Intertwining of exactly solvable Dirac equations with one-dimensional potentials

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The method of intertwining is used to construct transformations between one-dimensional electric potentials or one-dimensional external scalar fields for which the Dirac equation is exactly solvable. The transformations are analogous to the Darboux transformations between Schrödinger potentials. It is shown that a class of exactly solvable Dirac potentials corresponds to soliton solutions of the modified Korteweg–deVries (MKdV) equation, just as certain Schrödinger potentials are solitons of the Korteweg–deVries equation. It is also shown that the intertwining transformations are related to Bäcklund transformations for MKdV. The structure of the intertwining relations is shown to be described by an $N = 4$ superalgebra, generalizing supersymmetric quantum mechanics to the Dirac case.

I. INTRODUCTION

In recent years, much attention has been focused on exactly solvable problems in quantum mechanics. These problems transcend the usual boundaries between disciplines, finding application in essentially all areas of physics. It is a truism that one hears grumbled in graduate schools that everything soluble in physics is just the harmonic oscillator in disguise. This may not be far from the truth, though it might be more accurate to say that everything soluble is just the free particle. The method of intertwining[1] gives a unified approach to constructing transformations between solvable problems, and one finds that most (if not all) exactly solvable potentials can be constructed by intertwining operator transformations.

Two operators L_0 and L_1 are said to be intertwined by an operator D if

$$L_1 D = D L_0. \tag{1}$$

If the eigenfunctions ϕ_0 of L_0 are known, then the intertwining relation gives the (unnormalized) eigenfunctions of L_1 as $\phi_1 = D\phi_0$. The object of the method of intertwining is to construct the operator D, which performs an intertwining between operators L of a given form.

The results of the application of intertwining to the Schrödinger equation are well known, though they are not generally recognized as such. For example, if one makes the *Ansatz* that D is a first-order differential operator intertwining between two Hamiltonian operators of potential form $-\partial_x^2+V(x)$, one finds (see below) that D is what is commonly known as a Darboux transformation.[2] From this, one recovers the factorization method of Infeld and Hull,[3] which generalizes the notion of raising and lowering operators to equations other than that of the harmonic oscillator. One also finds supersymmetric quantum mechanics,[4] which is essentially the same thing but in fancier language.

If one makes the *Ansatz* that D is an integral operator, one finds equations familiar from the theory of inverse scattering.[5] It is well known from inverse scattering that the soliton solutions of the nonlinear Korteweg–deVries (KdV) equation form a class of solvable Schrödinger potentials. These soliton solutions can be constructed by Darboux transformations as well as with the integral operators of inverse scattering.[6,7]

There has been a recent resurgence of interest in exactly solvable Dirac equations with one-dimensional electric potential.[8–10] A closely related problem, with important application to solitons in conducting polymers, is the solution of the Dirac equation in a one-dimensional external scalar field.[11–13] This paper will give the intertwining construction of the exactly solvable Dirac potentials in these two problems, emphasizing the parallels with results obtained by intertwining for Schrödinger potentials.

The equations considered here will be 2×2 matrix differential equations in one variable x. They can be regarded as time-independent (1+1) Dirac equations or as reductions from the four-dimensional Dirac equation. These equations have the Ablowitz-Kaup-Newell-Segur (AKNS) form of the Lax eigenvalue equation for the modified Korteweg–deVries (MKdV) equation.[14] This is analogous to the observation that the time-independent Schrödinger equation is the Lax eigenvalue equation for KdV.

It is well known that MKdV can be solved by inverse scattering,[15] and indeed Campbell and Bishop[12] use inverse scattering to construct the kink and polaron solutions for conducting polymers. However, just as the soliton solutions of KdV can be constructed using differential operators, or Darboux transformations, so too can the soliton solutions of MKdV. This will be derived by the method of intertwining.

II. INTERTWINING OF SCHRÖDINGER POTENTIALS

For background and completeness, the treatment of intertwining between solvable Schrödinger potentials will be reviewed. Suppose that $H_0 = -\partial_x^2 + V_0$ and $H_1 = -\partial_x^2 + V_1$ are Hamiltonian operators in one dimension and that H_0 is exactly solvable. (N.B. It is not necessary that H_0 be solvable for there to be intertwining. It does increase the usefulness of having an intertwining.) H_1 and H_0 are said to be intertwined if there is an operator D for which

$$H_1 D = D H_0. \quad (2)$$

Given such a D, the (unnormalized) eigenfunctions ϕ_1 of H_1 are given in terms of the known eigenfunctions ϕ_0 of H_0 by

$$\phi_1 = D\phi_0. \quad (3)$$

The operator D has transformed one integrable problem into another.

By proposing an *Ansatz* for D and solving the consistency conditions obtained from the intertwining relation (2) by equating like powers of derivatives, transformations between exactly solvable potentials are constructed. The *Ansatz* that D is a first-order differential operator

$$D = \partial_x + g \quad (4)$$

leads to the consistency conditions

$$2g' = V_1 - V_0, \quad (5)$$

$$-g' + g^2 = V_0 + c, \quad (6)$$

where c is a constant and the prime indicates differentiation with respect to x.

Equation (6) is a Riccati equation which is linearized by the substitution

$$g = \frac{-\psi'}{\psi} \quad (7)$$

to give

$$-\psi'' + V_0\psi = -c\psi. \quad (8)$$

Thus ψ must be an eigenfunction of H_0. This is the first important conclusion: every eigenfunction of H_0 (without regard to boundary conditions or normalizability) generates a transformation to a new solvable Hamiltonian H_1 where the change in the potential is given by (5). Such a first-order differential intertwining is known as a Darboux transformation.[2]

Sometimes one reads that only nonvanishing eigenfunctions can be used to generate transformations. The reason given is that otherwise singularities will be introduced into the potential. While certainly true, this is not cause for concern, and it can be important in practical applications.[16] There is absolutely no restriction on the eigenfunctions used to generate intertwining transformations.

As an example of intertwining, suppose $V_0 = 0$, so that $H_0 = -\partial_x^2$. Taking

$$\psi_0 = \cosh kx \quad (9)$$

as an eigenfunction of H_0, one finds

$$g = -k \tanh kx \quad (10)$$

and, therefore,

$$V_1 = 2g' = -2k\,\mathrm{sech}^2 kx. \quad (11)$$

This is one of the Pöschl-Teller potentials.

The eigenfunctions of H_1 are obtained by applying

$$D = \partial_x - \frac{\psi_0'}{\psi_0} = \partial_x - k \tanh kx \quad (12)$$

to the eigenfunctions of H_0. The eigenfunction ψ_0 that generates the intertwining is annihilated by this transformation, so the eigenfunction ψ_1 of H_1 that corresponds to it must be reconstructed separately. Since H_0 is second order, there are two independent eigenfunctions for each eigenvalue. Given one, say ψ_0, the independent one is

$$\phi_0 = \psi_0 \int \psi_0^{-2}\, dx. \quad (13)$$

In this example, one finds $\phi_0 = \sinh kx$ as one would expect. The intertwining transformation generated by ψ_0 applied to ϕ_0 gives

$$\phi_1 = \left(\partial_x - \frac{\psi_0'}{\psi_0}\right)\phi_0 = \frac{1}{\psi_0}. \quad (14)$$

Using this in (13) to get the independent solution ψ_1, one finds

$$\psi_1 = \psi_0^{-1} \int \psi_0^2\, dx. \quad (15)$$

This is the eigenfunction of H_1 corresponding to ψ_0.

Clearly, not all intertwinings have the simple form of the *Ansatz* (4), but the intertwining procedure can be iterated to give higher-order differential operators. One can prove that every second-order differential intertwining operator can be factored into a product of first-order intertwining operators. It is likely that this extends to the statement that all finite-order differential intertwining operators can be factored as a product of first-order differential intertwining operators. So, for many purposes, it is sufficient to consider first-order operators. The case of intertwining with integral operators is planned to be discussed in a separate publication.[5]

Adding (5) to (6), one obtains a third equation,

$$g' + g^2 = V_1 + c. \quad (16)$$

From this, one infers that the adjoint of D,

$$D^\dagger = -\partial_x + g, \quad (17)$$

intertwines in the other direction, taking solutions of H_1

to those of H_0

$$\phi_0 = D^\dagger \phi_1. \tag{18}$$

Furthermore, one sees that D and $D^\dagger$ give a factorization of H_1 and H_0

$$DD^\dagger = (\partial_x + g)(-\partial_x + g) = H_1 + c, \tag{19}$$

$$D^\dagger D = (-\partial_x + g)(\partial_x + g) = H_0 + c. \tag{20}$$

This observation fundamentally links differential intertwining to the factorization method of Infeld and Hull.[3] The link to supersymmetric quantum mechanics[4] is made by defining the supercharge

$$Q = \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix}. \tag{21}$$

One finds that $Q^2 = 0$ as required and that the super-Hamiltonian is

$$\mathcal{H} = \{Q, Q^\dagger\} = \mathrm{diag}(H_1 + c, H_0 + c). \tag{22}$$

The intertwining relation can be captured diagrammatically as

$$H_0 \xrightarrow{D} H_1. \tag{23}$$

The diagram may be read as beginning with an eigenfunction of H_0, action by D takes it into an eigenfunction of H_1. In this diagram, the adjoint operator would act in the reverse direction. This diagrammatic structure will be useful below in describing the structure of intertwining for the Dirac equation.

From the perspective of the Korteweg–deVries equation, the condition (6) is the Miura transformation. It is one of two equations defining the Bäcklund transformation from solutions of KdV to MKdV.[17] The condition (5) can be used to find the first of two equations defining the auto-Bäcklund transformation between solutions of KdV. The second equation of the Bäcklund transformation in each case carries information about how the time dependence of the nonlinear solutions transform. This time dependence does not enter the Schrödinger equation because the Schrödinger potential is a solution of the KdV equation evaluated at a fixed parameter time and is not sensitive to the time dependence of the nonlinear equation. Similar correspondences between Bäcklund transformations and intertwining will be found below for MKdV and the two-dimensional Dirac equation.

III. MATRIX INTERTWINING OF DIRAC POTENTIALS

The equation for a two-dimensional fermion in an external scalar field $w(x)$ has the form

$$[i\gamma^\mu \partial_\mu + w(x)]\Psi = 0. \tag{24}$$

Assuming that the fermion has energy k,

$$\Psi = \Phi(x) e^{-ikt}, \tag{25}$$

and choosing the γ matrices to be Pauli matrices, $\gamma_0 = \sigma_1$, $\gamma_1 = i\sigma_2$, the equation takes the form of a time-independent 2×2 matrix equation[11,12]

$$(-i\sigma_3\partial_x + \sigma_1 w)\Phi = -k\Phi. \tag{26}$$

This is the AKNS form of the Lax eigenvalue equation for the modified Korteweg–deVries equation[14]

$$w_\tau - 6w^2 w_x + w_{xxx} = 0. \tag{27}$$

(N.B. The time parameter τ in this equation is not the same as that in the time-dependent Dirac equation.)

The four-dimensional Dirac equation with one-dimensional electric potential $A_0(x)$ is

$$[i\gamma^\mu\partial_\mu - \gamma^0 A_0(x) + m]\Psi = 0. \tag{28}$$

Using the γ matrix representation,

$$i\gamma_0 = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}, \quad i\gamma_1 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix},$$
$$i\gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad i\gamma_3 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \tag{29}$$

Percoco and Villalba[10] show that for a fermion of energy E, and transverse momenta p_y and p_z, the four-dimensional equation reduces to the bispinor equation

$$(\sigma_2\partial_x - \sigma_1 v)\tilde{\Phi} = -k\tilde{\Phi}, \tag{30}$$

where $v = E - A_0(x)$, $k^2 = p_y^2 + p_z^2 + m^2$, and

$$\Psi = \begin{pmatrix} \dfrac{im + p_y}{k - p_z}\tilde{\Phi} \\ -\sigma_3\tilde{\Phi} \end{pmatrix} \exp[i(p_y y + p_z z - Et)]. \tag{31}$$

A similarity transformation makes this

$$(\sigma_3\partial_x + \sigma_1 v)\Phi = -k\Phi, \tag{32}$$

where

$$\Phi = \frac{1}{\sqrt{2}}(\sigma_2 + \sigma_3)\tilde{\Phi}. \tag{33}$$

This is the AKNS form of the Lax eigenvalue equation of the MKdV equation[14]

$$v_\tau + 6v^2 v_x + v_{xxx} = 0. \tag{34}$$

Under the change of variables $x \to ix$, $\tau \to -i\tau$, one MKdV equation (34) becomes the other (27). Also, the matrix equation (32) becomes (26). This means that it is sufficient to consider intertwining of one of the matrix equations: the results for the other will follow upon changing $x \to ix$.

A word of caution is in order. Intertwining works in the complex domain. For physical reasons, one is interested only in real potentials in both Dirac equations. The substitution $x \to ix$ will not in general take real potentials into real potentials, so one cannot simply make the substitution in the physical potentials to change from one equation to the other. The substitution is made in the intertwining formulas relating potentials, and then real potentials are constructed.

To emphasize the connection to the MKdV equation (34), the Dirac equation in an electric potential will be considered first. Let

$$M_0 = \sigma_3\partial_x + \sigma_1 v_0. \tag{35}$$

An operator D intertwines M_0 with M_1 with potential v_1 when

$$M_1 D = D M_0. \tag{36}$$

Make the *Ansatz* for D

$$D = \begin{pmatrix} a_1\partial_x + g_1 & b_1\partial_x + h_1 \\ b_2\partial_x + h_2 & a_2\partial_x + g_2 \end{pmatrix}. \tag{37}$$

Imposing the matrix intertwining relation (36), and equating like numbers of derivatives, one finds the equations

$$\begin{aligned} &b_1 = 0 = b_2, \\ &a_1' = 0 = a_2', \\ &g_1' + v_1 h_2 = v_0 h_1, \\ &2h_1 + v_1 a_2 = v_0 a_1, \\ &h_1' + v_1 g_2 = v_0' a_1 + v_0 g_1, \\ &v_1 a_1 = 2h_2 + v_0 a_2, \\ &v_1 g_1 - h_2' = v_0' a_2 + v_0 g_2, \\ &v_1 h_1 - g_2' = v_0 h_2 \end{aligned} \tag{38}$$

(where the prime indicates differentiation with respect to x). Setting $a_1 = 1$ is a choice of normalization. Taking $a_2 = 1$ as well simplifies the equations considerably, giving

$$\begin{aligned} h_1 &= -h_2, \\ g_1 &= g_2, \end{aligned} \tag{39}$$

and, dropping the subscripts on h and g,

$$2h = v_0 - v_1, \tag{40}$$

$$g' = (v_0 + v_1)h, \tag{41}$$

$$h' = v_0' + (v_0 - v_1)g. \tag{42}$$

Thus D has the form

$$D = \begin{pmatrix} \partial_x + g & h \\ -h & \partial_x + g \end{pmatrix}. \tag{43}$$

If one had taken $a_2 = -1$ instead, one would find the form

$$\overline{D} = \begin{pmatrix} \partial_x + g & h \\ h & -\partial_x - g \end{pmatrix}, \tag{44}$$

where

$$2h = v_0 + v_1, \tag{45}$$

$$g' = (v_0 - v_1)h, \tag{46}$$

$$h' = v_0' + (v_0 + v_1)g. \tag{47}$$

The effect has been to change the sign of v_1. This is explained by noting that $\overline{D} = \sigma_3 D$ and, therefore,

$$\overline{M}_0 = \sigma_3 M_0 \sigma_3 = \sigma_3\partial_x - \sigma_1 v_1. \tag{48}$$

Other choices of a_2 do not appear to give anything new.

To solve the intertwining consistency conditions, it is convenient to define

$$\begin{aligned} G_+ &= g + ih, \\ G_- &= g - ih. \end{aligned} \tag{49}$$

Straightforward manipulation gives the equations

$$2G_+' = iv_0' + v_0^2 - iv_1' - v_1^2, \tag{50}$$

$$2G_-' = -iv_0' + v_0^2 + iv_1' - v_1^2, \tag{51}$$

and

$$G_+^2 - G_-^2 = -iv_0' - iv_1'. \tag{52}$$

A fourth equation

$$G_+^2 + G_-^2 = -v_0^2 - v_1^2 + 2c, \tag{53}$$

where c is a constant, is also found, but the computation is more subtle, so it will be given.

First, note that by subtracting the derivative of (40) from (42) one has

$$-h' = v_1' + (v_0 - v_1)g. \tag{54}$$

Eliminating v_0 from this using (40) gives

$$-(v_1 + h)' = 2gh. \tag{55}$$

Eliminating v_0 from (41) and multiplying by g gives

$$gg' = 2gh(v_1 + h), \tag{56}$$

and one finds

$$g^2 = -(v_1 + h)^2 + c_1. \tag{57}$$

Repeating this, eliminating v_1 from (41) and (42), gives

$$g^2 = -(v_0 - h)^2 + c_0. \tag{58}$$

Averaging these, one has

$$g^2 - h^2 = -\tfrac{1}{2}v_0^2 - \tfrac{1}{2}v_1^2 + \frac{c_0 + c_1}{2}, \tag{59}$$

but this is just $(G_+^2 + G_-^2)/2$, giving (53).

The consistency conditions in terms of G_+ and G_- can be rearranged into

$$\begin{aligned} -G_+' + G_+^2 &= -iv_0' - v_0^2 + c, \\ -G_-' + G_-^2 &= iv_0' - v_0^2 + c, \\ G_+' + G_+^2 &= -iv_1' - v_1^2 + c, \\ G_-' + G_-^2 &= iv_1' - v_1^2 + c. \end{aligned} \tag{60}$$

These are Riccati equations analogous to (6) and (16) in the Schrödinger case. The analogy is closer than one might expect. Applying a similarity transformation to return to the original equation (30) of Percoco and Villalba, one has

$$\tilde{M}_0 = \sigma_2 \partial_x - \sigma_1 v_0. \tag{61}$$

The eigenfunctions Φ of M_0 are rotated to

$$\tilde{\Phi} = \frac{1}{\sqrt{2}}(\sigma_2 + \sigma_3)\Phi. \tag{62}$$

The intertwining operator is also transformed by the similarity transformation and simply becomes

$$\tilde{D} = \begin{pmatrix} \partial_x + G_+ & 0 \\ 0 & \partial_x + G_- \end{pmatrix}. \tag{63}$$

Squaring $\tilde{M}_0$, one has

$$\tilde{M}_0^2 = \begin{pmatrix} \partial_x^2 + iv_0' + v_0^2 & 0 \\ 0 & \partial_x^2 - iv_0' + v_0^2 \end{pmatrix}. \tag{64}$$

Since the intertwining relation holds for the squared operators

$$\tilde{M}_1^2 \tilde{D} = \tilde{D} \tilde{M}_0^2, \tag{65}$$

one sees that the intertwining consistency conditions are necessarily those for Schrödinger Hamiltonian operators.

A diagram expressing the full intertwining structure can be drawn. Let

$$\tilde{M}_0^2 = \mathrm{diag}(K_0^+, K_0^-), \qquad \tilde{M}_1^2 = \mathrm{diag}(K_1^+, K_1^-). \tag{66}$$

The (non-Hermitian) Hamiltonians on the diagonal of $\tilde{M}_0^2$ are themselves related by intertwining, and $\tilde{M}_0$ is composed of the intertwining operators

$$\tilde{M}_0 = \begin{pmatrix} 0 & -i\partial_x - v_0 \\ i\partial_x - v_0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & D_0 \\ D_0^\dagger & 0 \end{pmatrix}. \tag{67}$$

The eigenvalue condition $\tilde{M}_0 \tilde{\Phi} = -k\tilde{\Phi}$ is simply the two equations (3) and (18). Note that the adjoint $D_0^\dagger$ is found by integration by parts and does not include a complex conjugation. Similarly, $\tilde{M}_1$ is composed of the intertwining operators that relate the diagonal elements of $\tilde{M}_1^2$,

$$\tilde{M}_1 = \begin{pmatrix} 0 & -i\partial_x - v_1 \\ i\partial_x - v_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & D_1 \\ D_1^\dagger & 0 \end{pmatrix}. \tag{68}$$

The intertwining matrix can be written

$$\tilde{D} = \mathrm{diag}(\partial_x + G_+, \partial_x + G_-) = \mathrm{diag}(D_+, D_-). \tag{69}$$

Then, the following diagram expresses the structure of the intertwining:

$$\begin{array}{ccc} K_0^+ & \xrightarrow{D_+} & K_1^+ \\ \uparrow^{D_0} & & \uparrow^{D_1} \\ K_0^- & \xrightarrow{D_-} & K_1^- \end{array} \tag{70}$$

The adjoint operators act in the reverse direction.

One also observes that

$$\tilde{D}\tilde{D}^\dagger = -\tilde{M}_1^2 + cI \tag{71}$$

and

$$\tilde{D}^\dagger \tilde{D} = -\tilde{M}_0^2 + cI, \tag{72}$$

where I is the 2×2 identity matrix. This is the manifestation of factorization in the Dirac case.

Requiring that the diagram (70) commute gives the consistency relations,

$$D_1 D_- = D_+ D_0 \tag{73}$$

and

$$D_0 D_-^\dagger = D_+^\dagger D_1. \tag{74}$$

Using the *Ansätze* for D_+ and D_- and equating like powers of derivatives in these equations, one finds equations that are equivalent to (40)–(42). This means that the diagram carries the same information as the original matrix intertwining relation (36).

Solving the consistency conditions proceeds as in the Schrödinger case. First, the Riccati equations based on v_0 are converted to linear form by the substitutions

$$G_+ = \frac{-\tilde{\phi}_1'}{\tilde{\phi}_1} \tag{75}$$

and

$$G_- = \frac{-\tilde{\phi}_2'}{\tilde{\phi}_2}. \tag{76}$$

One finds that $\tilde{\phi}_1$ and $\tilde{\phi}_2$ are the components of a spinor eigenfunction of $\tilde{M}_0^2$. However, not every eigenfunction of $\tilde{M}_0^2$ can be used. It is necessary that the $\tilde{M}_1^2$ truly be the square of an $\tilde{M}_1$. Since the diagram (70) is equivalent to the intertwining relation (36), it is clear that the components of the spinor eigenfunction must be related by the intertwining that factors $\tilde{M}_0^2$.

It is sufficient that $\tilde{\phi}_1$ and $\tilde{\phi}_2$ be the components of a spinor eigenfunction of $\tilde{M}_0$. This is not a necessary condition, however, because the intertwining needs to go only in one direction: if one of the components happens to be the generator of the intertwining, it will be annihilated, while the other component will intertwine to give it. The result is that the spinor is not an eigenfunction of $\tilde{M}_0$, but the components are linked by intertwining. This condition takes the form

$$\tilde{M}_0 \begin{pmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \end{pmatrix} = k \begin{pmatrix} 0 \\ \tilde{\phi}_2 \end{pmatrix}. \tag{77}$$

The spinor may be termed a "partial eigenfunction."

From

$$h = \frac{1}{2i}(G_+ - G_-), \tag{78}$$

one has

$$h = \frac{1}{2i}\frac{(\tilde{\phi}_2/\tilde{\phi}_1)'}{\tilde{\phi}_2/\tilde{\phi}_1}. \tag{79}$$

Using the similarity transformation, this can be written in terms of the ratio $\Gamma = \phi_1/\phi_2$ of the components of the corresponding spinor eigenfunction of M_0^2 as

$$h = \frac{-\Gamma'}{1+\Gamma^2} \tag{80}$$

This gives the transformation between potentials

$$v_1 = v_0 + 2(\arctan\Gamma)'. \tag{81}$$

This is the analog of (5). It may be used to find the first of the two equations defining the auto-Bäcklund transformation between solutions of the modified Korteweg–deVries equation.[18]

To find the analog of (6), we must express v_0 in terms of Γ. This is possible because the components of the spinor $\tilde{\Phi}$ generating the intertwining are themselves related by intertwining. In the case that this spinor is an eigenfunction, the similarity-transformed matrix equation

$$M_0\Phi = -k\Phi \tag{82}$$

can be solved as two simultaneous equations for v_0 in terms of $\Gamma = \phi_1/\phi_2$. One finds

$$v_0 = -\frac{\Gamma' + 2k\Gamma}{1+\Gamma^2}. \tag{83}$$

This is the analog of the Ricatti equation (6). Using this in (42), g is found to be

$$g = -k + \frac{2k\Gamma^2}{1+\Gamma^2}. \tag{84}$$

The intertwining operator takes the final form

$$D = \begin{pmatrix} \partial_x - k + \dfrac{2k\Gamma^2}{1+\Gamma^2} & \dfrac{-\Gamma'}{1+\Gamma^2} \\ \dfrac{\Gamma'}{1+\Gamma^2} & \partial_x - k + \dfrac{2k\Gamma^2}{1+\Gamma^2} \end{pmatrix}. \tag{85}$$

This agrees with a result given by Flaschka and McLaughlin[19] in a discussion of Bäcklund transformations.

In the case that $\tilde{\Phi}$ is a partial eigenfunction, a similarity transformation of (77) with $(\sigma_2 + \sigma_3)/\sqrt{2}$ gives

$$M_0\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \frac{k}{2}\begin{pmatrix}\phi_1 + i\phi_2\\ -i\phi_1 + \phi_2\end{pmatrix}. \tag{86}$$

This can be solved for v_0

$$v_0 = -\frac{\Gamma' + k\Gamma + (ik/2)(1-\Gamma^2)}{1+\Gamma^2}, \tag{87}$$

which, in turn, leads to

$$g = \frac{-k}{2} + \frac{k\Gamma^2 + ik\Gamma}{1+\Gamma^2}. \tag{88}$$

To illustrate the use of the intertwining operator, it is useful to consider an example. Suppose $v_0 = 0$. Equation (83) can be solved for Γ,

$$\Gamma = e^{-2kx} + c. \tag{89}$$

For simplicity, take $c = 0$. Using (81), one finds

$$v_1 = -2k\,\mathrm{sech}2kx. \tag{90}$$

This can be checked to be a one-soliton solution of the MKdV equation (34) at parameter time $\tau = 0$.

Alternatively, one could start from a spinor eigenfunction of $\tilde{M}_0^2$. Choosing $\tilde{\phi}_1 = \cosh kx + i\sinh kx$ as an eigenfunction of ∂_x^2, the second component is found from the eigenfunction condition $\tilde{M}_0\tilde{\Phi} = -k\tilde{\Phi}$,

$$\tilde{\phi}_2 = \frac{-i}{k}\partial_x\tilde{\phi}_1 = \cosh kx - i\sinh kx. \tag{91}$$

From

$$v_1 = -2h = \frac{1}{i}\left(\frac{\tilde{\phi}_1'}{\tilde{\phi}_1} - \frac{\tilde{\phi}_2'}{\tilde{\phi}_2}\right), \tag{92}$$

one finds

$$v_1 = 2k\,\mathrm{sech}2kx. \tag{93}$$

An important conclusion follows from this second example: if one constructs new potentials by starting from eigenfunctions of the Schrödinger Hamiltonian operators in $\tilde{M}_0^2$, one must consider complex as well as real eigenfunctions. This is a new feature compared to the Schrödinger case. There, only real eigenfunctions needed to be considered because an intertwining based on a complex eigenfunction would produce a complex potential. Here, a complex Schrödinger potential may correspond to a real electric potential in the Dirac equation.

IV. DIRAC EQUATION IN AN EXTERNAL SCALAR FIELD

The two-dimensional Dirac operator for a fermion in an external scalar field has the form

$$L_0 = -i\sigma_3\partial_x + \sigma_1 w_0. \tag{94}$$

The intertwining relation is $L_1 D = DL_0$. As mentioned above, the change of variables $x \to ix$, will convert the intertwining formulas above to those for the present case. Making this change, one finds

$$\tilde{L}_0 = -i\sigma_2\partial_x - \sigma_1 w_0 = \begin{pmatrix} 0 & D_0 \\ D_0^\dagger & 0\end{pmatrix} \tag{95}$$

and

$$\tilde{L}_0^2 = \begin{pmatrix} -\partial_x^2 + w_0' + w_0^2 & 0 \\ 0 & -\partial_x^2 - w_0' + w_0^2\end{pmatrix} = \mathrm{diag}(H_0^+, H_0^-). \tag{96}$$

Dividing out an overall factor of $-i$, the intertwining operator $\tilde{D}$ takes the form

$$\tilde{D} = \begin{pmatrix} \partial_x + F_+ & 0 \\ 0 & \partial_x + F_-\end{pmatrix} = \begin{pmatrix} D_+ & 0 \\ 0 & D_-\end{pmatrix}, \tag{97}$$

where F_+ and F_- satisfy the Riccati equations

$$
\begin{aligned}
-F_+' + F_+^2 &= w_0' + w_0^2 + c, \\
-F_-' + F_-^2 &= -w_0' + w_0^2 + c, \\
F_+' + F_+^2 &= w_1' + w_1^2 + c, \\
F_-' + F_-^2 &= -w_1' + w_1^2 + c.
\end{aligned} \tag{98}
$$

Observe that

$$\tilde{D}\tilde{D}^\dagger = \tilde{L}_1^2 + cI \tag{99}$$

and

$$\tilde{D}^\dagger \tilde{D} = \tilde{L}_0^2 + cI. \tag{100}$$

Again, this is how factorization makes its appearance in the Dirac case. The same diagram as above describes the structure of the intertwining

$$
\begin{array}{ccc}
H_0^+ & \xrightarrow{D_+} & H_1^+ \\
\uparrow D_0 & & \uparrow D_1 \\
H_0^- & \xrightarrow{D_-} & H_1^-
\end{array} \tag{101}
$$

And, again one has the consistency relations

$$D_1 D_- = D_+ D_0 \tag{102}$$

and

$$D_0 D_-^\dagger = D_+^\dagger D_1. \tag{103}$$

In the no-tilde basis, the intertwining operator can be expressed in terms of the ratio of components $\Gamma = \phi_1/\phi_2$ of the spinor eigenfunction $L_0\Phi = -k\Phi$ generating the transformation,

$$D = \begin{pmatrix} \partial_x - ik + \dfrac{2ik\Gamma^2}{1+\Gamma^2} & \dfrac{-\Gamma'}{1+\Gamma^2} \\ \dfrac{\Gamma'}{1+\Gamma^2} & \partial_x - ik + \dfrac{2ik\Gamma^2}{1+\Gamma^2} \end{pmatrix}. \tag{104}$$

In terms of Γ, the original potential is

$$w_0 = \frac{i\Gamma' - 2k\Gamma}{1+\Gamma^2}. \tag{105}$$

The new potential is

$$w_1 = w_0 - 2i(\arctan\Gamma)'. \tag{106}$$

In the case that the intertwining is generated by a partial eigenfunction (77), one has

$$D = \begin{pmatrix} \partial_x + \dfrac{-ik - 2k\Gamma + ik\Gamma^2}{2(1+\Gamma^2)} & \dfrac{-\Gamma'}{1+\Gamma^2} \\ \dfrac{\Gamma'}{1+\Gamma^2} & \partial_x + \dfrac{-ik - 2k\Gamma + ik\Gamma^2}{2(1+\Gamma^2)} \end{pmatrix}, \tag{107}$$

while the original potential is given by

$$w_0 = -\frac{-i\Gamma' + k\Gamma + (ik/2)(1-\Gamma^2)}{1+\Gamma^2}, \tag{108}$$

and the new potential is given by (106).

The form (96) of $\tilde{L}_0^2$ shows that there is a correspondence between factorizable Schrödinger potential problems and two-dimensional Dirac equations in an external scalar field. This has been remarked before by Cooper, Khare, Musto, and Wipf,[13] and the correspondence can be used to produce exactly solvable potentials for the Dirac equation. Given a solvable Schrödinger equation with potential, any eigenfunction ψ can be used to give a factorization

$$-\partial_x^2 + V_0 - c = \left(-\partial_x - \frac{\psi'}{\psi}\right)\left(\partial_x - \frac{\psi'}{\psi}\right), \tag{109}$$

where c is the eigenvalue of ψ. Using the intertwining between Schrödinger Hamiltonians generated by this eigenfunction

$$D_0 = \partial_x - \frac{\psi'}{\psi}, \tag{110}$$

the partner Hamiltonian can be found. The pair form the diagonal entries of $\tilde{L}_0^2$.

Factorization of the Schrödinger equation requires that the eigenvalue be shifted, so the eigenvalues of $\tilde{L}_0^2$ are $E - c$. This makes the eigenvalues of $\tilde{L}_0$ equal to $\pm(E-c)^{1/2}$. The known eigenfunctions of the initial Schrödinger equation give one component of the spinor eigenfunction of $\tilde{L}_0$ and application of the intertwining operator $\pm(E-c)^{-1/2}\tilde{D}$ to them gives the other. The resulting potential in the Dirac equation is $w_0 = \psi'/\psi$.

This approach is convenient when one has a collection of solvable Schrödinger equations, as one has, for example, in the soliton solutions of the KdV equation. From the intertwining of Schrödinger potentials, it was found that one of the consistency conditions was the Miura transformation, which relates solutions of KdV to solutions of MKdV. Given a KdV soliton, solving this Riccati equation gives an MKdV soliton, which is then an exactly solvable Dirac potential.

For example, the trivial Hamiltonian $H_1^- = -\partial_x^2 + k^2$ has the solution $\cosh kx$, which generates the intertwining operator $D_1 = \partial_x - k\tanh kx$. Since $w_1 = -k\tanh kx$ solves the Riccati equation (6), it is a soliton solution of MKdV at parameter time $\tau = 0$ and also an exactly soluble Dirac potential. The partner Hamiltonian operator is easily shown to be $H_1^+ = -\partial_x^2 - 2k^2\,\mathrm{sech}^2 kx + k^2$. Starting from the solutions of H_1^-, the intertwining operator D_1 gives those of H_1^+. Together these are (up to a normalization factor) the components of the spinor eigenfunctions of $\tilde{L}_1$ with the potential w_1.

This potential can be also be constructed by intertwining from $\tilde{L}_0$ with $w_0 = -k$. This is a case where $\tilde{\Phi}$ is a partial eigenfunction of $\tilde{L}_0$. Choosing the spinor

$$\tilde{\Phi} = \begin{pmatrix} \cosh kx \\ -e^{kx} \end{pmatrix}, \tag{111}$$

it is easy to see that (77) is satisfied. Performing the similarity transformation to Φ, one finds

$$\Gamma = \frac{1 + 2i + e^{-2kx}}{i + 2 + ie^{-2kx}}. \tag{112}$$

Using (108), one verifies that $w_0 = -k$. From (106), one finds

$$w_1 = -k \tanh kx. \tag{113}$$

Alternatively, one can derive w_1 by starting with $w_0 = -k$ and solving the Riccati equation (108) for Γ.

V. SUPERSYMMETRY ALGEBRA FOR DIRAC INTERTWINING

Just as the structure of intertwining for Schrödinger potentials can be represented by a supersymmetry algebra, so can the structure of transformations between Dirac potentials captured in the diagram (101). The algebra given here will be that for the case of an external scalar field. The same algebra holds for the case of an electric potential with the qualification that the adjoint does not include complex conjugation. Without this qualification, the Hamiltonians appearing in Sec. III would not be self-adjoint, and the intertwining operators acting in opposite directions would not be related by the adjoint.

There are four fundamental intertwining operators D_0, D_1, D_+, and D_- involved rather than the single one in the Schrödinger case. This signfies that the algebra will be $N = 4$. The basic requirements behind the algebra are that the supersymmetry generators be nilpotent and that the intertwining relations and intertwining consistency conditions (102) and (103) be reproduced.

A 4×4 matrix representation of the algebra can be constructed. Using the notation that e_{ij} is 1 in the i,jth entry of the matrix and zero everywhere else, the intertwining operators can be assigned to matrix locations as $e_{12}D_0$, $e_{34}D_1$, $e_{31}D_+$, and $e_{42}D_-$. (The adjoint operators are placed in the matrix adjoint locations.) Four supersymmetry generators $Q_{\alpha i}$ can be defined in terms of these as

$$\begin{aligned} Q_{00} &= i\sigma_2 \otimes (e_{31}D_+ + e_{42}D_-), \\ Q_{10} &= \sigma_1 \otimes (e_{31}D_+ - e_{42}D_-), \\ Q_{01} &= i\sigma_2 \otimes (e_{12}D_0 - e_{34}D_1), \\ Q_{11} &= \sigma_1 \otimes (e_{12}D_0 + e_{34}D_1), \end{aligned} \tag{114}$$

where the Pauli matrices are tensored on so that all of the Q's will satisfy anticommutation relations. Two super-Hamiltonians will appear in the algebra,

$$\mathcal{H}_1 = 1 \otimes \mathrm{diag}(H_0^+, H_0^-, H_1^+, H_1^-) \tag{115}$$

and

$$\mathcal{H}_0 = \sigma_3 \otimes \mathrm{diag}(-H_0^+, H_0^-, H_1^+, -H_1^-), \tag{116}$$

and two central charges

$$c_1 = 1 \otimes \mathrm{diag}(c, c, c, c), \tag{117}$$

$$c_0 = \sigma_3 \otimes \mathrm{diag}(-c, c, c, -c), \tag{118}$$

where c is the constant shift appearing in (99) and (100).

The superalgebra is then given by

$$\{Q_{\alpha i}, Q_{\beta j}\} = 0, \tag{119}$$

$$[Q_{\alpha i}, \mathcal{H}_j] = 0, \tag{120}$$

and

$$\begin{aligned} \{Q_{\alpha i}, Q^\dagger_{\beta j}\} = {} & \delta_{\alpha\beta}\delta_{ij}\mathcal{H}_1 + (-1)^j(1 - \delta_{\alpha\beta})\delta_{ij}\mathcal{H}_0 \\ & + \delta_{i0}\delta_{j0}[\delta_{\alpha 1}\delta_{\beta 1}c_1 + (1 - \delta_{\alpha\beta})c_0]. \end{aligned} \tag{121}$$

(Note that the adjoint does not affect the indices of the supersymmetry generators.)

VI. CONCLUSIONS

Applying the method of intertwining to 2×2 matrix differential operators, transformations between one-dimensional Dirac potentials have been derived. One-dimensional electric potentials and one-dimensional external scalar fields in which the Dirac equation is exactly solvable may be constructed by starting from an electric potential or external scalar field with known solution. In particular, one may begin from the free-field case. The intertwining transformations are generated by spinor eigenfunctions of the initial solvable problem or by certain special spinors which are partly eigenfunctions.

These intertwining transformations are analogous to the Darboux transformations between Schrödinger potentials. Darboux transformations are closely related to Bäcklund transformations for the KdV equation and may be used to construct KdV solitons. The intertwining transformations constructed here for the Dirac equation bear a similar relation to Bäcklund transformations for the MKdV equation, and they may be used to construct MKdV solitons.

Though not discussed here, it is well known[20] that sequential application of Darboux transformations in the Schrödinger case can be used to construct potentials with any desired collection of bound states. The same is true for the Dirac potentials. The bound-state energies that are added correspond to the eigenvalues of the eigenfunctions in the initial potential used in constructing the transformations. This may be useful in applications where the spectrum is known experimentally, but the Dirac potential is not known.

In the Schrödinger case, Darboux transformations may be understood in terms of the factorization they give of the intertwined Hamiltonian operators. In the Dirac case, in an appropriate basis, the square of the Dirac operator takes the form of a diagonal matrix composed of two Hamiltonian operators, and the Dirac operator itself is a factorization of them. The intertwining operator and its adjoint provide a second factorization of the square of each of the intertwined Dirac operators, shifted by a constant.

The structure of the intertwining relations in the Schrödinger case is described by an $N = 1$ supersymmetry. It has been shown here that an $N = 4$ superalgebra describes the structure of the intertwining relations for the Dirac case. At present this is only a formal observation, but perhaps it can be used to give further insight into the intertwining.

Elsewhere intertwining has been successfully used to construct exactly solvable potentials for the Schrödinger equation in two dimensions,[1] and contact was made with a class of potentials known to be soluble in higher dimensions. The present work encourages the hope that intertwining can also be used to solve the Dirac equation with higher-dimensional potentials.

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