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Generalized commutation relations for a single-mode oscillator

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A generalized commutation relation of a single-mode oscillator is proposed. Bose-Einstein, Fermi-Dirac, and Greenberg's infinite statistics are shown to be special cases of this commutation relation. The Fock states, coherent states, and analogs of squeezed coherent states for this commutation relation are constructed. Finally, the quantization of a free infinite-statistics field is discussed using the Umezawa-Takahashi procedure [H. Umezawa and Y. Takahashi, Prog. Theor. Phys. **9**, 14 (1953); Nucl. Phys. **51**, 193 (1964)].

Recently Greenberg[1] constructed an example of infinite statistics in which all the representations of the symmetric group can occur. Unlike its counterparts, namely Bose-Einstein (BE) and Fermi-Dirac (FD) statistics or parastatistics, the number operator in this case is not bilinear in a and $a^\dagger$ but an infinite series. In this paper, we show that we can write down a generalized commutation relation (CR), in the quantum-mechanical context, for a single mode for which the number operator is again an infinite series and contains BE, FD, and Greenberg's infinite statistics (IS) as special cases. We construct the Fock states, coherent states,[2] and analogs of familiar bosonic squeezed states[3] and show that they reduce in the appropriate limits to the known results for BE and FD commutation relations. We calculate the uncertainty products for these states in the case of infinite statistics. Finally we discuss the quantization of free fields with infinite statistics following the procedure of Umezawa and Takahashi.[4]

GENERALIZED COMMUTATIONS RELATIONS, NUMBER OPERATOR, AND FOCK STATES

Consider the commutation relation

$$aa^\dagger - qa^\dagger a = 1 . \tag{1}$$

By taking the Hermitian conjugate of this expression it is easily seen that q must be real. Starting with (1) it is readily verified that the number operator N which satisfies

$$[a,N] = a , \tag{2}$$

$$[a^\dagger,N] = -a^\dagger , \tag{3}$$

is

$$\begin{aligned} N &= \sum_{n=1}^{\infty} \frac{(1-q)^n}{(1-q^n)} a^{\dagger n} a^n \\ &= a^\dagger a + \left(\frac{1-q}{1+q} \right) a^{\dagger 2} a^2 + \cdots \\ &\quad + \frac{(1-q)^{n-1}}{1+q+\cdots+q^{n-1}} a^{\dagger n} a^n + \cdots . \end{aligned} \tag{4}$$

The normalized Fock states for the CR of (1) are

$$\begin{gathered} |0\rangle,\ a^\dagger|0\rangle,\ \frac{a^{\dagger 2}}{(1+q)^{1/2}}|0\rangle, \\ \frac{a^{\dagger 3}}{[(1+q)(1+q+q^2)]^{1/2}}|0\rangle, \ldots . \end{gathered} \tag{5}$$

For $q=1$, the number operator terminates after the first term and we recover both CR's and the number operator of BE statistics. Also the Fock states turn into the well-known Fock states of BE statistics, namely, $(a^\dagger)^n/\sqrt{n!}|0\rangle$. For $q=-1$, Eq. (1) turns into the anticommutator of FD statistics. Further, with the additional requirement $a^2=a^{\dagger 2}=0$, the expression in (4) reduces to the known result and for Fock states one obtains the two states $|0\rangle$ and $a^\dagger|0\rangle$. For $q=0$, we obtain from (4)

$$N_{\rm IS} = \sum_{n=1}^{\infty} a^{\dagger n} a^n , \tag{6}$$

which is the result due to Greenberg for a single mode obeying $aa^\dagger = I$. The normalized Fock states in this are $|0\rangle, a^\dagger|0\rangle, a^{\dagger 2}|0\rangle, \ldots$. We note that the expression for N given here has been obtained purely algebraically by only making use of the commutation relations (1) and has the virtue of reducing to the expression given by Greenberg in the $q=0$ limit. Other constructions of N which have been considered in the literature[5] that rely on bosonic Fock states do not have this property and lead to a singular expression in the $q=0$ limit.

DISPLACEMENT OPERATOR

For the CR of (1), we now construct a generator $A^\dagger$ for displacements satisfying

$$[a,A^\dagger] = 1 . \tag{7}$$

It is easily checked that the $A^\dagger$ given by

$$A^\dagger = a^\dagger \left(1 + \frac{1-q}{1+q} a^\dagger a + \frac{(1-q)^2}{1+q+q^2} a^{\dagger 2} a^2 + \cdots \right) \tag{8}$$

satisfies (7) for all q except $q=-1$. From (7), it follows that

$$\exp(A^{\dagger}\alpha)a\exp(-A^{\dagger}\alpha)=a-\alpha\,. \tag{9}$$

For $q=1$, $A^{\dagger}$ reduces to $a^{\dagger}$. For $q=0$, we get

$$A^{\dagger}=a^{\dagger}(N_{\rm IS}+1)=N_{\rm IS}a^{\dagger}\,. \tag{10}$$

For $q=-1$, with $a^2=a^{\dagger 2}=0$, $A^{\dagger}$ reduces to $a^{\dagger}$. In this case, as noted above, although (7) is no longer valid, (9) still holds with α regarded as a Grassmann number anticommuting with a and $a^{\dagger}$.

COHERENT STATES

Having constructed $A^{\dagger}$, the generator for displacements, satisfying (7), the construction of coherent states for arbitrary q, satisfying

$$a|\alpha\rangle=\alpha|\alpha\rangle\,, \tag{11}$$

is immediate. The normalized states satisfying (11) are given by

$$|\alpha\rangle=[e_q(|\alpha|^2)]^{-1/2}\exp(A^{\dagger}\alpha)|0\rangle \tag{12}$$

where

$$e_q(x)=\sum_{n=0}^{\infty}\frac{x^n}{n_q!}\,,\quad n_q=\frac{1-q^n}{1-q}\,. \tag{13}$$

For $q=1$, we get the usual Sudarshan-Glauber coherent states.[2] For $q=-1$, with α regarded as a Grassmann number, we get the fermionic coherent states of Ohnuki.[6] For $q=0$, the constant C is found to be $(1-|\alpha|^2)^{1/2}$ and the normalized coherent states for the Greenberg CR are found to be

$$|\alpha\rangle_{\rm IS}=\sqrt{1-|\alpha|^2}\sum_{n=0}^{\infty}\alpha^n(a^{\dagger})^n|0\rangle\,. \tag{14}$$

The coherent states for Greenberg CR's can also be derived in an alternate simpler way. Expanding $|\alpha\rangle$ in terms of the number states

$$|\alpha\rangle_{\rm IS}=\sum_{n=0}^{\infty}C_n|n\rangle\,, \tag{15}$$

and substituting it into (11), one obtains, using $a|n\rangle=|n-1\rangle$, the following recursion relation

$$C_n=\alpha C_{n-1}\,, \tag{16}$$

which can easily be solved to yield

$$C_n=\alpha^n C_0\,, \tag{17}$$

and hence

$$|\alpha\rangle_{\rm IS}=C_0\sum_n\alpha^n|n\rangle\,. \tag{18}$$

Fixing C_0 by normalizing $|\alpha\rangle_{\rm IS}$ to 1 we get the expression in (14).

In the many mode case of infinite statistics, one finds that the generators of displacement $A_i^{\dagger}$ satisfying

$$[a_i,A_j^{\dagger}]=\delta_{ij}\,, \tag{19}$$

are given by

$$A_i^{\dagger}=a_i^{\dagger}+\sum_k a_k^{\dagger}a_i^{\dagger}a_k+\sum_{k,l}a_k^{\dagger}a_l^{\dagger}a_i a_l a_k+\cdots\,. \tag{20}$$

One, however, has

$$[A_i^{\dagger},A_j^{\dagger}]\neq 0\,.$$

The construction of the coherent states in many mode case is immediate.

REALIZATION OF $[x,p]=i$

It is interesting to note that by defining

$$x=\frac{a+A^{\dagger}}{(2)^{1/2}}\,,\quad p=\frac{a-A^{\dagger}}{(2)^{1/2}i}\,, \tag{21}$$

we obtain, by virtue of (7), a realization (albeit non-Hermitian) of the CR's

$$[x,p]=i\,. \tag{22}$$

It is also easy to see that

$$\tfrac{1}{2}(x^2+p^2)=(N+\tfrac{1}{2})\,, \tag{23}$$

which is the usual result for harmonic oscillator. The surprise is that it is valid for all q's (except $q=-1$) of the CR proposed here. Further if the canonical density matrix is taken to be $e^{-\beta N}$ then $\langle N\rangle=(e^{\beta}-1)^{-1}$ for all q except $q=-1$.

SOME SPECIAL STATES

Having constructed the generator $A^{\dagger}$ of displacement, it is elementary to construct analogs of the bosonic squeezed states[3] for the CR equation (1). Using (1) it follows that

$$b=\exp\left[\frac{z}{2}(a^2-A^{\dagger 2})\right]a\exp\left[-\frac{z}{2}(a^2-A^{\dagger 2})\right],$$
$$=(\cosh z)a+(\sinh z)A^{\dagger}\,. \tag{24}$$

From $a|0\rangle=0$ we get

$$b|z\rangle=0\,, \tag{25}$$

where

$$|z\rangle=\exp\left[\frac{z}{2}(a^2-A^{\dagger 2})\right]|0\rangle\,. \tag{26}$$

The analogs of the Yuen-type squeezed states[3] are given by

$$|z,\alpha\rangle=\exp\left[\frac{z}{2}(a^2-A^{\dagger 2})\right]\exp(A^{\dagger}\alpha)|0\rangle\,. \tag{27}$$

It is easily checked that they satisfy

$$b|z,\alpha\rangle=\alpha|z,\alpha\rangle\,. \tag{28}$$

It should be noted that for $q=-1$, the Fermi case, since $a^2=a^{\dagger 2}=0$, Yuen-type states do not exist.

UNCERTAINTY PRODUCTS

If we define $a = X + iP$, then one finds that for coherent states of infinite statistics

$$\Delta X \Delta P = \tfrac{1}{4}(1 - |\alpha|^2) . \tag{29}$$

The product $\Delta X \Delta P$ could, therefore, be made as small as desired. For the Fock states of infinite statistics

$$\bar{X} = 0, \ \bar{P} = 0, \ \overline{X^2} = \overline{P^2} = \tfrac{1}{4}(2 - \delta_{n,0}) , \tag{30}$$

and hence

$$\Delta X \Delta P = \begin{cases} \tfrac{1}{4} \ \text{for} \ n = 0 , \\ \tfrac{1}{2} \ \text{for} \ n > 0 . \end{cases} \tag{31}$$

QUANTIZATION OF FREE INFINITE-STATISTICS FIELDS

We now quantize the infinite-statistics field theory proposed by Greenberg using the procedure of Umezawa and Takahashi. In this procedure the field equations are postulated. For a given commutation relation of a and $a^\dagger$, a Hamiltonian is postulated, so that the Heisenberg equations of motion are the same as the postulated field equations. Such a procedure is noncanonical as a Lagrangian is not postulated. Consider the quantization of the free Schrödinger field,

$$\left[i\frac{\partial}{\partial t} + \frac{1}{2m}\nabla^2 \right] \psi(\mathbf{x},t) = 0 . \tag{32}$$

Let $\psi(\mathbf{x},t)$ be expanded as

$$\psi(\mathbf{x},t) = \sum_k a_k(t) \exp(i\mathbf{k}\cdot\mathbf{x}) , \tag{33}$$

with

$$a_k(t) = e^{-i\omega_k t} a_k, \ \ \omega_k = \frac{k^2}{2m} ;$$

also

$$a_k a_l^\dagger = \delta_{kl} . \tag{34}$$

Postulate the Hamiltonian

$$H = \sum_k \omega_k N_k \tag{35}$$

with

$$N_k = a_k^\dagger a_k + \sum_l a_l^\dagger a_k^\dagger a_k a_l + \cdots . \tag{36}$$

One can easily verify that

$$i\frac{\partial}{\partial t}\psi = [\psi(\mathbf{x},t), H] \tag{37}$$

reproduces (32) using (34) and (36). It must be emphasized that the Hamiltonian (35) cannot be derived from the Lagrangian which gives (32), and, therefore, this theory cannot be quantized within the canonical procedure.

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