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Interface fluctuations in random media

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We study a stochastic model of an interface moving through a random medium. This model differs from the standard Kardar-Parisi-Zhang equation by the fact that the fluctuations are quenched random variables. We find an intermediate scaling regime with roughness exponent approximately equal to 0.75; this compares favorably with recent experiments [M. A. Rubio, C. A. Edwards, A. Dougherty, and J. P. Gollub, Phys. Rev. Lett. **63**, 1685 (1989)] on multiphase flow through a bead pack.

There has been much recent interest in the scaling behavior of nonequilibrium interfaces broadened by noise. Progress has occurred on both the theoretical and simulational fronts towards understanding the various scaling regimes as functions of the substrate dimensionality and the strength and correlation properties of the noise. In this context, a recent experiment on the displacement of one fluid by another in a porous medium[1] is worth noting. Specifically, the observed roughness exponent of the two-fluid interface does not correspond to any known theoretical model. In this paper, we propose a simple model that reproduces the qualitative features of the experiment and shows the observed unusual scaling, $w \sim L^{\alpha}$ with $\alpha \sim 0.73$, over some range of length scales. We shall argue, though, that the true asymptotic scaling behavior is however not anomalous, and is that of the Kardar-Parisi-Zhang[2] (KPZ) equation (or equivalently, ballistic aggregation[3] or the Eden model[4]) in $1+1$ dimensions.

The KPZ equation is the simplest continuum description of a stable nonequilibrium interface roughened by noise. The equation governs the time development of the height $y(\mathbf{x},t)$ of a surface above a d-dimensional substrate,

$$\dot{y}(\mathbf{x}) = D\nabla^2 y + \lambda |\nabla y|^2 + \eta_{\mathrm{KPZ}} , \tag{1}$$

where the noise η_{KPZ} is δ-function correlated in substrate position and time:

$$\langle \eta_{\mathrm{KPZ}}(\mathbf{x},t)\eta_{\mathrm{KPZ}}(\mathbf{x}',t')\rangle = S\delta^d(\mathbf{x}-\mathbf{x}')\delta(t-t') . \tag{2}$$

The interface width, averaged over a region of linear dimension L,

$$w(L,t) \equiv (\langle y^2\rangle - \langle y\rangle^2)^{1/2} , \tag{3}$$

saturates at long times to a value which scales as a power law in L:

$$w(L, t\rightarrow\infty) \sim L^{\alpha} . \tag{4}$$

The exponent α depends on the dimension d, but is independent of the coupling constants (at least over some finite range). This scaling behavior has been shown to agree with that observed for a number of other interface models, most notably the Eden and ballistic aggregation models. In the case of $d=1$ the exponent α is exactly $\frac{1}{2}$.

The physics of the experiment differs from that embodied in the KPZ equation in two regards. First, the noise is *quenched*, as the irregularities of the bead pack constituting the porous medium are fixed during any particular experiment. This is in contrast to the noise in the KPZ equation, which is uncorrelated in time. This effect should serve to broaden the interface; if the interface encounters a point where it becomes temporarily pinned, it will tend to experience that particular (large) value of the random force for a relatively long time, in fact until it becomes depinned. This is clearly different than being subject to a noise field which is completely uncorrelated in time. The second major difference is the nonlocal nature of the flow field. The result of this is a relative stabilization of the interface. This can be seen in the linear stability of the planar interface where the decay rate of a perturbation is proportional to the magnitude of the wave vector of the perturbation, as opposed to the square magnitude of the wave vector in the KPZ case. This effect would tend to make the interface less broad, decreasing the exponent. This can be seen explicitly in a simulation of the interface width using a time-reversed diffusion-limited aggregation[5] (DLA) algorithm. Since we are interested in understanding how the exponent can be larger than the KPZ value of $\frac{1}{2}$, we shall concentrate in this paper on the first mechanism, namely the effect of the quenched noise.

The model we study in this paper is

$$\frac{dy}{dt} = D\nabla^2 y + F + \eta(\mathbf{x},y) , \tag{5}$$

where $\eta(\mathbf{x},y)$ represents the quenched noise with $\langle\eta\rangle = 0$ and the correlation function $\langle \eta(\mathbf{x},y)\eta(\mathbf{x}',y')\rangle = g^2\delta(\mathbf{x}-\mathbf{x}')\delta(y-y')$. F is the pushing force, and D is the surface tension parameter. This model is exactly the KPZ equation, except for the nature of the noise and the absence of the nonlinear λ term. The latter term is, in fact, induced automatically in a renormalization-group sense since the noise is a nonlinear function of the interface position, y. The above model has, in fact, been written down previously in two different contexts, domain walls in magnetic systems[6] and segregating fluids in porous media.[7] In both of those studies, the emphasis was on the pinning-depinning transition. Let us first analyze the model in two different limits:

(1) The strong pushing regime, where $F \gg \langle \eta^2 \rangle^{1/2}$. Here we can write: $y = Ft + \tilde{y}$, and the equation becomes approximately

$$\frac{d\tilde{y}}{dt} = D\nabla^2 \tilde{y} + \eta(\mathbf{x}, Ft) .$$

In this limit the model reduces to the case of uncorrelated noise, so that

$$\tilde{y}(\mathbf{k},\omega) = \frac{G(\mathbf{k},\omega)\eta(\mathbf{k},\omega)}{\sqrt{F}}$$

with $G(\mathbf{k},\omega) = (i\omega - Dk^2)^{-1}$. Given the correlation of the noise, it is obvious that

$$\langle \tilde{y}(\mathbf{x},t)\tilde{y}(\mathbf{x}',t)\rangle \sim \frac{g^2}{DF}|\mathbf{x}-\mathbf{x}'|^{(2-d)} .$$

Also, the width, for $d=1$, satisfies $w(L) \sim (g^2/DF)^{1/2} L^{1/2}$, which is the same result as that given by a more sophisticated calculation.[8] The prefactor is proportional to $1/\sqrt{F}$, which coincides with the observed experimental behavior.[1] However, the L exponent α equals $\frac{1}{2}$ and hence is too small when compared with the experimentally observed 0.73.

(2) The static regime, when the interface gets pinned. Here the problem reduces to the one considered by Grinstein and Ma[9] in the magnetic domain problem, where one can write down the Hamiltonian:

$$H = \int d^d x \frac{D}{2} |\nabla y|^2 + \int d^d x \int_0^y dy' \eta(\mathbf{x}, y') .$$

For interface width w, the energy is estimated to be $E/L^d \sim Dw^2/L^2 + gw^{1/2}L^{-d/2}$. Minimizing it, one gets $w \sim L^{(4-d)/3}$. Thus in the case $d=1$, $w \sim L$. Grinstein and Ma argued that this is the critical dimension and below this dimension, the system is unstable against the formation of magnetic droplets.

To study the scaling behavior in the intermediate regime we turn to computer simulation of the equation. We discretize the equation (5) in x direction

$$y_i(t+\Delta t) = y_i(t) + \Delta t \{ y_{i+1}(t) + y_{i-1}(t) - 2y_i(t) + F + \eta(i, [y_i(t)]) \}, \quad i = 1,2,\ldots,N \tag{6}$$

with free boundary conditions: $y_0(t) = y_1(t)$ and $y_{N+1}(t) = y_N(t)$, $[y_i(t)]$ represents the nearest integer smaller than $y_i(t)$. The noise $\eta(i,j)$ is defined on a two-dimensional lattice; according to Eq. (6), the noise is correlated over the distance of a lattice spacing, which thus should correspond to the size of the glass bead in the experiment.

Before turning to the simulation results, we would like to point out several interesting features in Fig. 1 of Ref. 1. First, the interface has a macroscopic structure of several parabolas, each with a typical length ~2–4 cm, which is about the same length scale at which the scaling behavior saturates. Second, the correlation between subsequent images is extremely high. These qualitative features are indicative of a fairly broad Gaussian distribution, arranged so that there will be on average a small number of tightly pinned sites.

Based on the above, we distributed noise sites randomly with density p, i.e., $\eta(i,j)$ is nonzero with probability p. The pinning forces on these sites were chosen from a Gaussian distribution with width Δ. We worked with a lattice 900 sites long in the x direction, and up to 4000 lattice spacings in the growing direction. The interface width $w(L)$ averaged over all segments of horizontal length L, was obtained from 300 time slices from a single run. We also calculated the time correlation of the interface in one long run

$$C_{\text{auto}}(\tau) = \left\langle \int_{-\infty}^{\infty} [y(x,t+\tau) - \bar{y}(t+\tau)][y(x,t) - \bar{y}(t)] dt \right\rangle_x .$$

We found that within the experiment measuring time $C_{\text{auto}}(\tau)$ was finite, which means that all the time slices being averaged over were in fact correlated; this is just what occurs in the experimental data. This correlation forces us to do an ensemble average over different noise realizations to obtain more reliable results. After averaging over ten samples with the same initial condition and parameters, but different seeds of the random-number generator, we obtained the data presented in Fig. 1 for $p=0.1$, $\Delta=1.1$, $D=5.0$, and $F=0.4$. There is a region extending slightly over one decade of length where we see a power-law behavior best fit by $w(L) \sim L^{0.75 \pm 0.02}$, a result which agrees very well with the experiment. In Fig. 2 we show a typical time evolution of the model interface, which has the same large scale structure as seen in the experiment, although ours is somewhat noisier at small scales.

A more detailed examination of Fig. 1 reveals that the effective exponent (the slope of the curve on the log-log

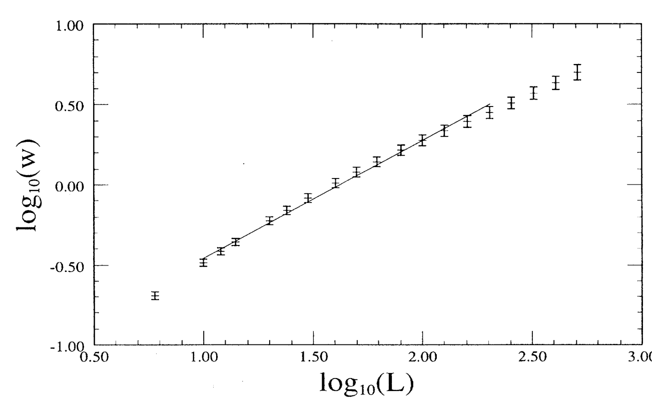

FIG. 1. Ensemble average of width vs length over ten samples with parameter values of $D=5.0$, $F=0.4$, $\Delta=1.1$, and $p=0.1$. The data are best fitted by $w \sim L^{\alpha}$ with $\alpha=0.75 \pm 0.02$.

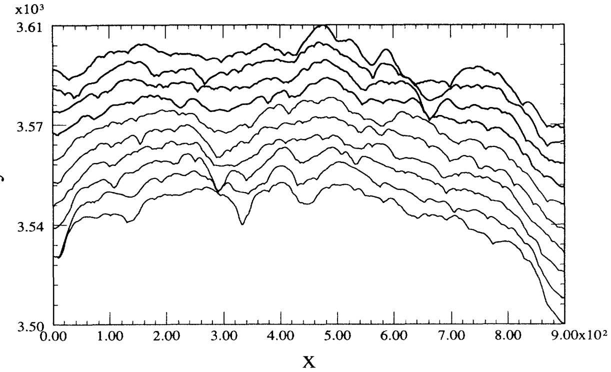

FIG. 2. Typical time sequence of interface with the same parameter as in Fig. 1.

plot) seems to decrease at the largest length scales. To investigate this more systematically, we have studied the effect of varying the forcing F. This is due to the simple observation that Eq. (5) for $d=1$ can be written as

$$\frac{d(yF)}{d(tF^2)}=D\frac{d^2(yF)}{d(xF)^2}+1+\eta(xF,yF)\,.$$

So, enlarging F by some factor is equivalent to enlarging the size of the system by the same factor with fixed F. This allows us to study the dependence of the scaling exponent on the length scale by just changing the value of F, which is computationally very convenient. The results of this study are presented in Fig. 3, where the best fit exponent is graphed versus the asymptotic velocity (on a log scale). This asymptotic velocity, defined as $v_{\rm eff}\equiv\lim_{t\to\infty}\langle y(t)\rangle_x/t$, is a monotonically increasing function of F. We see that while over a large range of velocities (approximately half a decade) the exponent is roughly constant at ~ 0.76, it falls off at the largest velocities to a value near $\frac{1}{2}$. This behavior is in accord with our expectations for the large F regime. It moreover suggests that the experimentally observed value of α is not a true asymptotic exponent. At small velocities, near the pinning threshold, the exponent seems to be rising. This also is consistent with our theoretical expectations. The picture

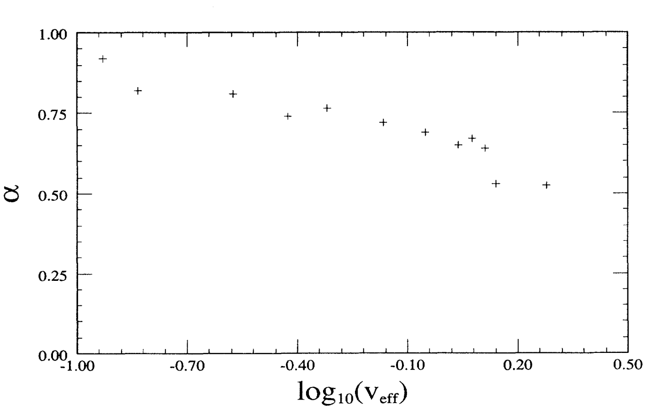

FIG. 3. Roughness exponent vs the effective moving velocity.

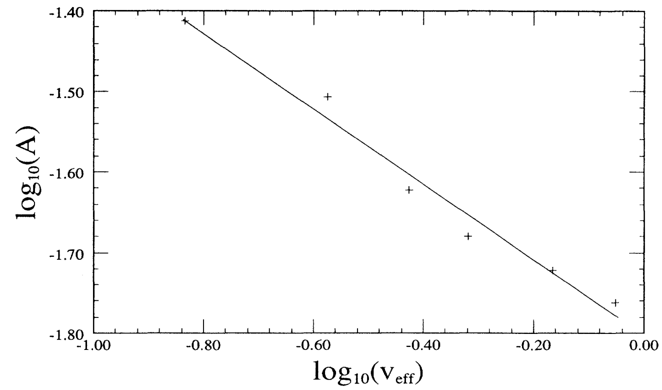

FIG. 4. The prefactor of the second to the seventh points (with exponent range from 0.81 to 0.69) in Fig. 3 vs the effective moving velocity, the slope is -0.47 ± 0.03.

that emerges, then, is that the measured exponent should be a continually varying function of the velocity, with a fairly large plateau region where the value of the exponent is ~ 0.75, roughly halfway between its small and large velocity limits.

Studying the time history of the profile, it appears that the above crossover is due to the presence of a second length scale in the problem, namely the distance between strong pinning centers. On scales shorter than this, the exponent is close to its pinned value, $\alpha\approx 1$. On length scales much larger than this, however, the exponent appropriate to a random moving interface is seen. To put it another way, it is known that using time-correlated noise in the KPZ equation can increase the roughness exponent, *if* the correlations decay as a power law in time and so extend over indefinitely long times. The random pinning forces in our problem, in the presence of finite surface tension D, are only capable of pinning the interface, and thereby correlating it, for a finite time.

It is interesting to study the prefactor in the power law, defined as $A\equiv w/L^{\alpha}$, where α is the measured best fit exponent for the particular parameters. This data, summarized in Figs. 4, 5, and 6, indicate that, as a rough esti-

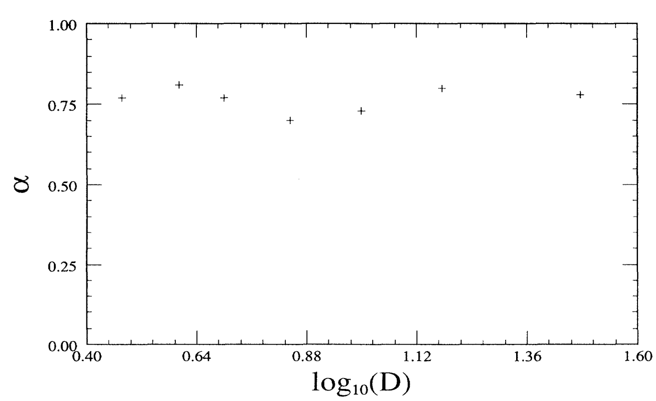

FIG. 5. Roughness exponent vs the surface tension over one decade. The average is about 0.76.

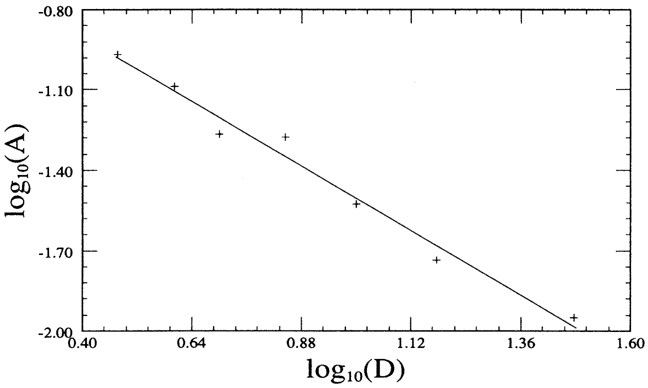

FIG. 6. The prefactor vs the surface tension, the slope is -1.00 ± 0.06.

mate, $A \sim D^{\mu} v_{\rm eff}^{\nu}$, with $\mu=-1.00\pm0.06$ and $\nu=-0.47\pm0.03$. This dependence is consistent with the experiment, if we interpret $v_{\rm eff}$ as proportional to the capillary number in the experiment. We should also note that varying D did not have much effect on either the observed exponent or on the growth velocity.

In going to the pinning regime, our model exhibited a surprising feature. In the finite system in which we did our simulation (which was quite close to the actual size of the experiment if we relate lattice spacing and bead size), the pinning-depinning transition is a first-order transition. In other words, when F is just higher than the threshold, the interface moves with a finite velocity. In the infinite volume limit, this transition is probably second order.[10] This might be an interesting point to examine experimentally.

In summary, we have argued in some intermediate parameter regime, the simple stochastic differential equation model does produce a 0.75 power-law roughness behavior over slightly more than one decade of length. But in the model, we can see a crossover to lower exponents at length scale $L > L_M$, where L_M is the average size of the parabolas that make up the interface. It will be interesting to see what happens if the experiment can be pushed to the two limits, namely, the static and rapidly moving regimes.

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