]\right| . \tag{5.4}$$

In principle, the functions $\mathbf{u}$ and $\mathbf{v}$ may be independent of one another. In particular, the right-hand side of the expression (5.4) can even vanish, which means that the uncertainty relation, in that case, does not impose any restrictions on the observables $X[\mathbf{u}]$ and $Y[\mathbf{v}]$. On the other hand, the restrictions imposed by the uncertainty relation will be strongest if the functions $\mathbf{u}$ and $\mathbf{v}$ correspond to the same photon wave packets.

If we expand both fields $\mathbf{D}$ and $\mathbf{B}$ in terms of plane waves (as done in Sec. III), we may easily show that the creation (or, alternatively, annihilation) parts of X and Y become

$$X_{\rm cr}[\mathbf{u}]=-i\sum_k\left[\frac{\hbar\omega_k}{2}\right]^{1/2}\hat{\mathbf{e}}_k^*\cdot\tilde{\mathbf{u}}(\mathbf{k})b_k^\dagger , \tag{5.5a}$$

$$Y_{\rm cr}[\mathbf{v}]=-ic\sum_k\left[\frac{\hbar}{2\omega_k}\right]^{1/2}(\mathbf{k}\times\hat{\mathbf{e}}_k^*)\cdot\tilde{\mathbf{v}}(\mathbf{k})b_k^\dagger , \tag{5.5b}$$

where $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$ denote the spatial Fourier transforms of $\mathbf{u}$ and $\mathbf{v}$.

The operators X and Y will thus describe the same photon wave packet if the operators $X_{\rm cr}$ and $Y_{\rm cr}$ are proportional to one another, i.e., if we choose $\tilde{\mathbf{v}}(\mathbf{k})$ to be $\tilde{\mathbf{v}}_u(\mathbf{k})$, defined by

$$\frac{\omega_k}{c}\tilde{\mathbf{v}}_{\rm u}(\mathbf{k})=i\alpha\mathbf{k}\times\tilde{\mathbf{u}}(\mathbf{k}) , \tag{5.6}$$

where α is a proportionality factor, independent of $\mathbf{k}$. In free space, the creation and annihilation parts of X and Y have definite signs of the frequency, i.e., they describe creation and annihilation of a physical photon wave packet. Thus it is natural to normalize $\mathbf{u}$ and $\mathbf{v}$ so that

$$[X_{\rm an}[\mathbf{u}],X_{\rm cr}[\mathbf{u}]]=\hbar/2 , \tag{5.7a}$$

$$[Y_{\rm an}[\mathbf{v}_{\rm u}],Y_{\rm cr}[\mathbf{v}_{\rm u}]]=\hbar/2 , \tag{5.7b}$$

where $X_{\rm an}=X_{\rm cr}^\dagger$ and $Y_{\rm an}=Y_{\rm cr}^\dagger$ are the annihilation parts of X and Y, respectively. That requirement leads to the conditions

$$\sum_k\frac{\hbar\omega_k}{2}|\tilde{\mathbf{u}}(\mathbf{k})|^2=\frac{\hbar}{2} , \tag{5.7c}$$

$$\sum_k\frac{\hbar\omega_k}{2}|\tilde{\mathbf{v}}_{\rm u}(\mathbf{k})|^2=\frac{\hbar}{2} . \tag{5.7d}$$

These conditions, together with the expression (5.6), imply that $|\tilde{\mathbf{v}}_{\rm u}(\mathbf{k})|=|\tilde{\mathbf{u}}(\mathbf{k})|$. The conditions (5.7) provide also that the variances ΔX and ΔY in the vacuum state in free space must be equal to one another,

$$\Delta_{\rm free}X=\Delta_{\rm free}Y=\sqrt{\hbar/2} . \tag{5.8}$$

In the presence of a dielectric the creation and annihilation parts of X and Y do not have definite signs of the frequency. Nevertheless, the uncertainty principle provides the greatest lower bound for the product of the variances if the conditions (5.6) and (5.7) are fulfilled. In that case, $X[\mathbf{u}]$ and $Y[\mathbf{v}_{\rm u}]$ describe the same packet of plane-wave photons. If we choose $\mathbf{u}$ and $\mathbf{v}_{\rm u}$ in that way and normalize them as in Eq. (5.7), we find quite generally that the uncertainty relation becomes

$$\Delta X[\mathbf{u}]\Delta Y[\mathbf{v}_u]\geq\frac{\hbar}{2}\ . \tag{5.9}$$

This inequality must hold for any quantum state both in free space and in the presence of a dielectric.

Let us now calculate the variances of the observables X and Y in the ground state of the Hamiltonian (2.27) for a dielectric characterized by $\epsilon(\mathbf{r})$. The quantized fields possess the eigenmode expansions:

$$\mathbf{D}(\mathbf{r})=i\sum_k\left[\frac{\hbar\omega_k}{2}\right]^{1/2}\epsilon(\mathbf{r})[a_k^\dagger\mathbf{f}_k^*(\mathbf{r})-a_k\mathbf{f}_k(\mathbf{r})]\ , \tag{5.10a}$$

$$\mathbf{B}(\mathbf{r})=c\sum_k\left[\frac{\hbar}{2\omega_k}\right]^{1/2}\{a_k^\dagger[\nabla\times\mathbf{f}_k^*(\mathbf{r})]+a_k[\nabla\times\mathbf{f}_k(\mathbf{r})]\}\ , \tag{5.10b}$$

and all of the a_k's have eigenvalues zero in the ground state,

$$a_k|0\rangle_{\text{diel}}=0\ . \tag{5.11}$$

Elementary calculation yields

$$\Delta^2_{\text{diel}}X[\mathbf{u}]=\int\int\sum_k\frac{\hbar\omega_k}{2}[\epsilon(\mathbf{r})\mathbf{f}_k^*(\mathbf{r})\cdot\mathbf{u}(\mathbf{r})]\times[\epsilon(\mathbf{r}')\mathbf{f}_k(\mathbf{r}')\cdot\mathbf{u}(\mathbf{r}')]d^3\mathbf{r}\,d\mathbf{r}'\ , \tag{5.12a}$$

$$\Delta^2_{\text{diel}}Y[\mathbf{v}_\mathbf{u}]=\int\int\sum_k\frac{\hbar c^2}{2\omega_k}[\nabla\times\mathbf{f}_k^*(\mathbf{r})\cdot\mathbf{v}_\mathbf{u}(\mathbf{r})]\times[\nabla\times\mathbf{f}_k(\mathbf{r}')\cdot\mathbf{v}_\mathbf{u}(\mathbf{r}')]d\mathbf{r}\,d\mathbf{r}'\ . \tag{5.12b}$$

From Eqs. (5.12) it is clear that ΔX and ΔY depend on ϵ. We shall show that either of them may become greater or smaller than $\sqrt{\hbar/2}$.

A simple approximate way to show that these possibilities indeed can be realized follows from the assumption that the weight functions $\mathbf{u}$ and $\mathbf{v}_\mathbf{u}$ are chosen in such a way that their main contribution to the terms in the sums (5.10) is through spatial modes with wave vectors $\mathbf{k}$, which in free space have temporal frequencies close to some optical frequency ω_0, i.e., $\omega_k=c|\mathbf{k}|\simeq\omega_0$. Such wave packets $\mathbf{u}$ and $\mathbf{v}_\mathbf{u}$, although localized in the Fourier domain, may still be fairly well localized in space. In particular, in the region of their localization around, say, $\mathbf{r}_\mathbf{u}$, the dielectric permittivity $\epsilon(\mathbf{r})$ may be assumed to be approximately constant, i.e.,

$$\epsilon(\mathbf{r})\mathbf{u}(\mathbf{r})\simeq\epsilon(\mathbf{r}_\mathbf{u})\mathbf{u}(\mathbf{r})\ , \tag{5.13a}$$

$$\epsilon(\mathbf{r})\mathbf{v}_u(\mathbf{r})\simeq\epsilon(\mathbf{r}_\mathbf{u})\mathbf{v}_\mathbf{u}(\mathbf{r})\ . \tag{5.13b}$$

For such wave packets we may approximate the free-space variances (5.7) and (5.8) as

$$\Delta^2_{\text{free}}X[\mathbf{u}]=\sum_k\frac{\hbar\omega_k}{2}|\tilde{\mathbf{u}}(\mathbf{k})|^2\simeq\frac{\hbar\omega_0}{2}\sum_k|\tilde{\mathbf{u}}(\mathbf{k})|^2=\frac{\hbar\omega_0}{2}\int d\mathbf{r}|\mathbf{u}(\mathbf{r})|^2\ , \tag{5.14a}$$

$$\Delta^2_{\text{free}}Y[\mathbf{v}_\mathbf{u}]=\sum_k\frac{\hbar\omega_k}{2}|\tilde{\mathbf{v}}_\mathbf{u}(\mathbf{k})|^2\simeq\frac{\hbar\omega_0}{2}\int d\mathbf{r}|\mathbf{v}_\mathbf{u}(\mathbf{r})|^2\ . \tag{5.14b}$$

In the dielectric, Eqs. (5.12) can be simplified by using the completeness relations (2.14b) and Eq. (5.6). We have to take into account, however, the fact that since the functions $\mathbf{u}$ and $\mathbf{v}_\mathbf{u}$ correspond to wave vectors $c|\mathbf{k}|\simeq\omega_k$, and since they are spatially localized in the region where $\epsilon(\mathbf{r})=\epsilon(\mathbf{r}_\mathbf{u})$, they must at the same time, according to the dispersion relation within the dielectric medium, correspond to the frequency

$$\omega_k\simeq\frac{\omega_0}{\sqrt{\epsilon(\mathbf{r}_\mathbf{u})}}\ . \tag{5.15}$$

For example, by using the relation (5.15) we may rewrite the expression (5.12a) as

$$\Delta^2_{\text{diel}}X[\mathbf{u}]=\sum_k\frac{\hbar\omega_0}{2\sqrt{\epsilon(\mathbf{r}_\mathbf{u})}}\times\int\int[\epsilon(\mathbf{r})\mathbf{f}_k^*(\mathbf{r})\cdot\mathbf{u}(\mathbf{r})]\times[\epsilon(\mathbf{r}')\mathbf{f}_k(\mathbf{r}')\cdot\mathbf{u}(\mathbf{r}')]d\mathbf{r}\,d\mathbf{r}'\ . \tag{5.16}$$

By employing the properties of the distribution (2.14b) and the fact that the fraction $\mathbf{u}$ is transverse, we then obtain

$$\Delta_{\text{diel}}X[\mathbf{u}]\simeq\left[\frac{\hbar\omega_0}{2\sqrt{\epsilon(\mathbf{r}_\mathbf{u})}}\int d\mathbf{r}\,\epsilon(\mathbf{r})|\mathbf{u}(\mathbf{r})|^2\right]^{1/2}\simeq\left[\frac{\hbar\omega_0\sqrt{\epsilon(\mathbf{r}_u)}}{2}\int d\mathbf{r}|\mathbf{u}(\mathbf{r})|^2\right]^{1/2}\ . \tag{5.17a}$$

Quite analogously we derive

$$\Delta_{\text{diel}}Y[\mathbf{v}_\mathbf{u}]\simeq\left[\frac{\hbar\omega_0}{2\sqrt{\epsilon(\mathbf{r})}}\int d\mathbf{r}|\mathbf{u}(\mathbf{r})|^2\right]^{1/2}\ . \tag{5.17b}$$

Obviously, for the normalization (5.7), we then have

$$\left[\frac{\hbar\omega_0}{2}\int d\mathbf{r}|\mathbf{u}(\mathbf{r})|^2\right]^{1/2}=\frac{\hbar}{2}\ , \tag{5.18}$$

so that $\Delta_{\text{diel}}X$ or $\Delta_{\text{diel}}Y$, depending on the form of the function $\epsilon(\mathbf{r}_\mathbf{u})$, may easily become smaller or greater than $\hbar/2$. Equations (5.17) provide a worthwhile generalization of the single-mode relations (1.31), discussed in the Introduction.

It should be stressed that the vacuum state of the dielectric is not necessarily a minimum uncertainty state for the operators X and Y. The product $\Delta_{\text{diel}}X[\mathbf{u}]\Delta_{\text{diel}}Y[\mathbf{v}_\mathbf{u}]$, which according to the approximate expression (5.17) is equal to $\hbar/2$ for the special state considered, may and usually will exceed $\hbar/2$, when calculated exactly. The ground state of the dielectric may nevertheless be either subfluctuant or superfluctuant in the observables $X[\mathbf{u}]$ or $Y[\mathbf{v}_\mathbf{u}]$. On the other hand, the ground state of the dielectric does not show any squeezing, since the normally ordered variances vanish,

$$\langle:(\Delta_{\text{diel}}X[\mathbf{u}])^2:\rangle=\langle:(\Delta_{\text{diel}}Y[\mathbf{v}_\mathbf{u}])^2:\rangle=0\ . \tag{5.19}$$

The examples we have discussed show that the presence of the dielectric medium does indeed affect the quantum-statistical properties of the electromagnetic field. Although such modifications are not detectable in photoabsorption experiments, they may be revealed by other schemes of measurement. We shall discuss examples of such detection schemes in Sec. VI.

VI. MODIFICATIONS OF SPONTANEOUS EMISSION BY DIELECTRIC MEDIA

As we have seen, the presence of a dielectric can modify the quantum-statistical properties of the electromagnetic field by changing the mean values of the equal-time correlation functions. Another type of modification occurs because of changes in the two-time correlation functions of the electromagnetic field.

In free space the vacuum expectation value of the two-time correlation function of the electric field is

$$C_{\alpha\beta}(\mathbf{r},t-t')=\langle E_\alpha^{(+)}(\mathbf{r},t)E_\beta^{(-)}(\mathbf{r},t')\rangle_{\text{vac}} = \sum_k \frac{\hbar\omega_k}{2V^{1/2}}\hat{e}_{k\alpha}\hat{e}_{k\beta}\exp[-i\omega_k(t-t')] \ . \tag{6.1}$$

The distribution (6.1) does not depend on the position $\mathbf{r}$. In the presence of an dielectric, on the other hand, we have

$$C_{\alpha\beta}^{\text{diel}}(\mathbf{r},t-t')=\langle E_\alpha^{(+)}(\mathbf{r},t)E_\beta^{(-)}(\mathbf{r},t')\rangle_{\text{vac}} = \sum_k \frac{\hbar\omega_k}{2}f_{k\alpha}(\mathbf{r})f_{k\beta}^*(\mathbf{r}) \times\exp[-i\omega_k(t-t')] \ . \tag{6.2}$$

Evidently, the distribution (6.2) has a nontrivial $\mathbf{r}$ dependence.

Modifications of two-time and, in general, multitime correlations of the electromagnetic field can be expressed in the frequency domain, through temporal Fourier or complex Laplace transforms. As we shall see below, this fact has important physical consequences and leads to measurable modifications of spontaneous emission processes. Such situations have been noted in a series of papers by Agarwal which are reviewed in Ref. 20.

Let us consider a two-level atom that has the transition frequency ω_0 and is located at $\mathbf{r}_0$. We assume that initially the atom is in the excited state $|1\rangle$ while the field is in the vacuum state. The atom undergoes spontaneous decay to the ground state $|0\rangle$ by means of an electric dipole transition with the dipole moment $\mathbf{d}=\langle 1|\mathbf{r}|0\rangle$. We shall assume in the following that the interaction of the atom with the photon field is not too strong, so that the Weisskopf-Wigner[21] approximation holds and the decay process proceeds exponentially.

The Hamiltonian of such a system is given by

$$H=H_A+H_F+H_{AF} \ , \tag{6.3}$$

where the free atomic part is

$$H_A=\hbar\omega_0|1\rangle\langle 1| \ , \tag{6.4}$$

while the free field part is

$$H_F=\sum_k \hbar\omega_k a_k^\dagger a_k \ . \tag{6.5}$$

If we introduce the transition operators

$$\mathbf{d}^{(-)}=\mathbf{d}|1\rangle\langle 0| \ ,$$
$$\mathbf{d}^{(+)}=\mathbf{d}^*|0\rangle\langle 1| \ ,$$

the interaction term can be written in the electric dipole and rotating-wave approximations as

$$H_{AF}=-\mathbf{d}^{(-)}\cdot\mathbf{E}^{(+)}(\mathbf{r}_0)-\mathbf{E}^{(-)}(\mathbf{r}_0)\cdot\mathbf{d}^{(+)} \tag{6.6}$$

or, more explicitly,

$$H_{AF}=-i\sum_k\left[\frac{\hbar\omega_k}{2}\right]^{1/2}\mathbf{d}\cdot\mathbf{f}_k(\mathbf{r}_0)|1\rangle\langle 0|a_k+\text{H.c.} \tag{6.7}$$

Note that the Hamiltonian (6.3) conserves the total number of atom and field excitations,

$$N=|1\rangle\langle 1|+\sum_k a_k^\dagger a_k \ .$$

Thus assuming that the initial state at $t=0$ was $|1,\text{vac}\rangle$, i.e., that the atom was excited and that no photons were present, only one photon may be emitted in the course of the evolution governed by the Hamiltonian (6.3). The time-dependent solution of the Schrödinger equation for our system

$$i\frac{d}{dt}|\Phi\rangle=H|\Phi\rangle \tag{6.8}$$

can be therefore written in the form

$$|\Phi(t)\rangle=\alpha(t)|1,\text{vac}\rangle+\sum_k \beta(k,t)|0,1k\rangle \ , \tag{6.9}$$

where $\beta(k,t)$ describes the probability amplitude for emission of a single photon into the mode k. The solution to the Schrödinger equations for the amplitudes $\alpha(t)$ and $\beta(k,t)$ can be formulated exactly by using the Laplace transform technique. When the transforms are evaluated within the framework of the Weisskopf-Wigner approximation, the solution for the amplitude $\alpha(t)$ takes the form

$$\alpha(t)=e^{-\gamma_s t-i(\omega_0+\delta\omega_0)t}\alpha(0) \ , \tag{6.10}$$

where the natural line width is

$$\gamma_s=\lim_{\nu\to 0}\text{Re}\left[\frac{1}{\hbar^2}\sum_{\beta\delta}d_\beta d_\delta^*\tilde{C}_{\beta\delta}(\mathbf{r}_0,\nu-i\omega_0)\right] =\pi\sum_k\frac{\omega_k}{2\hbar}|\mathbf{d}\cdot\mathbf{f}_k(\mathbf{r}_0)|^2\delta(\omega_k-\omega_0) \ , \tag{6.11a}$$

while the radiative shift of the frequency is

$$\delta\omega_0=\lim_{\nu\to 0}\text{Im}\left[\frac{1}{\hbar^2}\sum_{\beta\delta}d_\beta d_\delta^*\tilde{C}_{\beta\delta}(\mathbf{r}_0,\nu-i\omega_0)\right] =-\sum_k\frac{\omega_k}{2\hbar}\text{P}\frac{|\mathbf{d}\cdot\mathbf{f}_k(\mathbf{r}_0)|^2}{(\omega_k-\omega_0)} \ . \tag{6.11b}$$

In the above expressions $\tilde{C}_{\beta\delta}(\mathbf{r}_0,z)$ denotes the Laplace transform of the correlation function (6.2), while d_β's are the components of the dipole transition matrix element. The symbol P in the expression (6.11b) denotes the principal value.

If the dielectric is bounded in space and the basis functions are chosen to correspond to outgoing-wave boundary conditions, then the individual terms in the sum (6.11a) show that the rate of emission of a photon with polarization μ and momentum $\mathbf{k}_0$ pointing in the direction Ω_0 is

$$\gamma(\mathbf{k}_0,\mu,\omega_0,\mathbf{r}) =\pi \sum_k \frac{\omega_k}{2\hbar}|\mathbf{d}\cdot\mathbf{f}_k(\mathbf{r})|^2\delta(\omega_k-\omega_0)\delta(\Omega_k-\Omega_0) , \quad (6.12)$$

where $c|\mathbf{k}|=\omega_k$. As we can see, the spontaneous emission rates (6.11a) and (6.12) depend on the local structure of the electric-field modes that have the transition frequency ω_0.

Let us illustrate this dependence by considering two examples.

(a) *Spontaneous emission in free space.* We obtain in this case the familiar result

$$\gamma_s^{\text{free}}=\frac{\omega_0^3 d^2}{3\pi\hbar c^3} , \quad (6.13)$$

and the spontaneous emission rate is $\mathbf{r}$ independent as we expect. The probability, furthermore, of emitting a photon with the momentum $\mathbf{k}$ and polarization μ is given by

$$\gamma_s^{\text{free}}(\mathbf{k},\mu,\omega_0)=\frac{\omega_0^3}{\hbar c^3}\left[\frac{\mathbf{d}\cdot\hat{\mathbf{e}}_{k\mu}}{4\pi}\right]^2 . \quad (6.14)$$

(b) *Spontaneous emission within a dielectric.* We assume that the excited (probe) atom is located within a uniform medium of dielectric constant ϵ. We then face the following question: which set of eigenmode solutions $\mathbf{f}_k(\mathbf{r})$ should we substitute into the expression (6.11a)? This question entails the familiar problem of determining the strength of the field E that acts locally on the atom.

An approximate answer to that question can be obtained by much the same technique as is used in the derivation of the Clausius-Mossotti law (see, for example, Ref. 22). The excited atom is assumed to feel the local electric field inside an empty spherical hole, cut out of the homogeneous dielectric medium. We assume that the hole has a radial dimension R much smaller than the relevant wavelength $\lambda=2\pi c/\omega_0$.

The eigenfunctions $\mathbf{f}_k(\mathbf{r})$ must therefore fulfill the wave equation

$$\epsilon\frac{\omega_k^2}{c^2}\mathbf{f}_k(\mathbf{r})-\nabla\times[\nabla\times\mathbf{f}_k(\mathbf{r})] -(\epsilon-1)\Theta(R-r)\frac{\omega_k^2}{c^2}\mathbf{f}_k(0)=0 . \quad (6.15)$$

The unit step function $\Theta(R-r)$ describes a hole at $\mathbf{r}=0$. In Eq. (6.15) we have substituted the approximation

$$\Theta(R-r)\mathbf{f}_k(\mathbf{r})\simeq\Theta(R-r)\mathbf{f}_k(0) , \quad (6.16)$$

since $R \ll \lambda$ for $\omega_k\simeq\omega_0$. Equation (6.15) may now be solved by treating this term as an inhomogeneity. We may then write the solution of Eq. (6.15) in the form of a sum of "homogeneous" and inhomogeneous" parts

$$f_{ki}(\mathbf{r})=\frac{\hat{e}_{\mathbf{k}\mu i}}{(\epsilon V)^{1/2}}e^{i\mathbf{k}\cdot\mathbf{r}} +(\epsilon-1)\frac{\omega_k^2}{c^2}\sum_j\int G_{ij}(\mathbf{r},\mathbf{r}') \times\Theta(R-|\mathbf{r}'|)f_{kj}(0)d\mathbf{r}' , \quad (6.17)$$

where $G_{ij}(\mathbf{r},\mathbf{r}')$ is an approximate matrix Green's function and the indices $i,j=1,2,3$ label the components of $\mathbf{f}_k$. As we shall see, the expression (6.17) implies the correct normalization of $\mathbf{f}_k(\mathbf{r})$ in the limit $R\to0$ so that the condition (2.14a) is automatically fulfilled. The matrix Green's function G is defined to satisfy the wave equation

$$\epsilon\frac{\omega_k^2}{c^2}G_{ij}(\mathbf{r},\mathbf{r}')-[\nabla\times\nabla\times G(\mathbf{r},\mathbf{r}')]_{ij}=\delta_{ij}\delta(\mathbf{r}-\mathbf{r}') . \quad (6.18)$$

It therefore has the plane-wave expansion

$$G_{ij}(\mathbf{r},\mathbf{r}')=\frac{c^2}{\epsilon\omega_k^2}\sum_{\mathbf{p}}\frac{p_ip_j}{p^2}\frac{e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}}{V} +\lim_{\nu\to0}\sum_p\frac{\hat{e}_{\mathbf{p}\mu i}\hat{e}_{\mathbf{p}\mu j}}{\epsilon(\omega_k^2/c^2)+i\nu-p^2}\frac{e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}}{V} . \quad (6.19)$$

By inserting the expression (6.19) into Eq. (6.16) we obtain for $\mathbf{f}_k(\mathbf{r})$ the expression

$$\mathbf{f}_k(\mathbf{r})=\frac{\hat{\mathbf{e}}_{k\mu}}{(\epsilon V)^{1/2}}e^{i\mathbf{k}\mathbf{r}}+\left[\frac{\epsilon-1}{\epsilon}\right]\Theta(R-r)\mathbf{f}_k(0) -\left[\frac{\epsilon-1}{\epsilon}\right]\lim_{\nu\to0}\sum_p\int d\mathbf{r}'\left[\frac{p^2}{p^2-\epsilon(\omega_k^2/c^2)-i\nu}\right]\frac{\hat{\mathbf{e}}_{\mathbf{p}\mu'}[\hat{\mathbf{e}}_{\mathbf{p}\mu'}\cdot\mathbf{f}_k(0)]}{V}e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}\Theta(R-|\mathbf{r}'|) . \quad (6.20)$$

Elementary calculation shows that for $R \to 0$ the inhomogeneous part of $\mathbf{f}_k(\mathbf{r})$ does not contribute to the norm, i.e., the condition (2.14a) is indeed fulfilled.

By inserting $\mathbf{r}=0$ on both sides of the expression (6.20) we obtain a self-consistent relation which requires $\mathbf{f}_k(0)$ to be

$$\mathbf{f}_k(0)=\frac{\hat{\mathbf{e}}_{\mathbf{k}\mu}}{(\epsilon V)^{1/2}}\frac{3\epsilon}{2\epsilon+1} . \quad (6.21)$$

In the limit $V \to \infty$, the mode summation (6.12) may be evaluated by noting that the mode density at frequency ω_k within the dielectric is $n(k)=(1/\pi^2)k^2(dk/d\omega)V=(\epsilon^{3/2}/\pi^2)(\omega_k^2/c^3)V$. We find in this way the total decay constant

$$\gamma_s=\frac{9\epsilon^{5/2}}{(2\epsilon+1)^2}\gamma_s^{\text{free}} , \quad (6.22)$$

as well as the partial decay constant

$$\gamma_s(\mathbf{k},\mu,\omega_0)=\frac{9\epsilon^{5/2}}{(2\epsilon+1)^2}\gamma_s^{\text{free}}(\mathbf{k},\mu,\omega_0) , \quad (6.23)$$

for emission into a given plane-wave mode. We can see that the emission in the dielectric has the same angular distribution as in free space. The rate of spontaneous emission is enhanced for $\epsilon > 1$ and inhibited for $\epsilon < 1$. The formula (6.23) has recently been derived and checked experimentally by Yablonovich, Gmitter, and Bhat.[23]

The methods we have described also can be used to find the modification of the spontaneous emission rate for magnetic dipole transitions. For this purpose we study the Zeeman coupling of the atomic magnetic moment to the local magnetic field $\mathbf{B}(0)$. The expression for the transition rate then takes a form analogous to that of Eq. (6.11a), except that the electric dipole transition matrix element must be replaced by the corresponding magnetic dipole matrix element,[24] and the electric-field amplitude $\mathbf{f}_k(0)$ by the magnetic-field amplitude $(c/\omega_k)\nabla\times\mathbf{f}_k(0)$.

It is easy to show by using Eqs. (6.17) and (6.19) that, when the presence of the hole at $\mathbf{r}=0$ affects the value of $\mathbf{f}_k(0)$ in the limit $R \to 0$, it does not alter the value of $\nabla\times\mathbf{f}_k(0)$. The effect of having dielectric constant $\epsilon\neq 1$ then is a global renormalization of the value of the magnetic field, according to the rule

$$[\nabla\times\mathbf{f}_k(0)]_{\text{diel}}=\frac{1}{\sqrt{\epsilon V}}\mathbf{k}\times\hat{\mathbf{e}}_{k\mu} .$$

We thus find the decay constant for magnetic dipole transitions to be

$$\gamma_s=\epsilon^{1/2}\gamma_s^{\text{free}} . \quad (6.24)$$

The differences between the decay constants (6.22)–(6.24) and their vacuum values, it is worth emphasizing, have two causes. One is the change in the zero-point fluctuations of the local electric and magnetic fields, and the other is the change in the spectral density of plane-wave modes available at the fixed frequency at which the atom is prepared to radiate.

It should be stressed that all the results of this section are based on the (Weisskopf-Wigner) approximation for spontaneous emission. Such an approach is valid only if the time scale of the emission process bears an appropriate relation to the other time scales of the system. In particular, in our example (b) if the medium is to be considered infinite, $1/\gamma_s$ must be smaller than $\epsilon^{1/2}V^{1/3}/c$, the approximate time it takes the photons to leave it. Otherwise the outer boundaries of the medium may substantially influence the decay process.

VII. QUANTUM THEORY OF TRANSITION RADIATION

The theory we have formulated in Sec. II can easily be generalized to the presence of external charges and currents. An immediate application of such a generalized theory is the formulation of a quantum description of the transition radiation[13] emitted by a charged particle that moves through a nonuniform medium. Such transition radiation, emitted at the boundaries of laminar media, has recently attracted attention in connection with the possibility of constructing coherent x-ray sources.[25]

The Maxwell equations in the presence of external charges and currents take the familiar form in Heaviside units:

$$\begin{aligned}
&\nabla\cdot\mathbf{D}=\rho ,\\
&\nabla\cdot\mathbf{B}=0 ,\\
&\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}=\nabla\times\mathbf{H}-\frac{\mathbf{j}}{c} ,\\
&\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}=-\nabla\times\mathbf{E} .
\end{aligned} \quad (7.1)$$

The charge density ρ and the current $\mathbf{j}$ are constrained to fulfill the continuity equation

$$\frac{\partial}{\partial t}\rho+\nabla\cdot\mathbf{j}=0 . \quad (7.2)$$

Once more we shall limit ourselves to considering the case of linear, isotropic dielectric media, for which the electric displacement vector and magnetic induction are given by Eq. (2.2). The scalar and the vector potentials are then introduced precisely as before in Eqs. (2.3). In the radiation gauge, however, we can no longer take the electric potential Φ to vanish. We take it instead to satisfy the relation

$$\nabla\cdot[\epsilon(\mathbf{r})\nabla\Phi]+\rho=0 , \quad (7.3a)$$

which suggests implementing again the transversality condition on the vector potential that we used earlier,

$$\nabla\cdot[\epsilon(\mathbf{r})\mathbf{A}]=0 . \quad (7.3b)$$

The equation of motion for the vector potential $\mathbf{A}$ thus becomes

$$\frac{\epsilon(\mathbf{r})}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}+\nabla\times(\nabla\times\mathbf{A})=\frac{\mathbf{j}_T}{c} , \quad (7.4)$$

where the transverse part of the current is defined by

$$\mathbf{j}_T(\mathbf{r},t)=\mathbf{j}(\mathbf{r},t)-\epsilon(\mathbf{r})\nabla\dot{\Phi}(\mathbf{r},t) . \quad (7.5)$$

The equation of motion (7.4) can be derived from

Hamilton's principle for the Lagrangian

$$\mathcal{L}=\int d\mathbf{r}\left[\frac{\epsilon(\mathbf{r})}{2c^2}\dot{\mathbf{A}}(\mathbf{r},t)^2-\frac{1}{2}[\nabla\times\mathbf{A}(\mathbf{r},t)]^2-\frac{1}{c}\mathbf{j}_T(\mathbf{r},t)\,\mathbf{A}(\mathbf{r},t)\right] . \tag{7.6}$$

Equivalently we may use, as in Sec. II, the Hamiltonian description of the motion with

$$\mathcal{H}=\int d\mathbf{r}\left[\frac{c^2\mathbf{\Pi}(\mathbf{r},t)^2}{2\epsilon(\mathbf{r})}+\frac{1}{2}[\nabla\times\mathbf{A}(\mathbf{r},t)]^2+\frac{1}{c}\mathbf{j}_T(\mathbf{r},t)\,\mathbf{A}(\mathbf{r},t)\right] . \tag{7.7}$$

We shall make explicit use of this Hamiltonian in the calculations that follow. By carrying out the same steps as in Sec. II, we may expand the vector potential in terms of photon creation and annihilation operators as

$$\mathbf{A}(\mathbf{r},t)=c\sum_k\left[\frac{\hbar}{2\omega_k}\right]^{1/2}[a_k\mathbf{f}_k(\mathbf{r})+a_k^\dagger\mathbf{f}_k^*(\mathbf{r})] , \tag{7.8}$$

where the eigenfunctions $\mathbf{f}_k(\mathbf{r})$ fulfill as before, the differential equations (2.12) and the appropriate boundary conditions.

When written in terms of annihilation and creation operators the Hamiltonian then takes the form, analogous to Eq. (2.27),

$$\mathcal{H}=\sum_k\hbar\omega_k a_k^\dagger a_k+C[\epsilon]+c\sum_k\left[\frac{\hbar}{2\omega_k}\right]^{1/2}[a_k j_T^*(k,t)+a_k^\dagger j_T(k,t)] , \tag{7.9}$$

in which

$$j_T(k,t)=\int d\mathbf{r}\,\mathbf{j}_T(\mathbf{r},t)\cdot\mathbf{f}_k^*(\mathbf{r}) . \tag{7.10}$$

By using the definition (7.5) and the transversality condition (2.12b), we see that the full current $\mathbf{j}(\mathbf{r},t)$ may equally well be substituted in the integrand of this expression,

$$j_T(k,t)=\int d\mathbf{r}\,\mathbf{j}(\mathbf{r},t)\cdot\mathbf{f}_k^*(\mathbf{r}) , \tag{7.11}$$

since the gradient term in the definition (7.5) makes no contribution to the integral (7.10).

The equation of motion for $a_k(t)$ that follows from the Hamiltonian (7.9) is

$$\dot{a}_k(t)=-i\omega_k a_k(t)-i\left[\frac{1}{2\hbar\omega_k}\right]^{1/2}j_T(k,t) . \tag{7.12}$$

If the current $\mathbf{j}(\mathbf{r},t)$ can be regarded as predetermined, this equation belongs to a general class for which the induced time dependence of the state vector in the Schrödinger picture is particularly simple. For such systems an initially coherent state, for example the vacuum state, evolves into a pure coherent state at all later times.[26] The occupation numbers of all the modes defined by the functions $\mathbf{f}_k(\mathbf{r})$, if they begin with the value zero, evolve in separate Poisson distributions. It follows that the total number of photons emitted in transition radiation also forms a Poisson distribution.

To find the radiated field more explicitly we introduce the interaction picture annihilation operator $\tilde{a}_k(t)$ via the relation

$$a_k(t)=e^{-i\omega_k t}\tilde{a}_k(t) . \tag{7.13}$$

We then find from the equation of motion (7.12) that

$$\tilde{a}_k(\infty)=\tilde{a}_k(-\infty)-i\left[\frac{1}{2\hbar\omega_k}\right]^{1/2}\int_{-\infty}^{\infty}j_T(k,t)e^{i\omega_k t}dt . \tag{7.14}$$

The scattering matrix $\hat{S}$, when regarded as an operator, is defined as the unitary transformation that carries the $\tilde{a}_k(-\infty)$ into the $\tilde{a}_k(\infty)$, i.e.,

$$\tilde{a}_k(\infty)=\hat{S}^\dagger\tilde{a}_k(-\infty)\hat{S} . \tag{7.15}$$

It is evident from Eq. (7.15) that this transformation simply effects a displacement of $\tilde{a}_k(-\infty)$ by a complex c number. If we write that complex amplitude as

$$\alpha_k=-\frac{i}{\sqrt{2\hbar\omega_k}}\int_{-\infty}^{\infty}j_T(k,t)e^{i\omega_k t}dt , \tag{7.16}$$

then we see that, apart from an undetermined phase factor, the operator $\hat{S}$ must be the unitary displacement operator

$$\hat{S}=\exp\left[\sum_k[\alpha(k)a_k^\dagger(-\infty)-\alpha^*(k)a_k(-\infty)]\right] \tag{7.17}$$

that maps the in states into out states, and transforms the vacuum state at $t=-\infty$ into the coherent state $|\{\alpha_k\}\rangle$ at $t=\infty$. The probability that there are n photons finally present in the kth mode is then

$$p_n=\frac{|\alpha_k|^{2n}}{n!}e^{-|\alpha_k|^2} . \tag{7.18}$$

To illustrate the foregoing results let us consider the simplest possible case[13] in which a charged particle moves uniformly along the z axis and crosses a perpendicular plane boundary between two different uniform media at $z=0$ and $t=0$. The dielectric constant in this case can be written as

$$\epsilon(z)=\Theta(z)\epsilon_2+\Theta(-z)\epsilon_1 , \tag{7.19}$$

where $\Theta(z)$ is the unit step function, and ϵ_1 and ϵ_2 are the constants characteristic of the two media, respectively. If the particle moves uniformly with velocity v, its charge density and current are given by

$$\rho(\mathbf{r},t)=e\delta(x)\delta(y)\delta(z-vt) , \tag{7.20a}$$

$$\mathbf{j}(\mathbf{r},t)=(0,0,ev\delta(x)\delta(y)\delta(z-vt)) . \tag{7.20b}$$

With this simple current distribution we find that its transverse Fourier components are given by

$$j_T(k,t)=e\mathbf{v}\cdot\mathbf{f}_k^*(\mathbf{v}t) , \tag{7.21a}$$

and the amplitudes α_k by

$$\alpha_k=-\frac{iev}{\sqrt{2\hbar\omega_k}}\int_{-\infty}^{\infty}f_{k,z}^*(\mathbf{v}t)e^{i\omega_k t}dt \ . \tag{7.21b}$$

This expression is a general one, valid for all variations of the dielectric function $\epsilon(\mathbf{r})$. The different ways in which $\epsilon(\mathbf{r})$ may vary, however, call for different forms of the mode functions $\mathbf{f}_k(\mathbf{r})$. We shall restrict consideration, for the present, to the simple two-medium variation expressed by Eq. (7.19).

In order to study the spectral and angular distributions of the emitted photons we must first find the mode functions appropriate to the discontinuity $\epsilon(z)$ at the plane boundary $z=0$. That problem is solved by the well-known Fresnel solutions[1] of the wave equation. The complete set of required eigenfunctions consists of two classes, which may be denoted by an incident wave vector $\mathbf{k}=(k_x,k_y,k_z)$ and by the accompanying polarization index $\mu=1,2$ for the incident wave. We choose the polarization $\mu=1$ to lie in the plane of incidence, i.e., the plane containing $\mathbf{k}$ which is perpendicular to the boundary plane. The polarization $\mu=2$ is then chosen to be parallel to the boundary plane.

The two classes of eigensolutions are defined then as follows.

(I) The first class of eigensolutions corresponds to waves incident from the half-space $z<0$. For these $\mathbf{f}_k(\mathbf{r})$ contains both incident and reflected waves for $z<0$ and these can be written as

$$\mathbf{f}_k(\mathbf{r})=\mathcal{N}[\mathbf{E}^i(k)e^{i\mathbf{k}\cdot\mathbf{r}}+\mathbf{E}^r(k)e^{i\mathbf{k}^r\cdot\mathbf{r}}] \ , \tag{7.22a}$$

where $\mathcal{N}$ is a normalization constant that remains to be specified. The same solution contains only a transmitted wave for $z>0$,

$$\mathbf{f}_k(\mathbf{r})=\mathcal{N}\mathbf{E}^t(k)e^{i\mathbf{k}^t\cdot\mathbf{r}} \ . \tag{7.22b}$$

For both of the solutions (7.22) the incident, transmitted, and reflected waves are taken to be transverse. The reflected wave then has a wave vector

$$\mathbf{k}^r=(k_x,k_y,-k_z) \ , \tag{7.23a}$$

while for the transmitted or refracted wave we can write

$$\mathbf{k}^t=(k_x,k_y,\tilde{k}_z) \ , \tag{7.23b}$$

These wave vectors fulfill the dispersion relations

$$k_x^2+k_y^2+k_z^2=\epsilon_1\frac{\omega_k^2}{c^2} \ , \tag{7.24a}$$

$$k_x^2+k_y^2+\tilde{k}_z^2=\epsilon_2\frac{\omega_k^2}{c^2} \ , \tag{7.24b}$$

which, together with Eqs. (7.23), express Snell's law. Note that for $\epsilon_1>\epsilon_2$ certain of the solutions correspond to $\tilde{k}_z^2<0$, i.e., to total internal reflection.

To fully satisfy the transversality condition (2.12b) the functions $\mathbf{f}_k(\mathbf{r})$ must also obey an appropriate boundary condition, that is, the condition that the normal component of $\epsilon(z)\mathbf{f}_k(\mathbf{r})$ be continuous at $z=0$. In addition, we have the usual continuity condition on the parallel component of the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ (note that we assume the media to have unit magnetic permeability, so that $\mathbf{B}=\mathbf{H}$). These require continuity of the parallel component of $\mathbf{f}_k(\mathbf{r})$ and of the vector function $\nabla\times\mathbf{f}_k(\mathbf{r})$ at $z=0$.

(II) The second class of eigensolutions corresponds to waves incident from the half-space $z>0$ and has wave vectors with $k_z<0$. Those solutions take a form analogous to the first class, except that we have to exchange ϵ_1 and ϵ_2 and let $z\to -z$.

According to the expression (7.21) for the amplitudes α_k, transition radiation can only lead to the emission of photons of the polarization $\mu=1$, i.e., polarized in the plane of incidence, since for $\mu=2$ the vector function $\mathbf{f}_k(\mathbf{r})$ is perpendicular to the z axis and to $\mathbf{v}$.

The boundary conditions to be satisfied by the class (I) eigenmodes are

$$\begin{aligned}&\epsilon_1(E_z^i+E_z^r)=\epsilon_2E_z^t \ ,\\&\mathbf{E}_\parallel^i+\mathbf{E}_\parallel^r=\mathbf{E}_\parallel^t \ ,\\&\mathbf{k}\times\mathbf{E}_\parallel^i+\mathbf{k}^r\times\mathbf{E}_\parallel^r=\mathbf{k}^t\times\mathbf{E}_\parallel^t \ ,\end{aligned} \tag{7.25}$$

Any pair of different solutions satisfying the conditions (7.22) of their analogs for class (II), together with the boundary conditions (7.25), can be shown to be orthogonal in the sense indicated by Eq. (2.14a). We can furthermore secure the normalization specified by that equation by choosing the constant $\mathcal{N}$ to satisfy the relation

$$4\pi^3\mathcal{N}^2\left\{|\mathbf{E}^i|^2+|\mathbf{E}^r|^2+|\mathbf{E}^t|^2\frac{\epsilon_1\tilde{k}_z}{\epsilon_2k_z}\right\}=1 \ , \tag{7.26a}$$

in the absence of total internal reflection. The coefficient multiplying $|\mathbf{E}^t|^2$ on the left-hand side of Eq. (7.26a) arises from the identity

$$\delta(\tilde{k}_z-\tilde{k}_z')=\left|\frac{dk_z}{d\tilde{k}_z}\right|\delta(k_z-k_z')=\frac{\epsilon_1\tilde{k}_z}{\epsilon_2k_z}\delta(k_z-k_z') \ .$$

When total internal reflection takes place in the medium (I) there is no transmitted wave and the normalization condition reduces to the form

$$4\pi^3\mathcal{N}^2(|\mathbf{E}^i|^2+|\mathbf{E}^r|^2)=1 \ . \tag{7.26b}$$

To find the amplitudes α_k for the class (I) modes we must introduce the expressions (7.22) into Eq. (7.21) and carry out the indicated integrations over t. Those integrals, though not strictly convergent, are easily summable by using the familiar device of letting ω_k have a small imaginary part and taking the limit as it goes to zero. In this way we find

$$\begin{aligned}\alpha_k^*=\lim_{\nu\to0}iev\left[\frac{1}{2\hbar\omega_k}\right]^{1/2}\Bigg[&E_z^i(k)\frac{1}{\nu+i(k_zv-\omega_k)}\\&+E_z^r(k)\frac{1}{\nu-i(k_zv+\omega_k)}\\&+E_z^t(k)\frac{1}{\nu-i(\tilde{k}_z\nu-\omega_k)}\Bigg] \ .\end{aligned} \tag{7.27}$$

Note that for $v>c_i=c/\sqrt{\epsilon_i}$, $i=1,2$, Čerenkov radiation will be emitted[27] for $k_z v = \omega_k$ or $\tilde{k}_z v = \omega_k$. We shall limit considerations here to cases in which $v < c_i$ for both $i=1$ and 2, so that no Čerenkov radiation is present.

For the class II solutions the expression for α_k^* reads

$$\alpha_k^* = \lim_{\nu\to 0} iev \left[\frac{1}{2\hbar\omega_k}\right]^{1/2} \left[E_z^t(k)\frac{1}{\nu - i(\tilde{k}_z v + \omega_k)} + E_z^r(k)\frac{1}{\nu - i(k_z v - \omega_k)} + E_z^i(k)\frac{1}{\nu + i(k_z v + \omega_k)}\right]. \quad (7.28)$$

We propose now to calculate the average number of photons of frequency near ω_k emitted in the forward hemisphere into the solid angle $d\Omega = \cos\theta\, d\theta\, d\phi$. This quantity, because of the axial symmetry of the problem, depends only on the angle θ and is given by the sum of two contributions, one from each class of mode functions. The contribution of the first class of modes is proportional to $|\alpha_{k\mathrm{I}}|^2$, where $\alpha_{k\mathrm{I}}$ is the amplitude given by Eq. (7.27), and to the frequency density of such modes in the solid angle $d\Omega$,

$$(k^{\mathrm{I}})^2 \frac{dk^{\mathrm{I}}}{d\omega_k} = n_1 \frac{\omega_k^2}{c^3} ,$$

where $n_1 = \sqrt{\epsilon_1}$ is the refractive index of the first medium. The waves of the first class that reach any point of observation in the forward hemisphere are those that have been transmitted by the plane interface, and so their contribution is proportional to the transmission coefficient, which we shall write as T^{I}. The contribution of the modes of the first class to the spectral density of photons is thus proportional to

$$T^{\mathrm{I}} |\alpha_{k\mathrm{I}}|^2 n_1^3 \frac{\omega_k^2}{c^3} .$$

The contribution of the solutions of the second class is likewise proportional to $|\alpha_{k\mathrm{II}}|^2$ and to the mode density $n_2^3 \omega_k^2 / c^3$. It is only the reflected part of these waves, however, that is observed in the forward hemisphere and so a factor of the reflection coefficient, which we write as R^{II}, is also included. The total expression we find for the photon distribution over angles and frequencies is thus

$$\frac{d^2 N(\omega_k, \theta)}{d\Omega\, d\omega_k} = T^{\mathrm{I}} |\alpha_{k\mathrm{I}}|^2 n_1^3 \frac{\omega_k^2}{c^3} + R^{\mathrm{II}} |\alpha_{k\mathrm{II}}|^2 n_2^3 \frac{\omega_k^3}{c^3} . \quad (7.29)$$

To evaluate this expression more explicitly we note that any contributing mode of class (I) and frequency ω_k has a propagation vector $\mathbf{k}^{\mathrm{I}}$ of magnitude

$$(\mathbf{k}^{\mathrm{I}})^2 = \epsilon_1 \frac{\omega_k^2}{c^2} . \quad (7.30)$$

If its transmitted wave, furthermore, is to fall into the solid angle $d\Omega$, it must also have

$$k_z^{\mathrm{I}} = n_1 \frac{\omega_k}{c} \cos\theta_i = \frac{\omega_k}{c} (n_1^2 - n_2^2 \sin^2\theta)^{1/2} , \quad (7.31\mathrm{a})$$

where θ_i is the angle of incidence, and

$$\tilde{k}_z^{\mathrm{I}} = n_2 \frac{\omega_k}{c} \cos\theta . \quad (7.31\mathrm{b})$$

Any contributing mode of class (II) and frequency ω_k will likewise have a propagation vector of magnitude

$$(\mathbf{k}^{\mathrm{II}})^2 = \epsilon_2 \frac{\omega_k^2}{c^2} . \quad (7.32)$$

In this case, however, it is the reflected wave that falls into the solid angle $d\Omega$. It follows then that

$$\tilde{k}_z^{\mathrm{II}} = n_1 \frac{\omega_k}{c} \cos\theta_t = \frac{\omega_k}{c} (n_1^2 - n_2^2 \sin^2\theta)^{1/2} , \quad (7.33\mathrm{a})$$

where θ_t is the angle of transmission of this wave [it is equal to θ_i for the class (I) mode]. The z component of the incident propagation vector for the class (II) mode is given by

$$k_z^{\mathrm{II}} = n_2 \frac{\omega_k}{c} \cos\theta . \quad (7.33\mathrm{b})$$

We should note that if $\sin\theta_i > n_2/n_1$, then the contribution of the class (I) modes to the intensity (7.29) will vanish due to the total internal reflection.

The coefficients T^{I} and R^{II} that enter the expression (7.29) for the photon distribution are the familiar transmission and reflection coefficients given by the expressions[1]

$$T^{\mathrm{I}} = \frac{2(n_1^2 - n_2^2 \sin^2\theta)^{1/2}}{(n_1^2 - n_2^2 \sin^2\theta)^{1/2} + n_2 \cos\theta)} , \quad (7.34\mathrm{a})$$

$$R^{\mathrm{II}} = \frac{n_2 \cos\theta - (n_1^2 - n_2^2 \sin^2\theta)^{1/2}}{n_2 \cos\theta + (n_1^2 - n_2^2 \sin^2\theta)^{1/2}} . \quad (7.34\mathrm{b})$$

We shall restrict considerations here to the case of angles θ small enough so that total internal reflection cannot occur. In that case we have for the class (I) solutions

$$E_z^i = |\mathbf{E}^i| \sin\theta_i , \quad E_z^r = |\mathbf{E}^r| \sin\theta_i , \quad E_z^t = |\mathbf{E}^t| \sin\theta , \quad (7.35)$$

with $\sin\theta_i / \sin\theta = n_2/n_1$. For the class (II) solutions, on the other hand, we have

$$E_z^i = |\mathbf{E}^i| \sin\theta , \quad E_z^r = |\mathbf{E}^r| \sin\theta , \quad E_z^t = |\mathbf{E}^t| \sin\theta_t , \quad (7.36)$$

with $\sin\theta / \sin\theta_t = n_1/n_2$. From the above expressions one easily derives the final result

$$\frac{d^2N(\omega_k,\theta)}{d\Omega d\omega_k}=\frac{e^2v^2\omega_k\sin\theta^2}{2\hbar\omega_k c^3}\left[\frac{T^{\rm I}n_1^3}{N^{\rm I}}\left|\frac{n_2}{n_1}\left[\frac{1}{k_z^{\rm I}v-\omega_k}-(R^{\rm I})^{1/2}\frac{1}{k_z^{\rm I}v+\omega_k}\right]-(T^{\rm I})^{1/2}\frac{1}{\tilde{k}_z^{\rm I}v-\omega_k}\right|^2\right.$$
$$\left.+\frac{R^{\rm II}n_2^3}{N^{\rm II}}\left|\left[\frac{1}{k_z^{\rm II}v+\omega_k}-(R^{\rm II})^{1/2}\frac{1}{k_z^{\rm II}v-\omega_k}\right]-\frac{n_2}{n_1}(T^{\rm II})^{1/2}\frac{1}{\tilde{k}_z^{\rm II}v+\omega_k}\right|^2\right]. \tag{7.37}$$

The transition and reflection coefficients obey the identities $R^{\rm I}+T^{\rm I}=R^{\rm II}+T^{\rm II}=1$, while the normalization constants are given by

$$N^{\rm I}=\frac{1}{4\pi^3\left[1+R^{\rm I}+T^{\rm I}\dfrac{\epsilon_1\tilde{k}_z^{\rm I}}{\epsilon_2 k_z^{\rm I}}\right]}$$

and

$$N^{\rm II}=\frac{1}{4\pi^3\left[1+R^{\rm II}+T^{\rm II}\dfrac{\epsilon_2\tilde{k}_z^{\rm II}}{\epsilon_1 k_z^{\rm II}}\right]}$$

All our results are based, of course, on the assumption that the refractive indices are independent of frequency. In the limit in which the moving charge has velocity close to c, $v\simeq c$, and when ϵ_1 and ϵ_2 are close enough to 1, the expression for the average intensity may be simplified. The emission in that case takes place predominantly in the forward direction, i.e., with $\sin\theta\simeq\theta$. In the expression (7.37) the terms with denominators $k_zv-\omega_k$ and $\tilde{k}_zv-\omega_k$ give the principal contributions. In all the other places we use the approximate values

$$T^{\rm I}=T^{\rm II}=1\ ,$$
$$R^{\rm I}=R^{\rm II}=0\ ,$$
$$n_1=n_2=1\ .$$

The result we obtain is then the well known Frank-Ginsburg formula[13]

$$\frac{d^2N(\theta,\omega_k)}{d\Omega d\omega_k}=\frac{e^2v^2\theta^2}{4\pi^3\hbar\omega_k c^3}\left[\frac{1}{\theta^2+2(1-v/c)+2(1-n_1)}-\frac{1}{\theta^2+2(1-v/c)+2(1-n_2)}\right]^2 .$$

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