Wave kinetic equation in a fluctuating medium

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We derive the kinetic equation governing the evolution of the action density of a wave propagating in a weakly nonuniform, nonstationary medium characterized by fluctuations that are small in magnitude (but need not be slowly varying). The method of derivation is based on the Weyl representation of a general linear wave equation, which leads to a natural definition of the wave-action density and an analysis rooted in the ray phase space. The result is thus a *classical wave* derivation of the radiation transport equation for a general scalar wave equation. We illustrate our derivation with three example wave equations governing propagation in fluctuating media.

I. INTRODUCTION

The action density of a wave propagating in a medium is a concept that is central to the analysis of a variety of linear and nonlinear wave processes.[1,2] For example, the action density $J(\mathbf{k})$ for a plane wave with wave vector $\mathbf{k}$ and frequency $\omega(\mathbf{k})$ in a uniform, stationary medium is the wave analogy of the action defined for a classical harmonic oscillator: $J(\mathbf{k})$ is the energy density $U(\mathbf{k})$ divided by the frequency $J(\mathbf{k}) = U(\mathbf{k})/\omega(\mathbf{k})$. The action density is also the classical wave limit of the occupation number $n_{\mathbf{k}}$ concept of field-theoretic treatments of wave processes. In a weakly nonuniform, nonstationary medium, the action density $J(\mathbf{x},\mathbf{k};t)$ properly becomes a density on the $(\mathbf{x},\mathbf{k})$ phase space and its evolution is governed by the wave kinetic equation (WKE)

$$\frac{dJ(\mathbf{x},\mathbf{k};t)}{dt} \equiv \frac{\partial J}{\partial t} + \{J,\Omega\} = \Sigma(\mathbf{x},\mathbf{k};t) \ . \qquad (1)$$

Here, the curly brackets denote the Poisson bracket of classical mechanics, and $\Omega(\mathbf{x},\mathbf{k};t)$ is the local dispersion relation [obtained by requiring the local dispersion function $D_0(\mathbf{x},t,\mathbf{k},\omega)$ to vanish, $D_0 = 0 \Rightarrow \omega = \Omega(\mathbf{x},\mathbf{k};t)$]. Thus, (1) states that the wave action density is convected along rays in the $(\mathbf{x},\mathbf{k})$ phase space; the rays are generated by Hamilton's equations, with the local dispersion relation playing the role of the Hamiltonian. In the absence of sources $\Sigma(\mathbf{x},\mathbf{k};t)$ (which includes the effects of dissipation), the action density is conserved along these ray trajectories. In particular, we see that it is the action density (not the energy density) of the wave that is conserved in a weakly nonstationary medium.

We have derived (1) in previous work[3,4] for the general case of electromagnetic waves in a space- and time-varying medium. The method of derivation was based on the Weyl transform[5] of a general, linear wave equation, allowing for sources and dissipation; this procedure leads to a natural definition of the wave-action density and a treatment rooted in the ray phase space where $\mathbf{x}$ and $\mathbf{k}$ are considered as independent variables. In this paper we will extend that technique to the case where the medium is characterized by turbulent fluctuations which are small in magnitude but not necessarily slowly varying. We derive the kinetic equation for the ensemble-averaged wave action density, for which the right-hand side of (1) becomes

$$\Sigma(\mathbf{x},\mathbf{k};t) = -\frac{1}{(2\pi)^3}\int d^3k'\, F(\mathbf{x},t;\mathbf{k},\mathbf{k}') \times [J(\mathbf{x},\mathbf{k};t) - J(\mathbf{x},\mathbf{k}';t)] \qquad (2)$$

in the absence of other sources and dissipation. The local scattering cross section $F(\mathbf{x},t;\mathbf{k},\mathbf{k}')$ is given in terms of the local spectral density of the turbulence $S(\mathbf{x},t,\mathbf{k},\omega)$ and the local dispersion function $D_0(\mathbf{x},t,\mathbf{k},\omega)$ by

$$F(\mathbf{x},t;\mathbf{k},\mathbf{k}') \equiv \frac{S(\mathbf{x},t,\mathbf{k}-\mathbf{k}',\Omega(\mathbf{x},\mathbf{k};t)-\Omega(\mathbf{x},\mathbf{k}';t))}{(\partial D_0/\partial\omega)(\mathbf{x},\mathbf{k};t)\,(\partial D_0/\partial\omega)(\mathbf{x},\mathbf{k}';t)} \ . \qquad (3)$$

Thus, our method provides a *classical wave* derivation of the radiation transport equation for a general class of linear, scalar wave equations. The extension of our derivation to vector waves can be achieved with the techniques used by McDonald and Kaufman.[3]

Other derivations of this equation have been given[6] in the past for *particular* wave equations by appealing to a detailed analysis of multiple scattering, assumptions about the correlation properties of the fluctuations (Markov), or field-theoretic concepts. We will illustrate our general results by applying them to three example wave equations for which derivations have been given elsewhere.

The Schrödinger equation, with a randomly fluctuating component in the potential $V(\mathbf{x},t) = V_0(\mathbf{x}) + \delta V(\mathbf{x},t)$:

$$i\hbar\frac{\partial\psi(\mathbf{x},t)}{\partial t} + \frac{\hbar^2}{2m}\nabla^2\psi - [V_0(\mathbf{x}) + \delta V(\mathbf{x},t)]\psi = 0 \ . \qquad (4)$$

The equation for high-frequency electromagnetic waves in a plasma, with fluctuations in the plasma density $n(\mathbf{x},t) = n_0 + \delta n(\mathbf{x},t)$:

$$-\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2}+\nabla^2 A-\frac{\omega_p^2(\mathbf{x},t)}{c^2}A=-\frac{4\pi}{c}j(\mathbf{x},t)\ , \qquad (5)$$

where A is one component of the vector potential of the electromagnetic field, the electron plasma frequency is $\omega_p^2(\mathbf{x},t)=(4\pi e^2/m_e)[n_0+\delta n(\mathbf{x},t)]$, and $j(\mathbf{x},t)$ is a current source.

The equation governing waves in a medium with a fluctuating index of refraction $N(\mathbf{x},t)=1+\delta N(\mathbf{x},t)$ (such as electromagnetic waves in a dielectric medium):

$$-\frac{1}{c^2}\frac{\partial}{\partial t}\left([1+\delta N(\mathbf{x},t)]^2\frac{\partial A}{\partial t}\right)+\nabla^2 A=0\ . \qquad (6)$$

For cases (5) and (6) we shall also show how the paraxial (or parabolic, or small angle) approximation can be implemented at the level of the wave kinetic equation; this corresponds to the conversion of equations like (5) and (6) into an equation similar to (4) for waves propagating primarily in a single direction, with a transverse profile of finite extent.

II. THE WKE IN A FLUCTUATING MEDIUM

In this section we shall derive the wave kinetic equation (WKE) for a wave propagating in a fluctuating medium (such as a medium with turbulent fluctuations). We will allow the mean medium (without fluctuations) to be weakly nonuniform and nonstationary, in the sense that the wave under consideration has a short wavelength compared to the scale length of variation of the medium and a high frequency compared to the rate of variation of the medium. There will be no such restrictions placed on the variation of the turbulent fluctuations present in the medium, although they will be assumed to be small in magnitude. We shall also allow the medium to be weakly dissipative, and we permit the existence of weak source (or driving) terms in the underlying wave equation. The method we shall use is based on the Weyl transform, or Weyl representation of the wave equation; the details of this technique can be found in earlier work,[3,4] where the effects of nonuniformity, dissipation, and sources have been considered. In this paper we shall therefore concentrate on the treatment of the fluctuations in the Weyl formalism.

We consider a general, linear wave equation for a scalar wave ψ which in operator form can be written

$$(D_0+D_1)\psi=g\ . \qquad (7)$$

Here, the zeroth-order *dispersion operator* D_0 governs the propagation of the wave in the mean (nonfluctuating) medium, and is allowed to be non-Hermitian (to account for dissipation). The first-order dispersion operator D_1 represents the fluctuations. We assume that $\langle D_1\rangle=0$, where the angular brackets denote the ensemble average over realizations of the fluctuations. The right-hand side of (7) is the source field g.

Our starting point will be to derive from (7) an equation for the *mean spectral operator* $W\equiv\langle(\psi\psi^*)\rangle$. Our derivation follows that of Karal and Keller,[7] who derive the equation for the mean field $\langle\psi\rangle$ when D_1 is small:

$$(D_0-\langle D_1D_0^{-1}D_1\rangle)\langle\psi\rangle=g\ . \qquad (8)$$

As shown in Appendix A, the corresponding equation for W is

$$\begin{aligned}D_0W=&\ G(D_0^\dagger)^{-1}+\langle D_1D_0^{-1}D_1\rangle W\\&+D_0W\langle D_1^\dagger(D_0^\dagger)^{-1}D_1^\dagger\rangle(D_0^\dagger)^{-1}\\&+\langle D_1WD_1^\dagger\rangle(D_0^\dagger)^{-1}\ ,\end{aligned} \qquad (9)$$

where the source term is $G=(gg^*)$ (which is assumed to be deterministic).

The Weyl representation of the operator equation (9) is formed by replacing the abstract operators by their Weyl representations, and replacing the operator products by the Weyl product. For example, the Weyl representation of the zeroth-order dispersion operator D_0 is formed from its space-time kernel $D_0(\mathbf{x},t;\mathbf{x}',t')$ by

$$D_0(\mathbf{x},t,\mathbf{k},\omega)=\int d^3s\,d\tau\ D_0(\mathbf{x}+\tfrac{1}{2}\mathbf{s},t+\tfrac{1}{2}\tau;\mathbf{x}-\tfrac{1}{2}\mathbf{s},t-\tfrac{1}{2}\tau)\,e^{-i\mathbf{k}\cdot\mathbf{s}+i\omega\tau} \qquad (10)$$

and that of the mean spectral operator W is

$$W(\mathbf{x},t,\mathbf{k},\omega)=\int d^3s\,d\tau\ \langle\psi(\mathbf{x}+\tfrac{1}{2}\mathbf{s},t+\tfrac{1}{2}\tau)\psi^*(\mathbf{x}-\tfrac{1}{2}\mathbf{s},t-\tfrac{1}{2}\tau)\rangle\,e^{-i\mathbf{k}\cdot\mathbf{s}+i\omega\tau}\ . \qquad (11)$$

We observe that the Weyl representation of an operator is a function on the eight-dimensional extended phase space $z\equiv(x,k)\equiv(\mathbf{x},t,\mathbf{k},\omega)$. With these definitions, the Weyl representation of (9) is

$$\begin{aligned}D_0(z)\,e^{(i/2)\overleftrightarrow{\mathcal{L}}}\,W(z)=&\ G(z)\,e^{(i/2)\overleftrightarrow{\mathcal{L}}}\,(D_0^\dagger)^{-1}(z)+\langle D_1D_0^{-1}D_1\rangle(z)\,e^{(i/2)\overleftrightarrow{\mathcal{L}}}\,W(z)\\&+D_0(z)\,e^{(i/2)\overleftrightarrow{\mathcal{L}}}\,W(z)\,\langle D_1^\dagger(D_0^\dagger)^{-1}D_1^\dagger\rangle(z)\,e^{(i/2)\overleftrightarrow{\mathcal{L}}}\,(D_0^\dagger)^{-1}(z)+\langle D_1WD_1^\dagger\rangle(z)\,e^{(i/2)\overleftrightarrow{\mathcal{L}}}\,(D_0^\dagger)^{-1}(z)\ .\end{aligned}$$

(12)

The Weyl product is the action of the exponentiated, bidirectional differential operator $\overleftrightarrow{\mathcal{L}}$ defined as

$$\overleftrightarrow{\mathcal{L}} \equiv \overleftarrow{\partial_{\mathbf{x}}} \cdot \overrightarrow{\partial_{\mathbf{k}}} - \overleftarrow{\partial_{\mathbf{k}}} \cdot \overrightarrow{\partial_{\mathbf{x}}} + \overleftarrow{\partial_{\omega}}\, \overrightarrow{\partial_{t}} - \overleftarrow{\partial_{t}}\, \overrightarrow{\partial_{\omega}} \ . \tag{13}$$

We shall analyze this equation in the short-wavelength, high-frequency limit with a treatment consistent with the usual eikonal (or WKB, or semiclassical) method. With the details and justification given in McDonald and Kaufman,[3] we expand the exponentiated operator in its power series and order the terms as

$$D_0'(z)W(z) \sim O(1) \ , \tag{14}$$

$$\begin{aligned} D_0'(z)\overleftrightarrow{\mathcal{L}}W(z) &\sim D_0''(z)W(z) \\ &\sim G(z) \sim D_1^2(z)W(z) \sim O(\varepsilon) \ , \end{aligned} \tag{15}$$

where D_0' and D_0'' are the real (Hermitian) and imaginary (anti-Hermitian) parts of $D_0(z)$.

A. The $O(1)$ equation

$$\begin{aligned} D_0'(z)W(z) &= 0 \\ &\Rightarrow W(\mathbf{x},t,\mathbf{k},\omega) = 2\pi J(\mathbf{x},\mathbf{k};t)\, \delta(D_0'(\mathbf{x},t,\mathbf{k},\omega)) \ . \end{aligned} \tag{16}$$

At lowest order we see that $W(\mathbf{x},t,\mathbf{k},\omega)$ is required to be zero everywhere in the $(\mathbf{x},t,\mathbf{k},\omega)$ phase space except on the surface where the real part of the dispersion function $D_0'(\mathbf{x},t,\mathbf{k},\omega)$ vanishes. This *dispersion surface* is that region on which the waves obey the *local dispersion relation*

$$D_0'(\mathbf{x},t,\mathbf{k},\omega) = 0 \ \Rightarrow \ \omega = \Omega(\mathbf{x},\mathbf{k};t) \ . \tag{17}$$

In general, (17) may only be an implicit definition for the dispersion relation $\Omega(\mathbf{x},\mathbf{k};t)$ and there may be many branches, or solutions, to $D_0'(\mathbf{x},t,\mathbf{k},\omega) = 0$; for simplicity, we assume that there is only one branch. On this dispersion surface, (16) allows $W(\mathbf{x},t,\mathbf{k},\omega)$ to be nonzero; thus we have defined the *action density* $J(\mathbf{x},\mathbf{k};t)$ to be the "density" of W on this surface. For simplicity of notation, we shall henceforth denote the real part of the dispersion function by D_0, and explicitly refer to the imaginary part as D_0''.

Integrating (16) with respect to D_0 we have

$$\begin{aligned} J(\mathbf{x},\mathbf{k};t) &= \frac{1}{2\pi}\int dD_0\, W(\mathbf{x},t,\mathbf{k},\omega) \\ &= \frac{1}{2\pi}\int d\omega \frac{\partial D_0}{\partial \omega}(\mathbf{x},t,\mathbf{k},\omega)\, W(\mathbf{x},t,\mathbf{k},\omega) \ . \end{aligned} \tag{18}$$

Since by (16) $W(\mathbf{x},t,\mathbf{k},\omega)$ is singularly defined on the surface $D_0 = 0$ [or $\omega = \Omega(\mathbf{x},\mathbf{k};t)$], and since the integration over ω is transverse to this surface, (18) becomes

$$\begin{aligned} J(\mathbf{x},\mathbf{k};t) &= \frac{\partial D_0(\mathbf{x},\mathbf{k},t,\Omega(\mathbf{x},\mathbf{k};t))}{\partial \omega}\frac{1}{2\pi}\int d\omega W(\mathbf{x},t,\mathbf{k},\omega) \\ &= \frac{\partial D_0(\mathbf{x},\mathbf{k};t)}{\partial \omega}\frac{1}{2\pi}\int d\omega \int d^3s\, d\tau\, \langle \psi(\mathbf{x}+\tfrac{1}{2}\mathbf{s},t+\tfrac{1}{2}\tau)\psi^*(\mathbf{x}-\tfrac{1}{2}\mathbf{s},t-\tfrac{1}{2}\tau)\rangle\, e^{-i\mathbf{k}\cdot\mathbf{s}+i\omega\tau} \\ &= \frac{\partial D_0(\mathbf{x},\mathbf{k};t)}{\partial \omega}\int d^3s\, \langle \psi(\mathbf{x}+\tfrac{1}{2}\mathbf{s},t)\psi^*(\mathbf{x}-\tfrac{1}{2}\mathbf{s},t)\rangle\, e^{-i\mathbf{k}\cdot\mathbf{s}} \\ &\equiv \frac{\partial D_0(\mathbf{x},\mathbf{k};t)}{\partial \omega}\tilde{W}(\mathbf{x},\mathbf{k};t) \ . \end{aligned} \tag{19}$$

Here $\tilde{W}(\mathbf{x},\mathbf{k};t)$ is just the ensemble-averaged spatial *Wigner function*[8] corresponding to the wave $\psi(\mathbf{x},t)$.

To see that $J(\mathbf{x},\mathbf{k};t)$ defined by (16) indeed corresponds to the usual definition of wave-action density, consider the case of a plane wave (with wave vector $\mathbf{k}_0$ and frequency ω_0) in a uniform medium. For this example we have

$$\psi(\mathbf{x},t) = \tilde{\psi}\, e^{i\mathbf{k}_0\cdot\mathbf{x}-i\omega_0 t}$$

and (20)

$$D_0(\mathbf{x},t,\mathbf{k},\omega) = D_0(\mathbf{k},\omega) = 0 \ \ \Rightarrow \ \ \omega = \Omega(\mathbf{k})$$

for which (16) requires that $\omega_0 = \Omega(\mathbf{k}_0)$, just as in the usual treatment of waves in a uniform medium. Now (19) is

$$\begin{aligned} J(\mathbf{x},\mathbf{k};t) &= \frac{\partial D_0(\mathbf{k},\Omega(\mathbf{k}))}{\partial \omega}\tilde{W}(\mathbf{x},\mathbf{k};t) \\ &= \frac{\partial D_0(\mathbf{k}_0,\Omega(\mathbf{k}_0))}{\partial \omega}(2\pi)^3|\tilde{\psi}|^2\,\delta(\mathbf{k}-\mathbf{k}_0) \\ &\sim \frac{U(\mathbf{k}_0)}{\Omega(\mathbf{k}_0)} \end{aligned} \tag{21}$$

showing that in this case, the action density reduces to (except for a proportionality constant) the wave energy density $U(\mathbf{k}_0)$ divided by the frequency $\omega_0 = \Omega(\mathbf{k}_0)$. The definition (16) is simply the appropriate extension of the concept of wave-action density to the more general case of a weakly nonuniform and non-stationary medium.

The relation (21) between the wave energy density and the wave-action density implies that $D_0(z)$ be defined in such a way that the dimensions of the action density [defined by (18)] are correct. Indeed, depending on the form and interpretation of the particular wave equation under study, one must ensure that the dispersion kernel (and even the wave field itself) has been identified correctly on a physical basis. This will be illustrated in the examples of Sec. III.

B. The $O(\varepsilon)$ equation

$$D_0(z)\,(i/2)\overleftrightarrow{\mathcal{L}}\,W(z) = -iD_0''(z)W(z) + G(z)(D_0^\dagger)^{-1}(z) + \langle D_1 D_0^{-1} D_1\rangle(z)W(z) + D_0(z)W(z)\langle D_1^\dagger (D_0^\dagger)^{-1} D_1^\dagger\rangle(z)(D_0^\dagger)^{-1}(z) + \langle D_1 W D_1^\dagger\rangle(z)(D_0^\dagger)^{-1}(z)\ . \tag{22}$$

Here we have included all the terms from the right-hand side of (12) with the exponentiated differential operator $\exp[(i/2)\overleftrightarrow{\mathcal{L}}]$ replaced by unity, as well as the contribution due to the imaginary (anti-Hermitian) part D_0'' of the zeroth-order dispersion function D_0; all other occurances of D_0 in this expression are understood to be the real (Hermitian) part of D_0.

Before proceeding, we must compute the Weyl representation of the ensemble-averaged triple products in this expression. The details are given in Appendix B and the results are

$$\langle D_1 D_0^{-1} D_1\rangle(\mathbf{x},t,\mathbf{k},\omega) = \frac{1}{(2\pi)^4}\int d^3k'\,d\omega'\,S_1(\mathbf{x},t,\mathbf{k}-\mathbf{k}',\omega-\omega')\,D_0^{-1}(\mathbf{x},t,\mathbf{k}',\omega'),$$

$$\langle D_1^\dagger (D_0^\dagger)^{-1} D_1^\dagger\rangle(\mathbf{x},t,\mathbf{k},\omega) = \frac{1}{(2\pi)^4}\int d^3k'\,d\omega'\,S_1(\mathbf{x},t,\mathbf{k}-\mathbf{k}',\omega-\omega')\,(D_0^\dagger)^{-1}(\mathbf{x},t,\mathbf{k}',\omega')\ , \tag{23}$$

$$\langle D_1 W D_1^\dagger\rangle(\mathbf{x},t,\mathbf{k},\omega) = \frac{1}{(2\pi)^4}\int d^3k'\,d\omega'\,S_1(\mathbf{x},t,\mathbf{k}-\mathbf{k}',\omega-\omega')\,W(\mathbf{x},t,\mathbf{k}',\omega')\ .$$

In Appendix B1, we have assumed that the random operator D_1 represents a term in the wave equation which, in the x-space representation of (7), would contain no spatial or time derivatives. Thus, D_1 is assumed to have an x-space kernel of the form

$$D_1(\mathbf{x},t;\mathbf{x}',t') = d_1(\mathbf{x},t)\,\delta(\mathbf{x}-\mathbf{x}')\,\delta(t-t')\ , \tag{24}$$

where $d_1(\mathbf{x},t)$ represents the fluctuations in the medium (assumed to be a real-valued field). Such a form is obtained if D_1 represents fluctuations as in the example wave equations (4) and (5) in Sec. I; the form in (6), which includes the action of time derivatives, will be treated in Sec. III C and Appendix B2. The local spectral density S_1 of the fluctuations $d_1(\mathbf{x},t)$ is defined as

$$S_1(\mathbf{x},t,\mathbf{k},\omega) \equiv \int d^3s\,d\tau\,\langle d_1(\mathbf{x}+\tfrac{1}{2}\mathbf{s},t+\tfrac{1}{2}\tau)\,d_1(\mathbf{x}-\tfrac{1}{2}\mathbf{s},t-\tfrac{1}{2}\tau)\rangle\,e^{-i\mathbf{k}\cdot\mathbf{s}+i\omega\tau}\ . \tag{25}$$

The expressions in (23) involve the Weyl representations of the operators D_0^{-1} and $(D_0^\dagger)^{-1}$ as well as the spectral density of the wave W and the spectral density S_1 of the fluctuations $d_1(\mathbf{x},t)$. These expressions allow the fluctuations to have arbitrary spatial and temporal variation (i.e., not necessarily slow like the mean medium).

Substituting the lowest-order solution (16) and the expressions (23) into (22) we have

$$\begin{aligned}
&(i/2)D_0(\mathbf{x},t,\mathbf{k},\omega)\,\overleftrightarrow{\mathcal{L}}\,[2\pi J(\mathbf{x},\mathbf{k};t)\,\delta(D_0(\mathbf{x},t,\mathbf{k},\omega))]\\
&= -iD_0''(\mathbf{x},t,\mathbf{k},\omega)[2\pi J(\mathbf{x},\mathbf{k};t)\,\delta(D_0)] + G(\mathbf{x},t,\mathbf{k},\omega)(D_0^\dagger)^{-1}(\mathbf{x},t,\mathbf{k},\omega)\\
&\quad + 2\pi J(\mathbf{x},\mathbf{k};t)\,\delta(D_0)\,\frac{1}{(2\pi)^4}\int d^3k'\,d\omega'\,S_1(\mathbf{x},t,\mathbf{k}-\mathbf{k}',\omega-\omega')\,D_0^{-1}(\mathbf{x},t,\mathbf{k}',\omega')\\
&\quad + 2\pi J(\mathbf{x},\mathbf{k};t)\,\delta(D_0)\,D_0(\mathbf{x},t,\mathbf{k},\omega)\,(D_0^\dagger)^{-1}(\mathbf{x},t,\mathbf{k},\omega)\\
&\quad\times\frac{1}{(2\pi)^4}\int d^3k'\,d\omega'\,S_1(\mathbf{x},t,\mathbf{k}-\mathbf{k}',\omega-\omega')\,(D_0^\dagger)^{-1}(\mathbf{x},t,\mathbf{k}',\omega')\\
&\quad + (D_0^\dagger)^{-1}(\mathbf{x},t,\mathbf{k},\omega)\frac{1}{(2\pi)^4}\int d^3k'\,d\omega'\,S_1(\mathbf{x},t,\mathbf{k}-\mathbf{k}',\omega-\omega')\,[2\pi J(\mathbf{x},\mathbf{k}';t)\,\delta(D_0(\mathbf{x},t,\mathbf{k}',\omega'))]\ .
\end{aligned} \tag{26}$$

In view of the definition (13) of the operator $\overleftrightarrow{\mathcal{L}}$, the left-hand side of this equation is just the Poisson bracket in the extended phase space $(\mathbf{x},t,\mathbf{k},\omega)$; since this is a linear differential operator on the product $J(\mathbf{x},\mathbf{k};t)\,\delta(D_0(\mathbf{x},t,\mathbf{k},\omega))$, this results in two terms. The Poisson bracket of D_0 with $\delta(D_0)$ vanishes due to the asymmetry in $\overleftrightarrow{\mathcal{L}}$ so that the left-hand side becomes

$$
\begin{aligned}
&(i/2)D_0(\mathbf{x},t,\mathbf{k},\omega)\overleftrightarrow{\mathcal{L}}\,[2\pi J(\mathbf{x},\mathbf{k};t)\,\delta(D_0)]\\
&\quad = i\pi\delta(D_0)\,[D_0\overleftrightarrow{\mathcal{L}}\,J(\mathbf{x},\mathbf{k};t)]\\
&\quad = i\pi\delta(D_0)\left(\frac{\partial D_0}{\partial\omega}\frac{\partial J}{\partial t}+\{D_0,J\}\right)\,. \qquad (27)
\end{aligned}
$$

Here, the braces { , } denote the usual Poisson bracket in the $(\mathbf{x},\mathbf{k})$ phase space. Since this expression will be evaluated on the dispersion surface when the integration over D_0 is performed, we use the usual relations which hold where $D_0(\mathbf{x},t,\mathbf{k},\omega)=0$:

$$
\begin{aligned}
\left(\frac{\partial D_0}{\partial\mathbf{x}}\right)_{D_0=0} &= -\left(\frac{\partial D_0}{\partial\omega}\right)_{\mathbf{x},\mathbf{k},t}\left(\frac{\partial\Omega}{\partial\mathbf{x}}\right)_{\mathbf{k},t}\,,\\
\left(\frac{\partial D_0}{\partial\mathbf{k}}\right)_{D_0=0} &= -\left(\frac{\partial D_0}{\partial\omega}\right)_{\mathbf{x},\mathbf{k},t}\left(\frac{\partial\Omega}{\partial\mathbf{k}}\right)_{\mathbf{x},t}\,.
\end{aligned}
$$

With these, (27) becomes

$$
\begin{aligned}
&(i/2)D_0(\mathbf{x},t,\mathbf{k},\omega)\overleftrightarrow{\mathcal{L}}\,[2\pi J(\mathbf{x},\mathbf{k};t)\,\delta(D_0(\mathbf{x},t,\mathbf{k},\omega))]\\
&\quad = i\pi\delta(D_0)\,\frac{\partial D_0}{\partial\omega}\left(\frac{\partial J}{\partial t}+\{J,\Omega\}\right)\\
&\quad = i\pi\delta(D_0)\,\frac{\partial D_0}{\partial\omega}\,\frac{dJ}{dt} \qquad (28)
\end{aligned}
$$

which defines the usual total time derivative of the action density in phase space.

Having converted the left-hand side of (26) to (28), we next integrate over D_0. This removes the δ function from the left-hand side (28), while the terms on the right-hand side can be transformed as follows. The first term, due to the imaginary part of the dispersion function, is

$$
\begin{aligned}
&-2\pi i\int dD_0\;D_0''(\mathbf{x},t,\mathbf{k},\omega)J(\mathbf{x},\mathbf{k};t)\delta(D_0)\\
&\qquad = -2\pi i D_0''(\mathbf{x},\mathbf{k};t)J(\mathbf{x},\mathbf{k};t)\,. \qquad (29)
\end{aligned}
$$

Here, $D_0''(\mathbf{x},\mathbf{k};t)\equiv D_0''(\mathbf{x},t,\mathbf{k},\omega=\Omega(\mathbf{x},\mathbf{k};t))$ means that D_0'' is to be evaluated on the dispersion surface $D_0=0$; we will use this notation for all quantities which are to be evaluated as such.

In the remaining four terms, the Weyl representations of the inverse operators D_0^{-1} and $(D_0^\dagger)^{-1}$ need to be computed. It is easily shown that in the lowest approximation (required here) we have simply

$$
\begin{aligned}
(D_0^{-1})(z) &= \lim_{D''\to 0}\frac{1}{D_0(z)+iD_0''(z)}\,,\\
(D_0^\dagger)^{-1}(z) &= \lim_{D''\to 0}\frac{1}{D_0(z)-iD_0''(z)}\,.
\end{aligned} \qquad (30)
$$

Thus, the second term on the right-hand side of (26) is integrated over D_0 by standard methods to give

$$
\begin{aligned}
&\int dD_0\;G(\mathbf{x},t,\mathbf{k},\omega)(D_0^\dagger)^{-1}(\mathbf{x},t,\mathbf{k},\omega)\\
&\quad = \lim_{D_0''\to 0}\int dD_0\,\frac{G(\mathbf{x},t,\mathbf{k},\omega)}{D_0-iD_0''}\\
&\quad = i\pi G(\mathbf{x},t,\mathbf{k},\omega=\Omega(\mathbf{x},\mathbf{k};t))\\
&\quad \equiv i\pi G(\mathbf{x},\mathbf{k};t)\,. \qquad (31)
\end{aligned}
$$

Integration of the third term on the right-hand side of (26) with respect to D_0 simply removes the δ function and thereby forces evaluation on the dispersion surface $\omega\to\Omega(\mathbf{x},\mathbf{k};t)$. The remaining integral in that term is therefore

$$
\begin{aligned}
&2\pi\frac{1}{(2\pi)^4}\int d^3k'd\omega' S_1(\mathbf{x},t,\mathbf{k}-\mathbf{k}',\Omega(\mathbf{x},\mathbf{k};t)-\omega')\,D_0^{-1}(\mathbf{x},t,\mathbf{k}',\omega')\\
&\quad = 2\pi\frac{1}{(2\pi)^4}\int d^3k'd\omega'\frac{S_1(\mathbf{x},t,\mathbf{k}-\mathbf{k}',\Omega(\mathbf{x},\mathbf{k};t)-\omega')}{D_0+iD_0''}\\
&\quad = -i\pi\frac{1}{(2\pi)^3}\int d^3k'\frac{S_1(\mathbf{x},t,\mathbf{k}-\mathbf{k}',\Omega(\mathbf{x},\mathbf{k};t)-\Omega(\mathbf{x},\mathbf{k}';t))}{(\partial D_0/\partial\omega)(\mathbf{x},\mathbf{k}';t)}\,. \qquad (32)
\end{aligned}
$$

Due to the presence of a factor of $D_0\delta(D_0)$ in the fourth term on the right-hand side of (26), integration of this term gives zero. Finally, the fifth and last term integrated over D_0 requires

$$
\int dD_0(D_0^\dagger)^{-1}(\mathbf{x},t,\mathbf{k},\omega)S_1(\mathbf{x},t,\mathbf{k}-\mathbf{k}',\omega-\omega') = i\pi\,S_1(\mathbf{x},t,\mathbf{k}-\mathbf{k}',\Omega(\mathbf{x},\mathbf{k};t)-\omega') \qquad (33)
$$

so that the remaining integral can be computed to give

$$2\pi^2 i\frac{1}{(2\pi)^4}\int d^3k'd\omega' S_1(\mathbf{x},t,\mathbf{k}-\mathbf{k}',\Omega(\mathbf{x},\mathbf{k};t)-\omega')\,J(\mathbf{x},\mathbf{k}';t)\,\delta(D_0(\mathbf{x},t,\mathbf{k}',\omega'))$$

$$= i\pi\frac{1}{(2\pi)^3}\int d^3k'\frac{S_1(\mathbf{x},t,\mathbf{k}-\mathbf{k}',\Omega(\mathbf{x},\mathbf{k};t)-\Omega(\mathbf{x},\mathbf{k}';t))}{(\partial D_0/\partial\omega)(\mathbf{x},\mathbf{k}';t)}\,J(\mathbf{x},\mathbf{k}';t)\ . \quad (34)$$

Collecting (28), (29), (31), (32), and (34), the wave kinetic equation is finally

$$\frac{dJ(\mathbf{x},\mathbf{k};t)}{dt} = \frac{\partial J}{\partial t} + \{J,\Omega(\mathbf{x},\mathbf{k};t)\}$$
$$= 2\gamma(\mathbf{x},\mathbf{k};t)J(\mathbf{x},\mathbf{k};t) + \Gamma(\mathbf{x},\mathbf{k};t) - \frac{1}{(2\pi)^3}\int d^3k' F(\mathbf{x},t;\mathbf{k},\mathbf{k}')[J(\mathbf{x},\mathbf{k};t) - J(\mathbf{x},\mathbf{k}';t)]\ . \quad (35)$$

Here we have defined the usual linear growth rate $\gamma(\mathbf{x},\mathbf{k};t)$ as

$$\gamma(\mathbf{x},\mathbf{k};t) \equiv -\frac{D_0''(\mathbf{x},\mathbf{k};t)}{(\partial D_0/\partial\omega)(\mathbf{x},\mathbf{k};t)} \quad (36)$$

and the source term due to the right-hand side of the wave equation (7)

$$\Gamma(\mathbf{x},\mathbf{k};t) \equiv \frac{G(\mathbf{x},\mathbf{k};t)}{(\partial D_0/\partial\omega)(\mathbf{x},\mathbf{k};t)}\ . \quad (37)$$

The effect of the fluctuations [represented by D_1 in the wave equation (7)] is a scattering term which is defined in terms of the local scattering cross section

$$F(\mathbf{x},t;\mathbf{k},\mathbf{k}') \equiv \frac{S_1(\mathbf{x},t,\mathbf{k}-\mathbf{k}',\Omega(\mathbf{x},\mathbf{k};t)-\Omega(\mathbf{x},\mathbf{k}';t))}{(\partial D_0/\partial\omega)(\mathbf{x},\mathbf{k};t)\ (\partial D_0/\partial\omega)(\mathbf{x},\mathbf{k}';t)}\ , \quad (38)$$

where $S_1(\mathbf{x},t,\mathbf{k},\omega)$ is the local spectral density of the turbulent fluctuations. As it appears in the wave kinetic equation, the effect of the fluctuations is to cause a conversion of wave action at wave vector $\mathbf{k}$ [and frequency $\omega=\Omega(\mathbf{x},\mathbf{k};t)$] into the wave action at wave vector $\mathbf{k}'$ [and frequency $\omega'=\Omega(\mathbf{x},\mathbf{k}';t)$], provided that there are fluctuations in the medium at the difference wave vector $\mathbf{k}-\mathbf{k}'$ and difference frequency $\omega-\omega'$. By the same token, there is an increase of wave action $J(\mathbf{x},\mathbf{k};t)$ due to scattering from $J(\mathbf{x},\mathbf{k}';t)$. The form (38) of the scattering cross section is symmetric in $\mathbf{k}$ and $\mathbf{k}'$ [$S_1(\mathbf{x},t,\mathbf{k},\omega)=S_1(\mathbf{x},t,-\mathbf{k},-\omega)$ for real fluctuating fields $d_1(\mathbf{x},t)$, by (25)]; therefore, it is easy to show that the scattering term in (35) conserves the total action [the integral of $J(\mathbf{x},\mathbf{k};t)$ over all phase space].

We have derived the wave kinetic equation (35) (or radiation transport equation) for a very general class of linear wave equations [represented in operator form by (7)]. The Weyl technique employed here, beginning with the representation-free operator equation (9), has the advantage of immediately producing a classical equation in the extended phase space $(\mathbf{x},t,\mathbf{k},\omega)$, where all eight variables are independent. The systematic, order-by-order analysis of that equation then leads directly to the natural definition of the wave-action density (16), its governing equation (26) [which is reduced to the result (35)] and the natural identification of the local spectral density of the fluctuations as the scattering cross section. In Sec. III we will apply the general expression (35) to some of the specific wave equations that have been treated in the past in order to illustrate our method.

III. APPLICATIONS

In this section we will apply the wave kinetic equation derived in Sec. II to the example wave equations (4)–(6) given in Sec. I.

A. The Schrödinger equation with a fluctuating potential

Let us take as our first example the Schrödinger equation with a randomly fluctuating component in the potential

$$i\hbar\frac{\partial\psi(\mathbf{x},t)}{\partial t} + \frac{\hbar^2}{2m}\nabla^2\psi - [V_0(\mathbf{x}) + \delta V(\mathbf{x},t)]\psi = 0\ . \quad (39)$$

The first step in constructing the wave kinetic equation for this example is to identify the operators D_0 and D_1 in (39) so that it can be put in the form (7). Here it is clear that these operators (in x-space kernel form) are

$$D_0(\mathbf{x},t;\mathbf{x}',t') = i\hbar\delta(\mathbf{x}-\mathbf{x}')\frac{\partial}{\partial t}\delta(t-t') + \frac{\hbar^2}{2m}\delta(t-t')\nabla^2\delta(\mathbf{x}-\mathbf{x}') - V_0(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}')\delta(t-t')\ ,$$

$$D_1(\mathbf{x},t;\mathbf{x}',t') = -\delta V(\mathbf{x},t)\delta(\mathbf{x}-\mathbf{x}')\delta(t-t')\ , \quad (40)$$

$$g(\mathbf{x},t) = 0\ .$$

For this case there is no source term g. Furthermore, the random operator D_1 has the form just as we assumed in (24), so that the form (23) of the operator triple products applies.

The Weyl representations of D_0 and D_1 are constructed by (10) to be

$$D_0(\mathbf{x},t,\mathbf{k},\omega) = \hbar\omega - \frac{\hbar^2 k^2}{2m} - V_0(\mathbf{x}) \ , \qquad (41)$$

$$D_1(\mathbf{x},t,\mathbf{k},\omega) = -\delta V(\mathbf{x},t) \ .$$

The local dispersion relation obtained by setting $D_0 = 0$ is just the classical Hamiltonian

$$D_0(\mathbf{x},\mathbf{k},\omega) = 0 \ \Rightarrow \ \hbar\omega = \hbar\Omega(\mathbf{x},\mathbf{k}) = \frac{\hbar^2 k^2}{2m} + V_0(\mathbf{x}) \qquad (42)$$

if we take the energy $E = \hbar\omega$ and momentum $\mathbf{p} = \hbar\mathbf{k}$ as usual. The wave-action density $J(\mathbf{x},\mathbf{k};t)$ is therefore identified as a density on the classical energy surface. From (19) we have

$$J(\mathbf{x},\mathbf{k};t) = \hbar \int d^3s \ \langle \psi(\mathbf{x}+\tfrac{1}{2}\mathbf{s},t)\psi^*(\mathbf{x}-\tfrac{1}{2}\mathbf{s},t)\rangle \, e^{-i\mathbf{k}\cdot\mathbf{s}} \ . \qquad (43)$$

Here, we have used (42) to give $(\partial D_0/\partial\omega)(\mathbf{x},\mathbf{k};t) = \hbar$. Since $|\psi|^2$ is usually interpreted as a probability density on x space, we see that the action density is properly interpreted in this case to be a probability distribution in phase space [with correct units of action, as the $(\mathbf{x},\mathbf{k})$ phase-space volume is dimensionless]. Indeed, multiplying $J(\mathbf{x},\mathbf{k};t)$ in (43) by $\Omega(\mathbf{x},\mathbf{k})$ from (42) we obtain an expression for the quantum-mechanical energy density on phase space.

Since there is no dissipation in the Schrödinger equation (D_0 is real) and no sources ($g = 0$), the wave kinetic equation (35) is driven only by the fluctuations $\delta V(\mathbf{x},t)$ in the potential. Thus we have

$$\begin{aligned} \frac{dJ}{dt} &= \frac{\partial J}{\partial t} + \{J,\Omega\} \\ &= \frac{\partial J}{\partial t} + \frac{\hbar\mathbf{k}}{m}\cdot\frac{\partial J}{\partial\mathbf{x}} - \frac{1}{\hbar}\frac{\partial V_0}{\partial\mathbf{x}}\cdot\frac{\partial J}{\partial\mathbf{k}} \\ &= \frac{\partial J}{\partial t} + \dot{\mathbf{x}}\cdot\frac{\partial J}{\partial\mathbf{x}} + \dot{\mathbf{k}}\cdot\frac{\partial J}{\partial\mathbf{k}} \qquad (44) \\ &= -\frac{1}{(2\pi)^3}\int d^3k' F(\mathbf{x},t;\mathbf{k},\mathbf{k}')[J(\mathbf{x},\mathbf{k};t) \\ &\qquad -J(\mathbf{x},\mathbf{k}';t)] \ . \qquad (45) \end{aligned}$$

The first three lines show how the evaluation of the Poisson bracket yields the usual classical trajectories in the $(\mathbf{x},\mathbf{k} = \mathbf{p}/\hbar)$ phase space. The scattering cross section $F(\mathbf{x},t;\mathbf{k},\mathbf{k}')$ in the last line is

$$F(\mathbf{x},t;\mathbf{k},\mathbf{k}') = \frac{1}{\hbar^2} S_V(\mathbf{x},t,\mathbf{k}-\mathbf{k}',(\hbar/2m)(k^2-k'^2)) \ , \qquad (46)$$

where the spectral density $S_V(\mathbf{x},t,\mathbf{k},\omega)$ of the (real) potential fluctuations is

$$S_V(\mathbf{x},t,\mathbf{k},\omega) = \int d^3s\,d\tau \ \langle \ \delta V(\mathbf{x}+\tfrac{1}{2}\mathbf{s},t+\tfrac{1}{2}\tau) \times\delta V(\mathbf{x}-\tfrac{1}{2}\mathbf{s},t-\tfrac{1}{2}\tau)\rangle \times e^{-i\mathbf{k}\cdot\mathbf{s}+i\omega\tau} \qquad (47)$$

B. Electromagnetic waves in a fluctuating plasma

Consider Maxwell's equations

$$\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} \ , \quad \nabla\times\mathbf{B} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{J} \qquad (48)$$

in a plasma, where the (cold) electrons (with density n) carry the current with velocity $\mathbf{v}$ by the equation of motion

$$\mathbf{J} = -ne\mathbf{v} + \mathbf{j} \ , \quad \frac{\partial\mathbf{v}}{\partial t} = -\frac{e}{m_e}\mathbf{E} \ . \qquad (49)$$

Here, we have written the current $\mathbf{J}$ as the sum of that due to the linear response of the electrons, as well as a term $\mathbf{j}$ to account for other sources of current. We have also assumed the perturbed quantities (such as $\mathbf{E}$ and $\mathbf{v}$) to be small [hence the convective term $\mathbf{v}\cdot\nabla\mathbf{v}$ in the electron equation of motion (49) has been neglected]. We now introduce the vector potential $\mathbf{A}$ of the electromagnetic field in the standard way

$$\mathbf{B} = \nabla\times\mathbf{A} \ , \quad \mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} \qquad (50)$$

and use the Coulomb gauge ($\nabla\cdot\mathbf{A} = 0$); Faraday's law (48) is therefore identically satisfied. Noting that with (50) substituted for $\mathbf{E}$ in the equation of motion (49) we have $\mathbf{v} = e\mathbf{A}/m_e c$, Ampère's law becomes

$$-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} + \nabla^2\mathbf{A} - \frac{\omega_p^2}{c^2}\mathbf{A} = -\frac{4\pi}{c}\mathbf{j} \ , \qquad (51)$$

where $\omega_p = (4\pi ne^2/m_e)^{1/2}$ is the electron plasma frequency.

We now assume that the plasma density is uniform (in the mean) with small turbulent fluctuations; that is we take

$$n = n_0 + \delta n(\mathbf{x},t) \Rightarrow \omega_p^2 = \omega_{p0}^2 + \delta\omega_p^2(\mathbf{x},t) \quad (\delta\omega_p^2 \propto \delta n) \ . \qquad (52)$$

This wave equation for each component of $\mathbf{A} \equiv A\hat{\mathbf{a}}$ is just (5). If the fluctuations are small, then we have the form (7) with the identification of the operators (in kernel form):

$$\begin{aligned} D_0(\mathbf{x},t;\mathbf{x}',t') &= -\delta(\mathbf{x}-\mathbf{x}')\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\delta(t-t') \\ &\quad +\delta(t-t')\nabla^2\delta(\mathbf{x}-\mathbf{x}') \\ &\quad -\frac{\omega_{p0}^2}{c^2}\delta(\mathbf{x}-\mathbf{x}')\delta(t-t') \ , \\ D_1(\mathbf{x},t;\mathbf{x}',t') &= -\frac{\delta\omega_p^2(\mathbf{x},t)}{c^2}\delta(\mathbf{x}-\mathbf{x}')\delta(t-t') \ , \qquad (53) \\ g(\mathbf{x},t) &= -\frac{4\pi}{c}\,j(\mathbf{x},t) \ . \end{aligned}$$

Here, the scalar j in the source term is the component of that current in the direction $\hat{\mathbf{a}}$. Furthermore, the form of the random operator D_1 is just as we assumed in (24) so that the form (23) of the operator triple products applies.

The Weyl representation of the operators can be shown to be simply

$$D_0(\mathbf{x},t,\mathbf{k},\omega) = (\omega/c)^2 - k^2 - (\omega_{p0}/c)^2 \; ,$$

$$D_1(\mathbf{x},t,\mathbf{k},\omega) = -\delta\omega_p^2(\mathbf{x},t)/c^2 \; , \tag{54}$$

$$G(\mathbf{x},t,\mathbf{k},\omega) = 16\pi^2(jj^*)/c^2 \; .$$

The local dispersion relation, obtained by setting $D_0 = 0$, is therefore the usual one for high-frequency electromagnetic waves (such as radio waves) in a plasma

$$D_0(\mathbf{x},t,\mathbf{k},\omega) = 0 \Rightarrow \Omega(\mathbf{x},\mathbf{k};t) = \Omega(\mathbf{k}) = \pm\sqrt{c^2k^2+\omega_p^2} \; . \tag{55}$$

Although this dispersion function yields two branches (corresponding to positive and negative frequencies), we can choose just the positive branch to be governing our waves. This is because by definition (16) of the wave-action density $J(\mathbf{x},\mathbf{k};t)$, we can at this point choose J to be zero on the dispersion surface with negative frequency and still satisfy the requirements of the lowest-order solution.

In this case of electromagnetic waves, we must ensure that the definition (19) for the action density gives J with the correct units. In the system of units (represented by square brackets) used here, x-space energy density has the units of $U/L^3 = E^2(\mathbf{x})$, where U is energy, L is length and E is electric field. The units of action density should therefore be $[J] = U/\omega = UT = E^2L^3T$ [which is the units of action per dimensionless phase-space volume $d^3x\,d^3k(2\pi)^{-3}$]. Since our wave equation (51) has been derived in terms of the vector potential $\mathbf{A}$, the definition of J by either (16) or (19) involves the spectral density W of the vector potential. Thus, with the dispersion function D_0 given in (54), the units of J are correct in this case: $[J] = [D_0][W] = [\omega^2/c^2][A^2L^3T] = E^2L^3T$, since the units of A are EL. We note that the analysis of this example corrects the treatment beginning with Eq. (2) of our earlier paper,[3] where the electric field should be replaced by the vector potential.

Since the zeroth-order dispersion function D_0 is real, we have $D_0'' = 0$; thus $\gamma(\mathbf{x},t,\mathbf{k},\omega) = 0$, signifying no dissipation [which is evident from the wave equation (51)]. For the other terms in the wave kinetic equation (35), we need

$$\frac{\partial D_0(\mathbf{x},\mathbf{k};t)}{\partial\omega} = \frac{2\omega}{c^2}\bigg|_{\omega=\Omega(\mathbf{k})} = \frac{2}{c^2}\Omega(\mathbf{k}) = \frac{2}{c^2}\sqrt{c^2k^2+\omega_p^2} \; . \tag{56}$$

For example, the source term $\Gamma(\mathbf{x},\mathbf{k};t)$ becomes

$$\Gamma(\mathbf{x},\mathbf{k};t) = \frac{G(\mathbf{x},\mathbf{k};t)}{(\partial D_0/\partial\omega)(\mathbf{x},\mathbf{k};t)} = 8\pi^2\frac{(jj^*)(\mathbf{x},\mathbf{k};t)}{\Omega(\mathbf{k})} \; . \tag{57}$$

The local turbulent scattering cross section $F(\mathbf{x},t;\mathbf{k},\mathbf{k}')$ requires the spectral density $S(\mathbf{x},t,\mathbf{k},\omega)$ of the turbulent fluctuations which, from (54), are represented by fluctuations in the local plasma frequency $\delta\omega_p^2(\mathbf{x},t)$. In terms of the spectral density of the local plasma density fluctuations this is

$$S_1(\mathbf{x},t,\mathbf{k},\omega) = S_{\delta\omega_p^2/c^2}(\mathbf{x},t,\mathbf{k},\omega) = \left(\frac{4\pi e^2}{m_ec^2}\right)^2 S_n(\mathbf{x},t,\mathbf{k},\omega) \; , \tag{58}$$

which gives the scattering cross section

$$F(\mathbf{x},t;\mathbf{k},\mathbf{k}') = \frac{S_1(\mathbf{x},t,\mathbf{k}-\mathbf{k}',\Omega(\mathbf{x},\mathbf{k};t)-\Omega(\mathbf{x},\mathbf{k}';t))}{(\partial D_0/\partial\omega)(\mathbf{x},\mathbf{k};t)\;(\partial D_0/\partial\omega)(\mathbf{x},\mathbf{k}';t)} = \left(\frac{4\pi e^2}{m_e}\right)^2 \frac{S_n(\mathbf{x},t,\mathbf{k}-\mathbf{k}',\Omega(\mathbf{k})-\Omega(\mathbf{k}'))}{4\Omega(\mathbf{k})\Omega(\mathbf{k}')} \; . \tag{59}$$

Finally, the wave kinetic equation for this case can be written

$$\frac{\partial J(\mathbf{x},\mathbf{k};t)}{\partial t} + \frac{c^2\mathbf{k}}{\Omega(\mathbf{k})}\cdot\frac{\partial J}{\partial\mathbf{x}} = 8\pi^2\frac{(jj^*)(\mathbf{x},\mathbf{k};t)}{\Omega(\mathbf{k})} - \left(\frac{4\pi e^2}{m_e}\right)^2\frac{1}{(2\pi)^3}\int d^3k'\,\frac{S_n(\mathbf{x},t,\mathbf{k}-\mathbf{k}',\Omega(\mathbf{k})-\Omega(\mathbf{k}'))}{4\Omega(\mathbf{k})\Omega(\mathbf{k}')} \times[J(\mathbf{x},\mathbf{k};t)-J(\mathbf{x},\mathbf{k}',t)] \; . \tag{60}$$

Here, the Poisson bracket has been explicitly computed given the form (55) for the dispersion relation.

This equation can be simplified somewhat if we specialize to the case where the turbulent fluctuations are time independent; this can also be thought of as the case where the fluctuations are on a much slower time scale (or are at much lower frequency) than the time required for the wave to pass through the medium. In this case, with $\delta\omega_p^2(\mathbf{x},t) \propto \delta n(\mathbf{x},t) \to \delta n(\mathbf{x})$, the spectral density of the fluctuations has the form

$$S_n(\mathbf{x},t,\mathbf{k},\omega) = 2\pi\delta(\omega)\tilde{S}_n(\mathbf{x},\mathbf{k}) \tag{61}$$

so that (60) becomes

$$\frac{\partial J(\mathbf{x},\mathbf{k};t)}{\partial t}+\frac{c^2\mathbf{k}}{\Omega(\mathbf{k})}\cdot\frac{\partial J}{\partial \mathbf{x}}=8\pi^2\frac{(jj^*)(\mathbf{x},\mathbf{k};t)}{\Omega(\mathbf{k})}-\left(\frac{4\pi e^2}{m_e}\right)^2\frac{\pi}{2\Omega^2(\mathbf{k})}\frac{1}{(2\pi)^3}\int d^3k'\tilde{S}_n(\mathbf{x},\mathbf{k}-\mathbf{k}')\delta(\Omega(\mathbf{k})-\Omega(\mathbf{k}'))\times[J(\mathbf{x},\mathbf{k};t)-J(\mathbf{x},\mathbf{k}',t)]\ . \tag{62}$$

Finally, noting that the dispersion relation (55) yields the expression for the index of refraction

$$N\equiv\frac{ck}{\omega}=\left(1-\frac{\omega_{p0}^2}{\omega^2}\right)^{1/2}\ , \tag{63}$$

we see that the fluctuations in the index of refraction in a plasma are given by

$$\delta N=-\frac{1}{2}\frac{\delta\omega_p^2}{\omega^2}=-\frac{1}{2}\frac{4\pi e^2}{m_e}\frac{\delta n}{\omega^2}\ . \tag{64}$$

Thus, the spectral density of the refractive index fluctuations is related to the spectral density of the plasma density fluctuations by

$$S_N(\mathbf{x},t,\mathbf{k},\omega)=\frac{1}{4}\left(\frac{4\pi e^2}{m_e\omega^2}\right)^2 S_n(\mathbf{x},t,\mathbf{k},\omega)\ . \tag{65}$$

Therefore, (62) can be expressed in terms of the refractive index fluctuation spectrum as

$$\frac{\partial J(\mathbf{x},\mathbf{k};t)}{\partial t}+\frac{c^2\mathbf{k}}{\Omega(\mathbf{k})}\cdot\frac{\partial J}{\partial \mathbf{x}}=8\pi^2\frac{(jj^*)(\mathbf{x},\mathbf{k};t)}{\Omega(\mathbf{k})}-2\pi\Omega^2(\mathbf{k})\frac{1}{(2\pi)^3}\int d^3k'\tilde{S}_N(\mathbf{x},\mathbf{k}-\mathbf{k}')\delta(\Omega(\mathbf{k})-\Omega(\mathbf{k}'))\times[J(\mathbf{x},\mathbf{k};t)-J(\mathbf{x},\mathbf{k}',t)]\ . \tag{66}$$

Using the local dispersion relation to evaluate the δ function we have

$$\frac{\partial J(\mathbf{x},\mathbf{k};t)}{\partial t}+\frac{c^2\mathbf{k}}{\Omega(k)}\cdot\frac{\partial J}{\partial \mathbf{x}}=8\pi^2\frac{(jj^*)(\mathbf{x},\mathbf{k};t)}{\Omega(k)}-\frac{1}{4\pi^2}\frac{\Omega^4(k)}{c^4}\frac{c^2k}{\Omega(k)}\int d\Theta'\ \tilde{S}_N(\mathbf{x},\mathbf{k}-\mathbf{k}')[J(\mathbf{x},\mathbf{k};t)-J(\mathbf{x},\mathbf{k}',t)]\ , \tag{67}$$

where the remaining integral is over the solid angle $d\Theta$ in $\mathbf{k}'$ space. When the action transport is stationary ($\partial J/\partial t=0$) this becomes

$$\hat{\mathbf{k}}\cdot\frac{\partial J}{\partial \mathbf{x}}=\frac{8\pi^2}{kc^2}(jj^*)(\mathbf{x},\mathbf{k};t)-\frac{1}{4\pi^2}\frac{\Omega^4(k)}{c^4}\int d\Theta'\ \tilde{S}_N(\mathbf{x},\mathbf{k}-\mathbf{k}')[J(\mathbf{x},\mathbf{k};t)-J(\mathbf{x},\mathbf{k}',t)]\ . \tag{68}$$

When the characterisic frequency of the wave is much larger than the plasma frequency we have $\Omega(k)\approx ck$ so that (68) in the absence of sources $[(jj^*)=0]$ is

$$\hat{\mathbf{k}}\cdot\frac{\partial J}{\partial \mathbf{x}}=-\frac{k^4}{4\pi^2}\int d\Theta'\ \tilde{S}_N(\mathbf{x},\mathbf{k}-\mathbf{k}')\times[J(\mathbf{x},\mathbf{k};t)-J(\mathbf{x},\mathbf{k}',t)]\ , \tag{69}$$

which is the standard equation for steady-state radiation transport in a turbulent medium.

C. Waves in a fluctuating dielectric medium

In Sec. II we assumed that the random operator D_1 was of the special form (24), which represents a term in the wave equation that operates on the wave ψ by simple multiplication (as in the examples of the two previous sections). Another common form for the operator D_1 involves operation on ψ by both multiplication and differentiation. Consider, for example, Maxwell's equations in a nonuniform, time-dependent dielectric medium

$$\nabla\times\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}\ ,\quad \nabla\times\mathbf{B}=\frac{1}{c}\frac{\partial\epsilon(\mathbf{x},t)\mathbf{E}}{\partial t}\ , \tag{70}$$

where we have set the permeability $\mu=1$. Again introducing the vector potential $\mathbf{A}$ from (50), Ampère's equation becomes

$$-\frac{1}{c^2}\frac{\partial}{\partial t}\left(\epsilon(\mathbf{x},t)\frac{\partial\mathbf{A}}{\partial t}\right)+\nabla^2\mathbf{A}=\mathbf{0}\ . \tag{71}$$

With $\mu=1$, the index of refraction of the medium is defined as usual to be $N^2(\mathbf{x},t)\equiv\epsilon(\mathbf{x},t)$. Now suppose the nonuniformity is due to fluctuations $\delta N(\mathbf{x},t)$ or

$$N^2(\mathbf{x},t)=[1+\delta N(\mathbf{x},t)]^2\approx 1+2\delta N(\mathbf{x},t)\ . \tag{72}$$

In this case, (71) becomes

$$-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}+\nabla^2\mathbf{A}-\frac{2}{c^2}\frac{\partial}{\partial t}\left(\delta N(\mathbf{x},t)\frac{\partial\mathbf{A}}{\partial t}\right)=\mathbf{0}\ . \tag{73}$$

The operators D_0 and D_1 are now

$$D_0(\mathbf{x},t;\mathbf{x}',t') = -\delta(\mathbf{x}-\mathbf{x}')\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\delta(t-t') + \delta(t-t')\nabla^2\delta(\mathbf{x}-\mathbf{x}') , \tag{74}$$

$$D_1(\mathbf{x},t;\mathbf{x}',t') = \frac{2}{c^2}\frac{\partial}{\partial t}\frac{\partial}{\partial t'}\delta N(\mathbf{x},t)\delta(\mathbf{x}-\mathbf{x}')\delta(t-t') .$$

Thus, we see that the random operator D_1 is quite different from that considered in Sec. II. Since wave equations of the form (71) are quite common, we treat this case in this section.

The Weyl representation of the operators (74) is

$$D_0(\mathbf{x},t,\mathbf{k},\omega) = \frac{\omega^2}{c^2} - k^2 ,$$
$$D_1(\mathbf{x},t,\mathbf{k},\omega) = \frac{2}{c^2}\left(\omega^2\delta N(\mathbf{x},t) + \frac{1}{4}\frac{\partial^2\delta N}{\partial t^2}\right) . \tag{75}$$

In contrast to the case found in the previous two sections, the Weyl representation of the opeator D_1 in this case involves derivatives of the fluctuating component of the medium. If the characteristic frequency of the wave is much larger than the rate of variation of the fluctuations, then it would be appropriate to neglect the second term in the expression for D_1. Even in that case, however, the presence of the factor of ω^2 in the first term would invalidate the derivation given in Appendix B 1. Therefore, since we wish to allow for arbitrary rates of variation in the fluctuations, we shall keep both terms in the form of D_1. Consistent with the ordering (15), however, we shall neglect any terms that arise in the Weyl representation of the operator triple products that involve spatial derivatives of the "middle" operator [W, D_0^{-1}, or $(D_0^\dagger)^{-1}$]. With this assumption, it is shown in Appendix B 2 that for operators D_1 of the type in (75), the Weyl representation of the triple products are

$$\langle D_1 D_0^{-1} D_1\rangle(\mathbf{x},t,\mathbf{k},\omega) = \frac{4}{c^4}\frac{1}{(2\pi)^4}\int d^3k'd\omega'\mathcal{S}(\mathbf{x},t,\mathbf{k}-\mathbf{k}',\omega-\omega')\,D_0^{-1}(\mathbf{x},t,\mathbf{k}',\omega') ,$$

$$\langle D_1^\dagger (D_0^\dagger)^{-1} D_1^\dagger\rangle(\mathbf{x},t,\mathbf{k},\omega) = \frac{4}{c^4}\frac{1}{(2\pi)^4}\int d^3k'd\omega'\mathcal{S}(\mathbf{x},t,\mathbf{k}-\mathbf{k}',\omega-\omega')\,(D_0^\dagger)^{-1}(\mathbf{x},t,\mathbf{k}',\omega') , \tag{76}$$

$$\langle D_1 W D_1^\dagger\rangle(\mathbf{x},t,\mathbf{k},\omega) = \frac{4}{c^4}\frac{1}{(2\pi)^4}\int d^3k'd\omega'\mathcal{S}(\mathbf{x},t,\mathbf{k}-\mathbf{k}',\omega-\omega')\,W(\mathbf{x},t,\mathbf{k}',\omega') .$$

These expressions are similar in form to those given in (23) for operators D_1 which act by simple multiplication. Here however, the spectral density of the fluctuations $S_1(\mathbf{x},t,\mathbf{k},\omega)$ has been replaced by an effective spectral density $\mathcal{S}(\mathbf{x},t,\mathbf{k},\omega)$ defined by

$$\mathcal{S}(\mathbf{x},t,\mathbf{k}-\mathbf{k}',\omega-\omega') \equiv \omega'^2\left(\omega^2 + \frac{1}{4}\frac{\partial^2}{\partial t^2}\right) S_N(\mathbf{x},t,\mathbf{k}-\mathbf{k}',\omega-\omega') . \tag{77}$$

While this expression allows for arbitrary rates of variation of the fluctuations [in that the time-derivative term in (75) has been retained], we can at this point assume that the rate of variation of the spectral density of the fluctuations is much less than the characteristic frequency of the wave; thus, we shall neglect the time-derivative term in (77).

As in Sec. III B, this wave equation has no dissipation. Furthermore, we are neglecting sources, since the treatment of that term in the wave kinetic equation is no different than in previous sections. The treatment of the triple products in terms of $\mathcal{S}$ is now the same as that given in Sec. II for S_1, so that the scattering cross section in this case becomes

$$F(\mathbf{x},t;\mathbf{k},\mathbf{k}') \equiv \frac{4}{c^4}\frac{\mathcal{S}(\mathbf{x},t,\mathbf{k}-\mathbf{k}',\Omega(\mathbf{k})-\Omega(\mathbf{k}'))}{(\partial D_0/\partial\omega)(\mathbf{k})(\partial D_0/\partial\omega)(\mathbf{k}')} = \Omega(\mathbf{k})\Omega(\mathbf{k}')\,S_N(\mathbf{x},t,\mathbf{k}-\mathbf{k}',\Omega(\mathbf{k})-\Omega(\mathbf{k}')) , \tag{78}$$

where we have used (75) for the zeroth-order dispersion function.

If we now consider the case of stationary fluctuations (61) the scattering cross section becomes

$$F(\mathbf{x};\mathbf{k},\mathbf{k}') = 2\pi\Omega^2(\mathbf{k})\tilde{S}_N(\mathbf{x},\mathbf{k}-\mathbf{k}')\,\delta(\Omega(\mathbf{k})-\Omega(\mathbf{k}')) , \tag{79}$$

which is precisely the same as that in (66) of Sec. III B (where also the fluctuations were taken to be time independent, even though the form of D_1 was different). Thus, the results in (67)–(69) of that section apply to this example as well, as does the radiation transport equation (69).

D. The paraxial approximation

Since the wave kinetic equation is the same for both of the example wave equations of Secs. III B and III C (in the case of time-independent, or very slowly varying fluctuations), we can now consider the paraxial approximation for both of them simultaneously. It is often the case for both wave equations that we are interested in a wave that is propagating primarily in one direction, with an intensity profile of finite extent transverse to this direction. Taking the direction of propagation to be the positive z axis, this means that $k_z > 0$ is the dominant wave-vector component, much larger than the components $\mathbf{k}_\perp = (k_x, k_y)$ perpendicular to this direction. In this case, the dispersion relation (55) for the plasma waves becomes

$$\Omega(\mathbf{k}) = ck_z \left(1 + \frac{c^2k_\perp^2 + \omega_{p0}^2}{c^2k_z^2}\right)^{1/2} \approx ck_z + \frac{c^2k_\perp^2 + \omega_{p0}^2}{2ck_z} , \tag{80}$$

while for the waves in the dielectric medium we have

$$\Omega(\mathbf{k}) = ck_z \left(1 + \frac{k_\perp^2}{k_z^2}\right)^{1/2} \approx ck_z + \frac{ck_\perp^2}{2k_z} . \tag{81}$$

The turbulent scattering term in (66) and (79) can thus be written as

$$\begin{aligned} 2\pi\Omega^2(\mathbf{k})\frac{1}{(2\pi)^3} & \int d^3k' S_N(\mathbf{x}, \mathbf{k}-\mathbf{k}')\delta(\Omega(\mathbf{k}) - \Omega(\mathbf{k}'))[J(\mathbf{x},\mathbf{k};t) - J(\mathbf{x},\mathbf{k}',t)] \\ &= 2\pi c^2 k_z^2 \frac{1}{(2\pi)^3} \int d^3k' S_N(\mathbf{x}, \mathbf{k}-\mathbf{k}')\delta(ck_z - ck_z')[J(\mathbf{x},\mathbf{k};t) - J(\mathbf{x},\mathbf{k}',t)] \\ &= ck_z^2 \frac{1}{(2\pi)^2} \int d^2k_\perp' S_N(\mathbf{x}_\perp, z, \mathbf{k}_\perp - \mathbf{k}_\perp', 0)[J(\mathbf{x}_\perp, \mathbf{k}_\perp, z, k_z; t) - J(\mathbf{x}_\perp, \mathbf{k}_\perp', z, k_z; t)] . \end{aligned} \tag{82}$$

The argument of zero in the spectral density in the the last line indicates that it is to be evaluated at $k_z = 0$ due to the evaluation of the δ function that sets $k_z = k_z'$. We consider the propagation of a continuous, monochromatic beam wave (such as a laser); this is specified by taking

$$J(\mathbf{x},\mathbf{k};t) = 2\pi\delta(k_z - (\omega_0/c))\tilde{J}(\mathbf{x}_\perp, \mathbf{k}_\perp; z) , \tag{83}$$

where now we assume steady state (i.e., J is time-independent, but evolves along the propagation path z). Using (82) and (83) in either transport equation (66) or (79), we integrate over k_z and set the source term to be zero $[(jj^*) = 0]$ to obtain

$$\begin{aligned} \frac{\partial \tilde{J}}{\partial z} + \frac{\mathbf{k}_\perp}{k_0} \cdot \frac{\partial \tilde{J}}{\partial \mathbf{x}_\perp} &= -k_0^2 \frac{1}{(2\pi)^2} \int d^2k_\perp' S_N(\mathbf{x}_\perp, z, \mathbf{k}_\perp - \mathbf{k}_\perp', 0) \\ &\quad \times [\tilde{J}(\mathbf{x}_\perp, \mathbf{k}_\perp; z) - \tilde{J}(\mathbf{x}_\perp, \mathbf{k}_\perp'; z)] , \end{aligned} \tag{84}$$

where we have defined the carrier wave number $k_0 \equiv \omega_0/c$. This equation has been derived in various forms and by various techniques *beginning* with the paraxial equation in x-space by Klyatskin and Tatarskii[9] and by Besieris,[10] who also used the concept of the Weyl transform [although not in terms of the basic operator equation (9)]. As we have seen here, however, (84) arises as just a special case of the more general formalism presented in Sec. II applied to the particular wave equations in Secs. III B and III C.

IV. CONCLUSIONS

In this paper we have given a concise, classical wave derivation of the kinetic equation (or radiation transport equation) which governs the evolution of the wave-action density in a weakly nonuniform, nonstationary medium in the presence of small space- and time-dependent fluctuations (which are *not* assumed to be slowly varying compared with the wave frequency). The method used is based on the Weyl representation of the underlying linear wave equation, which has been assumed to have a very general form (7). This approach has the advantage that the concept of the ray phase space emerges naturally (with position $\mathbf{x}$ and wave vector $\mathbf{k}$ treated as independent variables), as does the Hamiltonian structure on the phase space. Furthermore, the wave-action density is defined in a natural way as the density of the wave spectral density on the dispersion surface (where waves obey the local dispersion relation).

The essential new feature of this derivation is the inclusion of the effect of fluctuations on the propagation of the action density. The new term in the wave kinetic equation gives the scattering of wave action from one local mode to another, mediated by fluctuations at the difference wave vector and difference frequency. The expression for the scattering cross section in terms of the spectral density of the fluctuations appears naturally in the Weyl formalism. Finally, we have illustrated our derivation by applying it to three wave equations (4)–(6) which are representative of a wide variety of wave systems characterized by the presence of fluctuations.

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APPENDIX A: THE MEAN SPECTRAL OPERATOR EQUATION

In this appendix we derive the mean spectral operator equation

$$D_0 W D_0^\dagger - \langle D_1 D_0^{-1} D_1\rangle W D_0^\dagger - D_0 W \langle D_1^\dagger (D_0^\dagger)^{-1} D_1^\dagger\rangle - \langle D_1 W D_1^\dagger\rangle = G \quad \text{(A1)}$$

following Karal and Keller.[7] The equation was given by Chen and Soong,[11] but we include the derivation here for completeness. Begin with the wave equation

$$(D_0 + D_1)\psi = g \quad \text{(A2)}$$

and its adjoint

$$\psi^*(D_0^\dagger + D_1^\dagger) = g^* \ , \quad \text{(A3)}$$

which combined give

$$(D_0 + D_1) W (D_0^\dagger + D_1^\dagger) = G \ , \quad \text{(A4)}$$

$$W \equiv (\psi\psi^*) \ , \quad G \equiv (gg^*) \ . \quad \text{(A5)}$$

Expanding (A4) we have

$$D_0 W D_0^\dagger + D_1 W D_0^\dagger + D_0 W D_1^\dagger + D_1 W D_1^\dagger = G \ . \quad \text{(A6)}$$

In the absence of the perturbation D_1, let the solution to (A6) be W_0

$$D_0 W_0 D_0^\dagger = G \Rightarrow W_0 = D_0^{-1} G (D_0^\dagger)^{-1} \ . \quad \text{(A7)}$$

Now we can write (A6) as

$$\begin{aligned} W &= W_0 - D_0^{-1}(D_1 W D_0^\dagger + D_0 W D_1^\dagger + D_1 W D_1^\dagger)(D_0^\dagger)^{-1} \\ &= W_0 - D_0^{-1} D_1 W - W D_1^\dagger (D_0^\dagger)^{-1} \\ &\quad - D_0^{-1} D_1 W D_1^\dagger (D_0^\dagger)^{-1} \ , \end{aligned} \quad \text{(A8)}$$

which can be verified by applying D_0 from the left and $D_0^\dagger$ from the right and using (A7) to obtain (A6). Iteration of this equation means substituting

$$W = W_0 - D_0^{-1} D_1 W_0 - W_0 D_1^\dagger (D_0^\dagger)^{-1} \quad \text{(A9)}$$

into the last three terms of (A8) to find

$$\begin{aligned} W = {} & W_0 - D_0^{-1} D_1 W_0 + D_0^{-1} D_1 D_0^{-1} D_1 W_0 + D_0^{-1} D_1 W_0 D_1^\dagger (D_0^\dagger)^{-1} \\ & - W_0 D_1^\dagger (D_0^\dagger)^{-1} + W_0 D_1^\dagger (D_0^\dagger)^{-1} D_1^\dagger (D_0^\dagger)^{-1} + D_0^{-1} D_1 W_0 D_1^\dagger (D_0^\dagger)^{-1} - D_0^{-1} D_1 W_0 D_1^\dagger (D_0^\dagger)^{-1} \ . \end{aligned} \quad \text{(A10)}$$

The expectation value (or ensemble average) of this is

$$\begin{aligned} \langle W\rangle = {} & W_0 - D_0^{-1}\langle D_1\rangle W_0 - W_0 \langle D_1^\dagger\rangle (D_0^\dagger)^{-1} \\ & + D_0^{-1}\langle D_1 D_0^{-1} D_1\rangle W_0 + W_0 \langle D_1^\dagger (D_0^\dagger)^{-1} D_1^\dagger\rangle (D_0^\dagger)^{-1} + D_0^{-1}\langle D_1 W_0 D_1^\dagger\rangle (D_0^\dagger)^{-1} \ . \end{aligned} \quad \text{(A11)}$$

The inverse of this to lowest order is

$$W_0 = \langle W\rangle + D_0^{-1}\langle D_1\rangle\langle W\rangle + \langle W\rangle\langle D_1^\dagger\rangle (D_0^\dagger)^{-1} \ . \quad \text{(A12)}$$

Finally, substituting this into (A11) we obtain

$$\begin{aligned} \langle W\rangle = {} & W_0 - D_0^{-1}\langle D_1\rangle\langle W\rangle - D_0^{-1}\langle D_1\rangle D_0^{-1}\langle D_1\rangle\langle W\rangle - D_0^{-1}\langle D_1\rangle\langle W\rangle\langle D_1^\dagger\rangle (D_0^\dagger)^{-1} \\ & - \langle W\rangle\langle D_1^\dagger\rangle (D_0^\dagger)^{-1} - D_0^{-1}\langle D_1\rangle\langle W\rangle\langle D_1^\dagger\rangle (D_0^\dagger)^{-1} - \langle W\rangle\langle D_1^\dagger\rangle (D_0^\dagger)^{-1}\langle D_1^\dagger\rangle (D_0^\dagger)^{-1} \\ & + D_0^{-1}\langle D_1 D_0^{-1} D_1\rangle\langle W\rangle + \langle W\rangle\langle D_1^\dagger (D_0^\dagger)^{-1} D_1^\dagger\rangle (D_0^\dagger)^{-1} + D_0^{-1}\langle D_1 \langle W\rangle D_1^\dagger\rangle\, (D_0^\dagger)^{-1} \ . \end{aligned}$$

Now, applying D_0 from the left and $D_0^\dagger$ from the right, and using (A7) we have the equation for $\langle W\rangle$

$$\begin{aligned} D_0\langle W\rangle D_0^\dagger = {} & G - \langle D_1\rangle\langle W\rangle D_0^\dagger - D_0\langle W\rangle\langle D_1^\dagger\rangle - \langle D_1\rangle D_0^{-1}\langle D_1\rangle\langle W\rangle D_0^\dagger \\ & - D_0\langle W\rangle\langle D_1^\dagger\rangle (D_0^\dagger)^{-1}\langle D_1^\dagger\rangle - 2D_0^{-1}\langle D_1\rangle\langle W\rangle\langle D_1^\dagger\rangle (D_0^\dagger)^{-1} \\ & + \langle D_1 D_0^{-1} D_1\rangle\langle W\rangle D_0^\dagger + D_0\langle W\rangle\langle D_1^\dagger (D_0^\dagger)^{-1} D_1^\dagger\rangle + \langle D_1\langle W\rangle D_1^\dagger\rangle \ . \end{aligned}$$

If we assume $\langle D_1 \rangle = 0$, only the last three terms are present; applying $(D_0^\dagger)^{-1}$ from the right then yields (A1) and (9) in the main text (where we have also written W for $\langle W \rangle$).

APPENDIX B: THE WEYL REPRESENTATION OF TRIPLE PRODUCTS

In this appendix we compute the Weyl representation of an ensemble-averaged operator triple product of the form

$$\langle ABA\rangle(z) = \langle A(z)\, e^{(i/2)\overleftrightarrow{\mathcal{L}}}\, B(z)\, e^{(i/2)\overleftrightarrow{\mathcal{L}}} A(z)\rangle \;, \tag{B1}$$

where the operator A [and hence its Weyl representation $A(z)$] is a fluctuating quantity. It can be shown that the order of differentiation indicated in (B1) is arbitrary; that is

$$\begin{aligned}\langle ABA\rangle(z) &= \langle [A(z)\, e^{(i/2)\overleftrightarrow{\mathcal{L}}}\, B(z)]\, e^{(i/2)\overleftrightarrow{\mathcal{L}}}\, A(z)\rangle \\ &= \langle A(z)\, e^{(i/2)\overleftrightarrow{\mathcal{L}}}\, [B(z)\, e^{(i/2)\overleftrightarrow{\mathcal{L}}}\, A(z)]\rangle \;.\end{aligned} \tag{B2}$$

1. Case 1

Let us specialize to the case where $A(z)$ depends only on the space-time variables $(\mathbf{x},t)$; this is the case where the space-time kernel corresponding to A contains no derivatives and is of the form

$$A(\mathbf{x},t;\mathbf{x}',t') = a(\mathbf{x},t)\,\delta(\mathbf{x}-\mathbf{x}')\delta(t-t') \Rightarrow A(z) = a(\mathbf{x},t) \;. \tag{B3}$$

The fluctuating field $a(\mathbf{x},t)$ is assumed to be real valued. We shall perform the computation in only one spatial dimension (and no time dependence) since the extrapolation to three dimensions and time is straightforward. Therefore, in this case (B1) becomes

$$\begin{aligned}\langle ABA\rangle(x,k) =\; &\langle a(x)\, e^{(i/2)\overleftrightarrow{\mathcal{L}}} B(x,k) \\ &\times e^{(i/2)\overleftrightarrow{\mathcal{L}}} a(x)\rangle \\ =\; &\langle a(x)\, e^{(i/2)\overleftarrow{\partial_x}\,\overrightarrow{\partial_k}} B(x,k) \\ &\times e^{-(i/2)\overleftarrow{\partial_k}\,\overrightarrow{\partial_x}} a(x)\rangle \;.\end{aligned} \tag{B4}$$

Expressing $B(x,k)$ in terms of its kernel $B(x,x')$ and using (B2) we have

$$\begin{aligned}a(x)\, e^{(i/2)\overleftarrow{\partial_x}\,\overrightarrow{\partial_k}} B(x,k)\, e^{-(i/2)\overleftarrow{\partial_k}\,\overrightarrow{\partial_x}} a(x) &= a(x)\, e^{(i/2)\overleftarrow{\partial_x}\,\overrightarrow{\partial_k}} \left(\int ds\; B(x+\tfrac{1}{2}s, x-\tfrac{1}{2}s)\, e^{-iks}\right) e^{-(i/2)\overleftarrow{\partial_k}\,\overrightarrow{\partial_x}} a(x) \\ &= \left(\int ds\; a(x)\, e^{(i/2)\overleftarrow{\partial_x}\,\overrightarrow{\partial_k}} e^{-iks} B(x+\tfrac{1}{2}s, x-\tfrac{1}{2}s)\right) e^{-(i/2)\overleftarrow{\partial_k}\,\overrightarrow{\partial_x}} a(x) \\ &= \left(\int ds\; a(x+\tfrac{1}{2}s)\, e^{-iks} B(x+\tfrac{1}{2}s, x-\tfrac{1}{2}s)\right) e^{-(i/2)\overleftarrow{\partial_k}\,\overrightarrow{\partial_x}} a(x) \\ &= \int ds\; a(x+\tfrac{1}{2}s)\, B(x+\tfrac{1}{2}s, x-\tfrac{1}{2}s)\, e^{-iks} e^{-(i/2)\overleftarrow{\partial_k}\,\overrightarrow{\partial_x}} a(x) \\ &= \int ds\; a(x+\tfrac{1}{2}s)\, B(x+\tfrac{1}{2}s, x-\tfrac{1}{2}s)\, e^{-iks} a(x-\tfrac{1}{2}s) \;.\end{aligned} \tag{B5}$$

In this procedure we have successively moved first the left-hand then the right-hand differentiation inside the integral over s; since $A(z)$ depends only on x (not k), there is no ambiguity in the grouping for the objects upon which the exponentiated differential operators act. The action of the exponentiated k derivatives on $\exp(-iks)$ is effected in terms of a Taylor series which, when coupled with the action of the exponentiated x derivatives on $a(x)$, produce the Taylor series for $a(x \pm \frac{1}{2}s)$. Thus, the ensemble average of (B5) gives

$$\begin{aligned}\langle ABA\rangle(x,k) &= \int ds\; \langle a(x+\tfrac{1}{2}s)a(x-\tfrac{1}{2}s)\rangle\, B(x+\tfrac{1}{2}s, x-\tfrac{1}{2}s)\, e^{-iks} \\ &= \int ds\; C_a(x,s)\, B(x+\tfrac{1}{2}s, x-\tfrac{1}{2}s)\, e^{-iks} \;,\end{aligned} \tag{B6}$$

where we have defined the local correlation function $C_a(x,s)$ of the random process $a(x)$ in the usual manner. This relation can now be expressed in terms of the Weyl representation $B(x,k)$ of B and the local spectral density $S_a(x,k)$ of $a(x)$

$$\begin{aligned}\langle ABA\rangle(x,k) &= \int ds\; e^{-iks} \left(\frac{1}{2\pi}\int dk' S_a(x,k')\, e^{ik's}\right)\left(\frac{1}{2\pi}\int dk'' B(x,k'')\, e^{ik''s}\right) \\ &= \frac{1}{2\pi}\int dk' S_a(x,k-k')\, B(x,k') \;.\end{aligned} \tag{B7}$$

This is our result, which in three dimensions and time becomes explicitly

$$\langle ABA\rangle(\mathbf{x},t,\mathbf{k},\omega) = \frac{1}{(2\pi)^4}\int d^3k'd\omega' S_a(\mathbf{x},t,\mathbf{k}-\mathbf{k}',\omega-\omega')\, B(\mathbf{x},t,\mathbf{k}',\omega') \;. \tag{B8}$$

2. Case 2

Another common form for the random operator A that arises in wave equations is

$$A(\mathbf{x},t;\mathbf{x}',t') = \frac{\partial}{\partial t}\frac{\partial}{\partial t'}[a(\mathbf{x},t)\delta(\mathbf{x}-\mathbf{x}')\delta(t-t')] \ , \tag{B9}$$

where we shall again assume that the fluctuating field $a(\mathbf{x},t)$ is real. The Weyl representation of this operator can be constructed as follows: from the definition (10), we have

$$A(\mathbf{x},t,\mathbf{k},\omega) = \int d^3s\, d\tau\ A(\mathbf{x}+\tfrac{1}{2}\mathbf{s},t+\tfrac{1}{2}\tau;\mathbf{x}-\tfrac{1}{2}\mathbf{s},t-\tfrac{1}{2}\tau)\, e^{-i\mathbf{k}\cdot\mathbf{s}+i\omega\tau} \ . \tag{B10}$$

Changing time variables in (B9) from (t,t') to $(T,\tau) = (\frac{1}{2}(t+t'),t-t')$ we have

$$A(\mathbf{x}+\tfrac{1}{2}\mathbf{s},t+\tfrac{1}{2}\tau;\mathbf{x}-\tfrac{1}{2}\mathbf{s},t-\tfrac{1}{2}\tau) = \left(\frac{1}{4}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial\tau^2}\right)[a(\mathbf{x}+\tfrac{1}{2}\mathbf{s},t+\tfrac{1}{2}\tau)\delta(\mathbf{s})\delta(\tau)] \ , \tag{B11}$$

where we have set $T=t$. Using this expression in (B10) and integrating by parts twice yields

$$A(\mathbf{x},t,\mathbf{k},\omega) = \omega^2 a(\mathbf{x},t) + \tfrac{1}{4}a''(\mathbf{x},t) \ , \tag{B12}$$

where $a''(\mathbf{x},t)$ denotes the second time-derivative of $a(\mathbf{x},t)$.

Let us now compute the Weyl representation of the operator triple product (B2). For simplicity, we shall discard all spatial and time derivatives of the "middle" operator $B(z)$; in our application, these triple-product terms occur at an order where the magnitude of $A^2(z)B(z)$ is already assumed to be small. Since $A(z)$ in (B12) has two terms, there will be four terms altogether in the evaluation of the triple product (B2). We first consider

$$T_1(z) \equiv \omega^2 a(x,t)\, e^{(i/2)\overleftrightarrow{\mathcal{L}}} B(x,t,k,\omega)\, e^{(i/2)\overleftrightarrow{\mathcal{L}}}\ \omega^2 a(x,t) \ , \tag{B13}$$

where, as in Appendix B1, we will work in only one spatial dimension but will now explicitly include the (t,ω) variables. First we have

$$\begin{aligned}\omega^2 a(x,t)\, e^{(i/2)\overleftrightarrow{\mathcal{L}}} B(x,t,k,\omega) &= \omega^2 a(x,t)\, e^{(i/2)(-\overleftarrow{\partial_t}\ \overrightarrow{\partial_\omega}+\overleftrightarrow{\mathcal{L}}_x)}\ B(x,t,k,\omega)\\ &= \omega^2 a(x,t)\, e^{(i/2)(-\overleftarrow{\partial_t}\ \overrightarrow{\partial_\omega}+\overleftrightarrow{\mathcal{L}}_x)} \int ds\, d\tau\ C(x,t;s,\tau)\, e^{-iks+i\omega\tau}\\ &= \omega^2 \int ds\, d\tau\ a(x+\tfrac{1}{2}s,t+\tfrac{1}{2}\tau)\, C(x,t;s,\tau)\, e^{-iks+i\omega\tau} \ .\end{aligned}$$

In the first line we have split the operator $\overleftrightarrow{\mathcal{L}}$ into its (t,ω) terms and its (x,k) part $\overleftrightarrow{\mathcal{L}}_x$; as stated above, we discard all space and time derivatives acting on B. In the second line we have expressed $B(x,t,k,\omega)$ in terms of its associated correlation function $C(x,t;s,\tau) = B(x+\frac{1}{2}s,t+\frac{1}{2}\tau;x-\frac{1}{2}s,t-\frac{1}{2}\tau)$. Finally, we have proceeded as in (B5) to evaluate the action of the exponentiated derivatives.

We now consider the remaining part of (B13)

$$T_1(z) = \omega^2 \int ds\, d\tau\ a(x+\tfrac{1}{2}s,t+\tfrac{1}{2}\tau)\, C(x,t;s,\tau)\, e^{-iks+i\omega\tau} e^{(i/2)(\overleftrightarrow{\mathcal{L}}_t+\overleftrightarrow{\mathcal{L}}_x)}\, \omega^2 a(x,t) \ . \tag{B14}$$

This is somewhat more complicated since both t and ω derivatives must be considered in both directions. Since it is clear that the effect of the spatial operator $\overleftrightarrow{\mathcal{L}}_x$ will be the same as in (B5) [i.e., it will produce $a(x-\frac{1}{2}s,t)$ inside the integral], we need only consider the action of the temporal operator $\overleftrightarrow{\mathcal{L}}_t = \overleftarrow{\partial_\omega}\ \overrightarrow{\partial_t} - \overleftarrow{\partial_t}\ \overrightarrow{\partial_\omega}$. We have

$$\omega^2 a(t+\tfrac{1}{2}\tau)C(t)e^{i\omega\tau}\ e^{(i/2)\overleftrightarrow{\mathcal{L}}_t}\omega^2 a(t) = \omega^2 a(t+\tfrac{1}{2}\tau)C(t)e^{i\omega\tau}\ e^{(i/2)\overleftarrow{\partial_\omega}\ \overrightarrow{\partial_t}}(\omega^2 - i\omega\overleftarrow{\partial_t} - \tfrac{1}{4}\overleftarrow{\partial_t}^2)a(t) \ .$$

The action of the remaining exponentiated operator is

$$\begin{aligned}\omega^2\, e^{i\omega\tau}\, e^{(i/2)\overleftarrow{\partial_\omega}\ \overrightarrow{\partial_t}} &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{2}\right)^n \partial_\omega^n(\omega^2 e^{i\omega\tau})\overrightarrow{\partial_t}^n\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{2}\right)^n \overrightarrow{\partial_t}^n \sum_{m=0}^{n}\binom{n}{m}(\partial_\omega^m\omega^2)(i\tau)^{n-m}e^{i\omega\tau}\\ &= e^{i\omega\tau}(\omega^2 + i\omega\overrightarrow{\partial_t} - \tfrac{1}{4}\overrightarrow{\partial_t}^2)\, e^{-(1/2)\tau\overrightarrow{\partial_t}} \ .\end{aligned}$$

Combining these two results we have

$$e^{i\omega\tau}\, a(t+\tfrac{1}{2}\tau)C(t)\,(\omega^2 + i\omega\overrightarrow{\partial_t} - \tfrac{1}{4}\overrightarrow{\partial_t}^2)\, e^{-(1/2)\tau\overrightarrow{\partial_t}}(\omega^2 - i\omega\overleftarrow{\partial_t} - \tfrac{1}{4}\overleftarrow{\partial_t}^2)\, a(t)$$

$$= e^{i\omega\tau}\, a(t+\tfrac{1}{2}\tau)C(t)\,(\omega^2 + i\omega\overrightarrow{\partial_t} - \tfrac{1}{4}\overrightarrow{\partial_t}^2)\,(\omega^2 - i\omega\overleftarrow{\partial_t} - \tfrac{1}{4}\overleftarrow{\partial_t}^2)\, a(t-\tfrac{1}{2}\tau)$$

$$= e^{i\omega\tau}C(t)\left(\omega^4 a_+a_- + i\omega^3(a_+a_-' - a_+'a_-) + \tfrac{1}{4}\omega^2(4a_+'a_-' - a_+''a_- - a_+a_-'') + \frac{i}{4}\omega(a_+'a_-'' - a_+''a_-') + \tfrac{1}{16}a_+''a_-''\right) .$$

Here the notation means $a_\pm' = \partial_t a(x \pm \frac{1}{2}s, t \pm \frac{1}{2}\tau)$. As assumed above, we have discarded time derivatives of the correlation function $C(t)$. Using this result in (B14) we find

$$T_1(z) = \int ds\, d\tau\; e^{-iks+i\omega\tau}\, C(x,t;s,\tau)\left(\omega^4 a_+a_- + i\omega^3(a_+a_-' - a_+'a_-) + \tfrac{1}{4}\omega^2(4a_+'a_-' - a_+''a_- - a_+a_-'') + \frac{i}{4}\omega(a_+'a_-'' - a_+''a_-') + \tfrac{1}{16}a_+''a_-''\right) . \quad \text{(B15)}$$

Now, as in (B6) and (B7) of the preceding section, we want to reexpress the correlation function C in terms of the Weyl representation B and introduce the spectral density S_a of the fluctuation $a(x,t)$. However, except for the first term in large parentheses in (B15), the bilinear products of a_+ and a_- involve time derivatives of those quantities. Thus, let us introduce the spectral density S_{mn} of these quantities as

$$\langle(\partial_t^m a_+)(\partial_t^n a_-)\rangle \equiv \frac{1}{(2\pi)^2}\int dk'd\omega'\; S_{mn}(x,t,k',\omega')\, e^{ik's - i\omega'\tau} . \quad \text{(B16)}$$

We shall now show how the spectral density S_{mn} can be expressed in terms of the fundamental spectral density S_a.

Focusing on just the temporal variation of the fluctuating field $a(t)$, we introduce the Fourier transform $\hat{a}(\omega)$; time-derivatives of $a(t)$ can then be expressed as

$$\partial_t^m a(t) = \frac{1}{2\pi}\int d\omega'(-i\omega')^m \hat{a}(\omega')\, e^{-i\omega' t} . \quad \text{(B17)}$$

Therefore, using this in the definition of the spectral density S_{mn} [the inverse of (B16)] we have

$$\begin{aligned}
S_{mn}(t,\omega) &= \int d\tau\; \langle \partial_t^m a(t+\tfrac{1}{2}\tau)\partial_t^n a(t-\tfrac{1}{2}\tau)\rangle\, e^{i\omega\tau} \\
&= \int d\tau\, e^{i\omega\tau}\frac{1}{(2\pi)^2}\int d\omega'd\omega''\langle\hat{a}(\omega')\hat{a}^*(\omega'')\rangle\; (-i\omega')^m(i\omega'')^n e^{-i\omega'[t+(1/2)\tau]}e^{i\omega''[t-(1/2)\tau]} \\
&= \frac{1}{(2\pi)^2}\int d\omega'd\omega''\langle\hat{a}(\omega')\hat{a}^*(\omega'')\rangle(-i\omega')^m(i\omega'')^n e^{-i(\omega'-\omega'')t}\int d\tau\, e^{i\tau[\omega-(1/2)(\omega'+\omega'')]} \\
&= \frac{1}{2\pi}\int d\nu\langle\hat{a}(\omega+\tfrac{1}{2}\nu)\hat{a}^*(\omega-\tfrac{1}{2}\nu)\rangle\; [-i(\omega+\tfrac{1}{2}\nu)]^m[i(\omega-\tfrac{1}{2}\nu)]^n\, e^{-i\nu t} \\
&= \left[-i\left(\omega+\frac{i}{2}\partial_t\right)\right]^m\left[i\left(\omega-\frac{i}{2}\partial_t\right)\right]^n\frac{1}{2\pi}\int d\nu\langle\hat{a}(\omega+\tfrac{1}{2}\nu)\hat{a}^*(\omega-\tfrac{1}{2}\nu)\rangle\, e^{-i\nu t} \\
&= [-i\omega+\tfrac{1}{2}\partial_t]^m[i\omega+\tfrac{1}{2}\partial_t]^n\, S_a(t,\omega) .
\end{aligned} \quad \text{(B18)}$$

It is straightforward to show that S_a can be defined in terms of the Fourier transform $\hat{a}(\omega)$ as in the last two lines.

Now consider a single ensemble-averaged term in the expression (B15) for $T_1(z)$:

$$\begin{aligned}
&\int ds\, d\tau\; e^{-iks+i\omega\tau}\, C(x,t;s,\tau)\langle(\partial_t^m a_+)(\partial_t^n a_-)\rangle \\
&\quad= \int ds\, d\tau\; e^{-iks+i\omega\tau}\frac{1}{(2\pi)^2}\int dk'd\omega'\; e^{ik's-i\omega'\tau}\, B(x,t,k',\omega') \\
&\qquad\times\frac{1}{(2\pi)^2}\int dk''d\omega''\; e^{ik''s-i\omega''\tau}\,(-i\omega''+\tfrac{1}{2}\partial_t)^m(i\omega''+\tfrac{1}{2}\partial_t)^n\, S_a(x,t,k'',\omega'') \\
&\quad= \frac{1}{(2\pi)^2}\int dk'd\omega'\{[-i(\omega-\omega')+\tfrac{1}{2}\partial_t]^m[i(\omega-\omega')+\tfrac{1}{2}\partial_t]^n\, S_a(x,t,k-k',\omega-\omega')\}B(x,t,k',\omega') .
\end{aligned}$$

Thus, the ensemble average of the first term $\langle T_1(z)\rangle$ [defined in (B13)] of the triple product is

$$\langle T_1(z)\rangle = \frac{1}{(2\pi)^2}\int dk'd\omega'\; B(x,t,k',\omega') \times\Big(\omega^4 + i\omega^3\left\{[i(\omega-\omega')+\tfrac{1}{2}\partial_t]-[-i(\omega-\omega')+\tfrac{1}{2}\partial_t]\right\}$$
$$+\tfrac{1}{4}\omega^2\left\{4[-i(\omega-\omega')+\tfrac{1}{2}\partial_t][i(\omega-\omega')+\tfrac{1}{2}\partial_t]-[-i(\omega-\omega')+\tfrac{1}{2}\partial_t]^2-[i(\omega-\omega')+\tfrac{1}{2}\partial_t]^2\right\}$$
$$+\frac{i}{4}\omega\left\{[-i(\omega-\omega')+\tfrac{1}{2}\partial_t][i(\omega-\omega')+\tfrac{1}{2}\partial_t]^2-[-i(\omega-\omega')+\tfrac{1}{2}\partial_t]^2[i(\omega-\omega')+\tfrac{1}{2}\partial_t]\right\}$$
$$+\tfrac{1}{16}[-i(\omega-\omega')+\tfrac{1}{2}\partial_t]^2[i(\omega-\omega')+\tfrac{1}{2}\partial_t]^2\Big)S_a(x,t,k-k',\omega-\omega')\ . \tag{B19}$$

The calculation of the remaining three terms in the triple product follows along the same lines as that used for $\langle T_1(z)\rangle$ above. The three terms are

$$T_2(z) \equiv \omega^2 a(x,t)\, e^{(i/2)\overleftrightarrow{\mathcal{L}}}\, B(x,t,k,\omega)\, e^{(i/2)\overleftrightarrow{\mathcal{L}}}\, \tfrac{1}{4}a''(x,t)$$
$$= \tfrac{1}{4}\int ds\, d\tau\; e^{-iks+i\omega\tau}\, C(x,t;s,\tau)\, \left(\omega^2 a_+ a''_- + i\omega a_+ a'''_- - \tfrac{1}{4}a_+ a''''_-\right)\ , \tag{B20}$$

$$T_3(z) \equiv \tfrac{1}{4}a''(x,t)\, e^{(i/2)\overleftrightarrow{\mathcal{L}}}\, B(x,t,k,\omega)\, e^{(i/2)\overleftrightarrow{\mathcal{L}}}\, \omega^2 a(x,t)$$
$$= \tfrac{1}{4}\int ds\, d\tau\; e^{-iks+i\omega\tau}\, C(x,t;s,\tau)\, \left[\omega^2 a''_+ a_- - i\omega a'''_+ a_- - \tfrac{1}{4}a''''_+ a_-\right]\ , \tag{B21}$$

$$T_4(z) \equiv \tfrac{1}{4}a''(x,t)\, e^{(i/2)\overleftrightarrow{\mathcal{L}}}\, B(x,t,k,\omega)\, e^{(i/2)\overleftrightarrow{\mathcal{L}}}\, \tfrac{1}{4}a''(x,t)$$
$$= \tfrac{1}{16}\int ds\, d\tau\; e^{-iks+i\omega\tau}\, C(x,t;s,\tau)\, a''_+ a''_-\ . \tag{B22}$$

Expressing C in terms of B, introducing the spectral density S_a as in (B16)–(B19) and adding all four terms together we have our result (now explicitly in three dimensions)

$$\langle ABA\rangle(z) = \frac{1}{(2\pi)^4}\int d^3k'd\omega'\left[\omega'^2\left(\omega^2+\frac{1}{4}\frac{\partial^2}{\partial t^2}\right)S_a(\mathbf{x},t,\mathbf{k}-\mathbf{k}',\omega-\omega')\right]B(\mathbf{x},t,\mathbf{k}',\omega')\ . \tag{B23}$$

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