Electron component of a plasma in a homogeneous electric field

Alexander V. Rokhlenko
Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903
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We investigate spatially homogeneous stationary solutions of the Boltzmann equation describing the electron component of a gas plasma in a homogeneous electric field. We consider both elastic and weakly inelastic collisions of electrons with the neutral-particle plasma component as well as the Coulomb interaction between charged particles, described by Landau's collision integral. The Boltzmann equation for the electron swarm is linearized around a Maxwellian with some unknown temperature. For some important cases the problem reduces to a single-parameter system of two integrodifferential equations. Its solution allows us to treat plasmas under a variety of conditions in a rather simple way. We solve this system with the help of the spline-collocation method, and find the first two corrections to the Maxwellian corresponding to symmetrical and flow components of the velocity distribution function. The equation for the electron temperature then comes from the energy balance condition and from the requirement that the solution of the kinetic equation be unique. The electron temperature and energy absorption are described by curves with hysteresis as functions of the electric-field intensity. The conductivity versus electron temperature has a single maximum. Its rising slope corresponds to Spitzer's well-known formula, when the electron temperature is not very high and electron–neutral-particle collisions are unimportant. All numerical results are obtained under the assumption that the cross section for electron–neutral-particle collisions is independent of electron energy but generalizations are possible, since the analytical solution of the problem is far advanced.

INTRODUCTION

In this work, we carry out a theoretical investigation of plasma properties in a homogeneous electric field based on a solution of the Boltzmann equation. The following assumptions are made.

(i) The plasma is spatially homogeneous and stationary.

(ii) The velocity distribution functions of the plasma neutral-particle component and of the positive ions are Maxwellians with some known temperature.

(iii) All of the kinetic processes that involve electrons are determined by the external field and electron-electron (*e-e*), electron-ion (*e-i*), and electron–neutral-particle (*e-n*) pair collisions. The interaction between charged particles is described by the Landau collision integral, and the collisions between the electrons and neutral particles are assumed to be elastic or "almost elastic."[1]

(iv) The concentration of charged particles is not very small, so the *e-e* collisions cause the electron distribution function to be, in zero order, a Maxwellian with some temperature $T(E)$ depending on the electric-field intensity E.

(v) Corrections of order higher than the first can be neglected. This requires some restrictions for the electric-field intensity, which will be specified later, and allows us to linearize the collision *e-e* integral and therefore our problem around the zeroth approximation of the distribution function in a way similar to the one presented in Ref. 2.

No calculations of the electron distribution function f^e and the temperature $T(E)$ in this setting are known to the author. Previous calculations of this kind, carried out by many authors; for example, Refs. 3–6, either ignored one of the interactions *e-e*, *e-i*, *e-n* or used some other approximations that enabled them to consider both direct and alternating electric fields, inelastic collisions, and the possibility of nonstationary solutions of the kinetic equation (see the review of this problem in Ref. 7). The method described in the article does not require any additional approximations or simplifications to the ones listed (except for trivial ones) and permits a natural extension to the case of a plasma in a constant homogeneous magnetic field and an alternating electric field. The idea of our method is very transparent: the linearization of the problem and the direct solving of the integrodifferential equations for the angular harmonics of the distribution function. Validity and generalization of the procedure are given serious attention.

The steady-state kinetic equation for the electrons under the above-mentioned conditions can be written down in the form

$$-\frac{e\mathbf{E}}{m}\cdot\nabla_v f^e = \mathcal{C} f^e \ . \tag{1}$$

Here, e is the charge, m is the mass, v is the velocity of the electron, $\mathbf{E}$ is the vector of the electric field intensity, and the right-hand side of the equation is the collision integral

$$\mathcal{C} f^e = \mathcal{T}(f^e,f^e) + \mathcal{T}(f^e,f^i) + \mathcal{T}(f^e,f^n) \ . \tag{2}$$

According to the accepted assumptions

$$f^i = C_i \exp\left[-\frac{M_i v_i^2}{2kT_i}\right], \quad f^n = C_n \exp\left[-\frac{M_n v_n^2}{2kT_n}\right], \tag{3}$$

where k is Boltzmann's constant and the meaning of indices is obvious. The electron swarm distribution function is now given by a first-order perturbed Maxwellian distribution, as it was done in Ref. 8,

$$f^e = C_e[e^{-mv^2/2kT} + f(\mathbf{v})] \equiv C_e[\Phi(v) + f(\mathbf{v})] , \quad T \equiv T_e , \qquad (4)$$

where the first term substantially exceeds the second one (the verification of this inequality will be discussed), but in (4) the Maxwellian does not correspond to the unperturbed state of the plasma, when $E=0$, and therefore $T_e = T_i = T_n$.

The normalization coefficients C_j $(j=e,i,n)$ must be determined from the equations

$$\int f^j d^3 p_j = N_j , \qquad (5)$$

where $p_j \equiv m_j v_j$ is the impulse of a j-kind particle, N_j $(j=e,i,n)$ are the concentrations of the corresponding particles.

Coulomb's collisions between electrons and charged particles of the j kind are given according to Landau[9,10] by

$$\mathcal{T}(f^e, f^j) = -\nabla_v \cdot \mathbf{s}/m, \quad j \neq n ,$$
$$s_\alpha = \sum_\beta 2\pi (ee_j)^2 L_j \int \left[f^e \frac{\partial f^j}{\partial p_{j\beta}} - f^j \frac{\partial f^e}{\partial p_{e\beta}} \right] \times \frac{\partial^2 |\mathbf{v}_e - \mathbf{v}_j|}{\partial v_{j\alpha} \partial v_{j\beta}} d^3 p_j . \qquad (6)$$

Here the Greek indices correspond to coordinates x,y,z of the three-dimensional space, and L_j is the Coulomb logarithm for collisions of electrons with the j-kind charged particles. According to Ref. 7, $L_e \approx L_i$ and because of its very weak dependence on plasma parameters L_e is supposed to be constant (near 10).

Calculation of $\mathcal{T}(f^e, f^i) \equiv \mathcal{T}_{e\text{-}i}$

To reduce the length of formulas we denote

$$\frac{\partial f^j(\mathbf{v}_j)}{\partial v_{j\alpha}} \equiv f^j_\alpha, \quad \frac{\partial^2 f^j(\mathbf{v}_j)}{\partial v_{j\alpha} \partial v_{j\beta}} \equiv f^j_{\alpha\beta}, \quad \mathbf{v}_e \equiv \mathbf{v}, \quad \mathbf{v}_i \equiv \mathbf{u} .$$

Considering for the sake of simplicity only one kind of single-charged ion we obtain from (6) the relationship

$$\mathcal{T}_{e\text{-}i} = -\frac{2\pi e^4 L_i M_i^3}{m} \sum_{\alpha,\beta} \frac{\partial}{\partial v_\alpha} \int \left[f^e(\mathbf{v}) \frac{f^i_\beta(\mathbf{u})}{M_i} - f^i(\mathbf{u}) \frac{f^e_\beta(\mathbf{v})}{m} \right] \frac{\partial^2 |\mathbf{u}-\mathbf{v}|}{\partial u_\alpha \partial u_\beta} d^3 u$$
$$= \frac{2\pi e^4 L_i M_i^3}{m} \sum_{\alpha,\beta} \int \left[\left[\frac{M_i}{m} - 1 \right] f^i_\alpha(\mathbf{u}) f^e_\beta(\mathbf{v}) + \frac{M_i}{m} f^i(\mathbf{u}) f^e_{\alpha\beta}(\mathbf{v}) - f^e(\mathbf{v}) f^i_{\alpha\beta}(\mathbf{u}) \right] \frac{\partial^2 |\mathbf{u}-\mathbf{v}|}{\partial u_\alpha \partial u_\beta} d^3 u .$$

It transforms after integration by parts into

$$\mathcal{T}_{e\text{-}i} = 16\pi^2 e^4 L_i \frac{C_e}{m} \left\{ \frac{1}{v^2} \frac{\partial}{\partial v} \left[j(v) \left[\frac{m}{M_i} \Phi + \frac{kT_i}{M_i v} \frac{\partial \Phi}{\partial v} \right] \right] + \frac{M_i^3}{8\pi} \sum_{\alpha,\beta} f_{\alpha\beta}(\mathbf{v}) h_{\alpha\beta}(v) - \left[1 - \frac{m}{M_i} \right] \frac{\mathbf{v} \cdot \nabla f}{v^3} j(v) \right\} .$$

To get this result we use (3), the identity[11]

$$\int d^3 u \, |\mathbf{u}-\mathbf{v}|^{-1} \Delta \mathcal{F}(\mathbf{u}) = -4\pi \mathcal{F}(\mathbf{v}) \qquad (7)$$

for twice differentiable functions $\mathcal{F}(\mathbf{u})$, and the obvious equality

$$\Delta |\mathbf{u}-\mathbf{v}| = 2/|\mathbf{u}-\mathbf{v}|$$

(Δ is the Laplacian). We also introduce temporarily the functions

$$j(v) = M_i^3 \int_0^v u^2 f^i(u) du \qquad (8)$$

and

$$h(v) = \int f^i(u) |\mathbf{u}-\mathbf{v}| d^3 u .$$

We will neglect the small value m/M_i in the last sum. Since

$$h_{\alpha\beta} \simeq \frac{4\pi}{M_i^3} j(v)(v^2 \delta_{\alpha\beta} - v_\alpha v_\beta) v^{-3}$$

we can get an expression for $\mathcal{T}_{e\text{-}i}$ in the final form

$$\mathcal{T}_{e\text{-}i} = 16 \frac{\pi^2 e^4}{m^2} L_i C_e \left\{ v^{-2} \frac{\partial}{\partial v} \left[j(v) \left[\frac{m}{M_i} \Phi + \frac{kT_i}{M_i v} \frac{\partial \Phi}{\partial v} \right] \right] + j(v) v^{-3} \hat{L} f(\mathbf{v}) \right\} , \qquad (9)$$

where we have used the angular momentum operator

$$\hat{L}=\sum_{\alpha,\beta}\frac{\partial}{\partial v_\alpha}(v^2\delta_{\alpha\beta}-v_\alpha v_\beta)\frac{\partial}{\partial v_\beta} . \tag{10}$$

The Legendre polynomials $\mathcal{P}_l(\xi)$ are the eigenfunctions of the operator $\hat{L}$:

$$\hat{L}\mathcal{P}_l(\xi)=-l(l+1)\mathcal{P}_l(\xi) , \tag{11}$$

where l is the order of the polynomial, $\xi=\cos(\mathbf{v},\mathbf{a})$, and $\mathbf{a}$ is some arbitrary fixed vector.

Calculation of $\mathcal{T}(f^e,f^e)\equiv\mathcal{T}_{e\text{-}e}$

After the substitution of (4) into (6) we throw away all terms involving squares of $f(\mathbf{v})$ according to the starting assumption and reach the linearization of Boltzmann's equation (1). The e-e collision integral is given by

$$\mathcal{T}_{e\text{-}e}=2\pi e^4L_emC_e^2\sum_{\alpha,\beta}\int[f(\mathbf{u})\Phi_{\alpha\beta}(\mathbf{v})+\Phi(u)f_{\alpha\beta}(\mathbf{v})-\Phi(v)f_{\alpha\beta}(\mathbf{u})-f(\mathbf{v})\Phi_{\alpha\beta}(\mathbf{u})]\frac{\partial^2|\mathbf{u}-\mathbf{v}|}{\partial u_\beta\partial u_\beta}d^3u .$$

From here by using identity (7) and integration by parts we get

$$\mathcal{T}_{e\text{-}e}=2\pi e^4L_eC_e^2m\left[16\pi\Phi(v)f(\mathbf{v})+\sum_{\alpha,\beta}f_{\alpha\beta}(\mathbf{v})\frac{\partial^2}{\partial v_\alpha\partial v_\beta}\int\Phi(u)|\mathbf{u}-\mathbf{v}|d^3u+\Phi_{\alpha\beta}(\mathbf{v})\frac{\partial^2}{\partial v_\alpha\partial v_\beta}\int f(\mathbf{u})|\mathbf{u}-\mathbf{v}|d^3u\right] .$$

It is easy to show the validity of the identity

$$\sum_{\alpha,\beta}\mathcal{H}_{\alpha\beta}(\mathbf{v})\frac{\partial^2\mathcal{F}(v)}{\partial v_\alpha\partial v_\beta}=\frac{d^2\mathcal{F}}{dv^2}\frac{\partial^2\mathcal{H}}{\partial v^2}+\frac{2}{v^2}\frac{d\mathcal{F}}{dv}\frac{\partial\mathcal{H}}{\partial v}+\frac{1}{v^3}\frac{d\mathcal{F}}{dv}\hat{L}\mathcal{H}(\mathbf{v}) , \tag{12}$$

where $\mathcal{F}(v),\mathcal{H}(\mathbf{v})$ are arbitrary twice differentiable functions, and to write $\mathcal{T}_{e\text{-}e}$ in the form

$$\mathcal{T}_{e\text{-}e}=2\pi e^4mL_eC_e^2\left[16\pi\Phi(v)f(\mathbf{v})+\frac{d^2G}{dv^2}\frac{\partial^2f}{\partial v^2}+\frac{2}{v^2}\frac{dG}{dv}\frac{\partial f}{\partial v}+\frac{d^2\Phi}{dv^2}\frac{\partial^2g}{\partial v^2}+\frac{2}{v^2}\frac{d\Phi}{dv}\frac{\partial g}{\partial v}+\frac{1}{v^3}\frac{dG}{dv}\hat{L}f(\mathbf{v})+\frac{1}{v^3}\frac{d\Phi}{dv}\hat{L}g(\mathbf{v})\right] , \tag{13}$$

where we have introduced the following notation:

$$G(v)=\int\Phi(u)|\mathbf{u}-\mathbf{v}|d^3u,\quad g(\mathbf{v})=\int f(\mathbf{u})|\mathbf{u}-\mathbf{v}|d^3u . \tag{14}$$

The function $G(v)$ and its derivatives can be expressed by means of the error function. Relationships (13) and (14) are useful for possible generalization and changing of the zeroth-order approximation.

Calculation of $\mathcal{T}(f^e,f^n)\equiv\mathcal{T}_{e\text{-}n}$

A simple way to obtain the collision integral of elastic scattering on neutral particles with mass $M_n>>m$ is to remember that the Coulomb transport cross section $s_{e\text{-}i}^t$ of electrons on ions is given[7] by

$$s_{e\text{-}i}^t=4\pi e^4L_i/m^2v^4 .$$

Now we must do the following: (i) take this value in (9) and replace it by the transport cross section of e-n scattering $s_{e\text{-}n}^t=1/N_nl(v)$, and (ii) use (5) to replace the value $j(v)$ (8) by

$$M_n^3\int_0^v u^2f^n(u)du\approx N_n/4\pi . \tag{15}$$

Thus we get

$$\mathcal{T}_{e\text{-}n}=C_e\left\{v^{-2}\frac{d}{dv}\left[\frac{v^4}{l(v)}\left[\frac{m}{M_n}\Phi+\frac{kT}{M_nv}\frac{d\Phi}{dv}\right]\right]+\frac{v}{2l(v)}\hat{L}f(\mathbf{v})\right\} , \tag{16}$$

where $l(v)$ is the mean free path of electrons between collisions with neutral particles of the plasma.

The approximate equality (15) is incorrect for only a very small number of electrons δN^e, whose velocities are less than or comparable to average ion velocities,

$$\frac{\delta N_e}{N_e}\sim\left[\frac{mT_n}{M_nT}\right]^{3/2} ,$$

but even this difference is suppressed by the term v. We must retain $j(v)$ in (9) to prevent the divergence of $\mathcal{T}_{e\text{-}i}$ at the point $v=0$. The importance of this step and the role of $j(v)$ will be seen later.

The collision integral (16) can be used for elastic collisions only, but we can generalize it according to Refs. 1 and 12 by the replacement

$$2m/M_n \rightarrow \chi(v) \; . \tag{17}$$

Here, $\chi(v) \ll 1$ is the fraction of the electron energy, which is lost in one "almost elastic" collision. The magnitude χ was studied by many authors. Its nonequality to $2m/M$ means the approximate treatment of nonelastic collisions. It was found to be very small not only for noble gases, if the electron energy ε does not exceed a few eV. For example, $\chi < 0.04$ in oxygen in the case $\varepsilon \leq 4$ eV.[13] Finally

$$\mathcal{T}_{e\text{-}n} = \tfrac{1}{2} C_e \left\{ \frac{1}{v^2} \frac{d}{dv} \left[\frac{v^4 \chi(v)}{l(v)} \left[\Phi + \frac{kT_n}{mv} \frac{d\Phi}{dv} \right] \right] + \frac{v}{l(v)} \hat{L} f(\mathbf{v}) \right\} . \tag{18}$$

SOLVING THE KINETIC EQUATION

After we substitute (4) and the partial collision integrals (9), (13), and (18), Eq. (1) can be written down as

$$-\frac{eE}{m} \nabla_v [\Phi(V) + f(\mathbf{v})] = 2\pi e^4 m L_e C_e \left[16\pi \Phi(v) f(\mathbf{v}) + \frac{d^2 G}{dv^2} \frac{\partial^2 f}{\partial v^2} + \frac{2}{v^2} \frac{dG}{dv} \frac{\partial f}{\partial v} + \frac{d^2 \Phi}{dv^2} \frac{\partial^2 g}{\partial v^2} + \frac{2}{v^2} \frac{d\Phi}{dv} \frac{\partial g}{\partial v} + \frac{1}{v^3} \left[\frac{dG}{dv} \hat{L} f(\mathbf{v}) + \frac{d\Phi}{dv} \hat{L} g(\mathbf{v}) \right] \right]$$
$$+ 16\pi^2 e^4 m^{-2} L_i \left\{ v^{-2} \frac{d}{dv} \left[\frac{m}{M_i} j(v) \left[\Phi + \frac{kT_i}{mv} \frac{d\Phi}{dv} \right] \right] + \frac{1}{2v^3} j(v) \hat{L} f(\mathbf{v}) \right\}$$
$$+ \frac{1}{2v^2} \frac{d}{dv} \left[\frac{v^4 \chi(v)}{l(v)} \left[\Phi + \frac{kT_n}{mv} \frac{d\Phi}{dv} \right] \right] + \frac{v}{2l(v)} \hat{L} f(\mathbf{v}) \; . \tag{19}$$

Let vector $\mathbf{E}$ be directed along the z axis and $\cos(\mathbf{E}, \mathbf{v}) = \xi = \mathcal{P}_1(\xi)$. The right-hand side of (19) can be written in the form

$$-\frac{eE}{m} \left[\xi \frac{d\Phi}{dv} + \left[\xi \frac{\partial}{\partial v} + \frac{1-\xi^2}{v} \frac{\partial}{\partial \xi} \right] f(\mathbf{v}) \right] . \tag{20}$$

Using the axial symmetry of our problem we expand $f(\mathbf{v})$ in terms of the Legendre polynomials[12]

$$f(\mathbf{v}) = \sum_{l=0}^{\infty} f_l(v) \mathcal{P}_l(\xi) \tag{21}$$

and from (14) and (21) we obtain

$$g(\mathbf{v}) = \sum_{l=0}^{\infty} g_l(v) \mathcal{P}_l(\xi) / (2l-1) \; , \tag{22}$$

where

$$g_l(v) = \int_0^v f_l(u) \left[\frac{1}{2l+3} \frac{u^{l+4}}{v^{l+1}} - \frac{1}{2l-1} \frac{u^{l+2}}{v^{l-1}} \right] du + \int_v^\infty f_l(u) \left[\frac{1}{2l+3} \frac{v^{l+2}}{u^{l-1}} - \frac{1}{2l-1} \frac{v^l}{u^{l-3}} \right] du \; . \tag{23}$$

The right-hand side of (19) is given by

$$-\frac{eE}{m} \left\{ \xi \frac{d\Phi}{dv} + \sum_{l=0}^{\infty} \mathcal{P}_l(\xi) \left[\frac{l}{2l-1} \left[\frac{da_{l-1}}{dv} - \frac{l-1}{v} a_{l-1} \right] + \frac{l+1}{2l+3} \left[\frac{da_{l+1}}{dv} + \frac{l+2}{v} a_{l+1} \right] \right] \right\}$$

by virtue of the known[14] formulas

$$(1-\xi^2) \frac{d\mathcal{P}_l}{2\xi} = \frac{l(l+1)}{2l+1} (\mathcal{P}_{l-1} - \mathcal{P}_{l+1}) \; ,$$

$$\xi \mathcal{P}_l(\xi) = \frac{l}{2l+1} \mathcal{P}_{l-1} + \frac{l+1}{2l+1} \mathcal{P}_{l+1}$$

and after the substitution of (21) into (20). The coefficients $f_l(v)$ in the series (21) and Eq. (4) determine the distribution function of electrons if the electron temperature $T_e = T(E)$ is known. The calculation of T and $f_l(v)$ is our direct task.

By substitution of relations (20)–(23) into (19) and using the orthogonality of the Legendre polynomials

$$\int_{-1}^{1} \mathcal{P}_l(\xi) \mathcal{P}_m(\xi) d\xi = 2\delta_{lm} / (2l+1)$$

we obtain the system of equations for the functions $f_l(v)$. It is solved in the Lorentz approximation, i.e., by neglecting all $f_l(v)$ with $l \geq 2$:

$$f(\mathbf{v}) \approx f_0(v) + \xi f_1(v) \; .$$

Let us set up the dimensionless variables and parameters

$$x = v\sqrt{m/kT} \, , \quad l(v) = l_0 \lambda(x), \quad \chi(v) = \Theta(x) 2m/M_n \; ,$$
$$\mu = \frac{m}{M_i} \left[1 - \frac{T_i}{T} \right], \quad \nu = \frac{m}{M_n} \left[1 - \frac{T_n}{T} \right] \; ,$$
$$s = 4\pi e^3 (L_e N_e / kTE), \quad R = \frac{(kT)^2}{4\pi e^4 L_e N_e l_0} \; ,$$
$$\alpha(x) = \frac{4\pi}{N_e} j(v) = \left[\frac{2}{\pi} \right]^{1/2} \int_0^{\sqrt{M_i T / m T_i}} y^2 \exp(-y^2/2) dy \; . \tag{24}$$

Equation (19) reduces to a system of the two integrodifferential equations for $f_l(v) = a_l(x)$ $(l = 0,1)$:

$$\frac{x}{s}e^{-x^2/2}=\left[\frac{2}{\pi}\right]^{1/2}\left\{2a_1(x)e^{-x^2/2}+x^{-2}\left[(a_1x^{-1})''\int_0^x e^{-y^2/2}y^2dy+x(a_1x^{-1})'\int_0^x\left[1+\frac{y^2}{x^2}\right]e^{-y^2/2}dy\right]\right.$$
$$\left.+\frac{x^3}{5}e^{-x^2/2}\left[\int_0^x a_1(y)\frac{y^5}{x^5}dy+\int_x^\infty a_1(y)dy\right]-\frac{x}{3}e^{-x^2/2}\left[\int_0^x a_1(y)\frac{y^3}{x^3}dy+\int_x^\infty a_1(y)dy\right]\right\}$$
$$-\left[\frac{Rx}{\lambda(x)}+\frac{\alpha(x)}{x^3}\right]a_1(x)\ , \tag{25}$$

$$-\frac{1}{3sx^2}\frac{d(x^2a_1)}{dx}=\frac{1}{x^2}\frac{d}{dx}\{e^{-x^2/2}[\mu\alpha(x)+R\nu\Theta(x)x^4/\lambda(x)]\}$$
$$+\left[\frac{2}{\pi}\right]^{1/2}\left\{2a_0(x)e^{-x^2/2}+x^{-2}\left[a_0''(x)\frac{1}{x}\int_0^x y^2e^{-y^2/2}dy+a_0'(x)\int_0^x\left[1-\frac{y^2}{x^2}\right]e^{-y^2/2}dy\right]\right.$$
$$+\frac{x^2}{3}e^{-x^2/2}\left[\int_0^x a_0(y)\frac{y^4}{x^3}dy+\int_x^\infty a_0(y)y\,dy\right]$$
$$\left.-e^{-x^2/2}\left[\int_0^x a_0(y)\frac{y^2}{x}dy+\int_x^\infty a_0(y)y\,dy\right]\right\}\ . \tag{26}$$

Here and in further notation a' and a'' mean derivatives of the first and the second order, respectively. The first equation (25) does not contain $a_0(x)$ and determines $a_1(x)$ in principle; the second one (26) is the equation for the symmetrical correction to the distribution function.

We will look for solutions in the class of twice differentiable functions with exponential decay at infinity ($0\le x<\infty$). As the result of some transformations we find the first integral for (26)

$$-\frac{x^2}{3s}a_1(x)=e^{-x^2/2}\left[\mu\alpha(x)+R\nu x^4\frac{\Theta(x)}{\lambda(x)}\right]$$
$$+\left[\frac{2}{\pi}\right]^{1/2}\left\{(a_0+a_0'/x)\int_0^x y^2e^{-y^2/2}dy-e^{-x^2/2}\left[\int_0^x a_0(y)y^2\left[\frac{y^2}{3}-1\right]dy+\frac{x^3}{3}\int_x^\infty a_0(y)y\,dy\right]\right\}\ . \tag{27}$$

The following replacement of unknown functions

$$a_0(x)=\mu e^{-x^2/2}\psi(x),\quad a_1(x)=-\frac{x}{s}e^{-x^2/2}\varphi(x) \tag{28}$$

simplifies system (25) and (27):

$$-\frac{d\psi}{dx}x^{-1}\int_0^x y^2e^{-y^2/2}dy+\int_0^x\psi(y)\left[\frac{y^2}{3}-1\right]y^2e^{-y^2/2}dy+\frac{x^3}{3}\int_x^\infty\psi(y)ye^{-y^2/2}dy=\Omega(x)\ , \tag{29}$$

$$\left[\alpha(x)+\frac{Rx^4}{\lambda(x)}\right]\varphi(x)-\left[\frac{2}{\pi}\right]^{1/2}\left[x^2\varphi'e^{-x^2/2}+\left[\varphi''+\frac{1-x^2}{x}\varphi'\right]\int_0^x y^2e^{-y^2/2}dy\right.$$
$$\left.+2\int_0^x[\varphi(y)-\varphi(x)]y^2e^{-y^2/2}dy+\int_0^x\left[\frac{y^5}{5}+2\frac{y^3}{3}\right]e^{-y^2/2}\varphi'(y)dy+\left[\frac{x^5}{5}-\frac{x^3}{3}\right]\int_x^\infty\varphi'(y)e^{-y^2/2}dy\right]=x^3\ , \tag{30}$$

where

$$\Omega(x)=\left[\frac{\pi}{2}\right]^{1/2}\left[\alpha(x)+R\frac{\nu\Theta(x)}{\mu\lambda(x)}x^4-\frac{x^3}{3\mu s^2}\varphi(x)\right]\ . \tag{31}$$

Equation (29) for the symmetrical correction can be solved analytically. After integration by parts it turns into

$$-\frac{\psi'(x)}{x}\int_0^x y^2e^{-y^2/2}dy+\tfrac{1}{3}\int_0^x\psi'(y)y^3e^{-y^2/2}dy$$
$$+\frac{x^3}{3}\int_x^\infty\psi'(y)e^{-y^2/2}dy=\Omega(x)\ .$$

We differentiate this relation and use the temporary notation $d\psi/dx=x\omega$. Thus we obtain

$$-\omega'(x)\int_0^x y^2 e^{-y^2/2}dy + x^2\int_x^\infty \omega'(y)e^{-y^2/2}dy = \Omega'(x) .$$

Let us multiply this equation by $e^{-x^2/2}$ and integrate the product from 0 to t,

$$\int_t^\infty \omega'(x)e^{-x^2/2}dx = \int_0^t \Omega'(x)e^{-x^2/2}dx \Big/ \int_0^t y^2 e^{-y^2/2}dy . \quad (32)$$

If we integrate from t to ∞, the result is given by

$$\int_t^\infty \omega'(x)e^{-x^2/2}dx = -\int_t^\infty \Omega'(x)e^{-x^2/2}dx \Big/ \int_0^t y^2 e^{-y^2/2}dy .$$

The left-hand sides of this equation and (32) are the same, therefore

$$\int_0^\infty \Omega'(x)e^{-x^2/2}dx = 0 . \quad (33)$$

We can find $\omega'(x)$ from (32) and subsequently $\psi(x)$:

$$\psi(x) = \mathcal{A}_1 + \mathcal{A}_2 x^2 + \int_0^x \frac{y^2 - x^2}{2} e^{y^2/2} \frac{d}{dy}\left[\frac{\int_0^y \Omega'(t)e^{-t^2/2}dt}{\int_0^y t^2 e^{-t^2/2}dt}\right] dy , \quad (34)$$

where $\mathcal{A}_1$ and $\mathcal{A}_2$ are arbitrary constants for the time being.

It is not difficult to show that Eq. (33) with definition (31) is the energy balance condition for electrons. It is just the equation that determines the electron temperature T and so deserves to be written once more in the form

$$\int_0^\infty e^{-x^2/2}\left[\mu x + R\nu \frac{\Theta(x)}{\lambda(x)} x^5 - \frac{x^4 \varphi(x)}{3s^2}\right] dx = 0 . \quad (35)$$

Here the equality $\alpha(0)=0$ [see (24)] is used, but $\alpha(x) \approx 1$ almost for all important values of the electron velocity. We want to pay attention to the $\alpha(x)$ behavior, because usually the inequality $\alpha(x) \not\equiv 1$ is ignored for collision integral calculations. Taking $\alpha(x) \equiv 1$ would lead to an incorrect relationship instead of (35).

The last item in (35) is directly proportional to the power capacity, which is transferred to the electron gas by the electric field. Indeed, the velocity of an electron changes between collisions by the low

$$\mathbf{v}(t+dt) = \mathbf{v}(t) - \frac{e\mathbf{E}}{m} dt ,$$

therefore

$$\frac{d\varepsilon}{dt} = \frac{d}{dt}\left[\frac{m\mathbf{v}^2}{2}\right] = -e\mathbf{v}\cdot\mathbf{E} .$$

The specific absorption capacity equations

$$P(E) = -\int f^e(\mathbf{v}) e\mathbf{v}\cdot\mathbf{E}\, d^3p = \frac{(kT/2\pi)^{3/2}}{\sqrt{m}} \frac{E^2}{3e^2 L_e} W(R) , \quad (36)$$

where

$$W(R) = \int_0^\infty x^4 \varphi(x) e^{-x^2/2} dx . \quad (37)$$

It is very easy to show that the first two terms in (35) describe the energy being transferred from electrons to ions and neutral particles of plasma, respectively. We will return to relationship (35) for the electron temperature and to formulas (34) and (28) for the symmetrical correction to the distribution function.

We must know the function $\varphi(x)$ and hence solve Eq. (30), which can be rewritten in the form

$$\left[1 + \frac{Rx^4}{\lambda(x)}\right]\varphi(x) - \left[\frac{2}{\pi}\right]^{1/2}\left[x^2\varphi' e^{-x^2/2} + \left[\varphi'' + \frac{1-x^2}{x}\varphi'\right]\int_0^x y^2 e^{-y^2/2}dy + 2\int_0^x [\varphi(y)-\varphi(x)] y^2 e^{-y^2/2}dy + \int_0^x \left[\frac{y^5}{5} + 2\frac{y^3}{3}\right]\varphi'(y)e^{-y^2/2}dy + \left[\frac{x^5}{5} - \frac{x^3}{3}\right]\int_x^\infty \varphi'(y)e^{-y^2/2}dy\right] = x^3 , \quad (38)$$

in order to find the flow term $\xi f_1(v)$ of the electron distribution function (3). The factorization of all parameters in this equation, central for our problem, is an extremely favorable fact. If the free-flight path with respect to *e-n* collisions (or the *e-n* cross section) can be regarded as a power function in the most essential domain [see (24)], Eq. (38) has only one parameter which determines the function $\varphi(x)$. In the important case, when

$$l(v) = l_0 = \text{const} , \quad (39)$$

this parameter is R and $\lambda(x)=1$. If the free-flight time $\tau_0 = l(v)/v$ is constant, the first term in (38) is given by $(1+Qx^3)\varphi(x)$ with

$$Q = \sqrt{mk^3T^3}/4\pi e^4 LN\tau_0 .$$

We solve the problem under condition (39), but our experience in modeling of solutions of Eq. (39) (see below) suggests some principles of the approximation in more general cases.

The solution of (38) must be a continuous function $\varphi(x)$ with acceptable integration properties, which permit us to determine the physical characteristics of plasma. The boundary conditions

$$\varphi(x) \underset{x\to 0}{\sim} \text{const}\times x^3, \quad \varphi(x) \underset{x\to\infty}{\sim} \frac{1}{Rx} \quad (R \neq 0) \quad (40)$$

can be found from the analysis of (38) in vicinities of $x=0$ and $x=\infty$. If Eq. (38) has a solution in the above outlined class of functions, then conditions (40) make this solution unique. We did not succeed in proving the solubility of (38) and (40), in general; nevertheless we are able to build the family of these solutions for each chosen value of R constructively with a numerical method.

First of all, let us study the asymptotic behavior of $\varphi(x)$ more accurately. If $x>5$, (38) degenerates into the second-order differential equation

$$\hat{S}\varphi \equiv \varphi'' + \frac{1-x^2}{x}\varphi' - (Rx^4+3)\varphi = -x^3 - B \ , \quad (41)$$

where

$$B = \left[\frac{2}{\pi}\right]^{1/2} \int_0^\infty y^4 \left[\frac{y^2}{5} - \frac{1}{3}\right] \varphi(y) e^{-y^2/2} dy$$

is a functional, which depends on the solution $\varphi(x)$ on its entire domain, but in fact is independent of its asymptotic behavior.

The general solution of (41) can be written in the form

$$\varphi = B_0\varphi_0(x) + \varphi_1(x) + B\varphi_2(x) \ ,$$

where

$$\hat{S}\varphi_0 = 0, \quad \hat{S}\varphi_1 = -x^3, \quad \hat{S}\varphi_2 = -1 \ .$$

We use the linearity of (41). B_0 is an arbitrary constant, and the other constant equals zero because of the diverging asymptotic behavior of the other homogeneous equation solution. The particular solutions must not grow faster than a power function when $x \to \infty$. If $R=0$, it is very easy to show that

$$\varphi_0(x) \sim x^3, \quad \varphi_1(x) \sim \frac{x^3}{6} + \frac{3x}{8}, \quad \varphi_2(x) \sim \tfrac{1}{3} \ .$$

If $R>1$, the WBK method[15] shows that $\varphi_0(x)$ is an extremely quickly decreasing function for large x. So we can consider $\varphi_0(x) \equiv 0$ in the domain that is of interest to us now. The asymptotic behavior of $\varphi_1(x)$ and $\varphi_2(x)$ can be found directly: $\varphi_1(x) \sim 1/Rx$, $\varphi_2(x) \sim 1/Rx^4$. When we solve (38) numerically, the $\varphi(x)$ asymptotic behavior is given by

$$\tilde{\varphi}(x) = \frac{x^3}{D(R)+Rx^4} + \frac{B}{3+Rx^4}, \quad x>5 \ , \quad (42)$$

where $D(R) = 6(1+490R)/(1+1130R)$ was obtained as the result of the preliminary calculation analysis. One can take $D=6$ simply, and it causes very slight changes in the final results because of the calculation procedure stability.

The spline-collocation method[16] is applied for the solution of Eq. (38). We build the spline approximation of $\varphi(x)$ in the domain $x \subset [0,X]$ with the help of defect 1 cubic splines on a homogeneous grid. The step of the grid h depends on value R and determines the calculation precision. It was chosen equal to 0.1 approximately and reduces to the value about 0.025 when R is large. The values to be found are φ_n and φ_n' in knots of the grid, where we require (38) to hold. Boundary conditions at the ends of the interval are given by

$$\varphi(0) = \varphi'(0) = 0, \quad \varphi(X) = \tilde{\varphi}(X), \quad \varphi'(X) = \tilde{\varphi}'(X) \ . \quad (43)$$

The value B can be expressed in the terms φ_n, φ_n' in the obvious way. We obtain the algebraic system of linear equations: one-half of them are the collocation conditions [that is to say, Eq. (38) in the grid knots], and the other half are the equations of continuity for the second derivative of the solution at the knots

$$3(\varphi_{n-1} - \varphi_{n+1}) + h(\varphi_{n-1}' + 4\varphi_n' + \varphi_{n+1}') = 0 \ . \quad (44)$$

The second half of the system matrix has a simple structure, and it is not too difficult to halve the number of equations, but this possibility is not realized in our calculation.

This method of solving Eq. (38) turns out to be steady and effective, though the solutions of the corresponding homogeneous equation are functions $\varphi_0 \to e^{\pm|\text{const}|x^3}$, $(x>5)$ of very fast change. The attempt to solve the problem as Cauchy's (with the boundary condition at $x=0$ or $x=X$) leads to a divergence process in a few steps.

After the calculation of φ_n, φ_n' we can find the function $\psi(x)$ with the help of numerical integration by formulas (34) and (31). The relationship

$$\psi(x) = \mathcal{A}_1 + \mathcal{A}_2 x^2 + \left[\frac{\pi}{2}\right]^{1/2} \int_h^x y \frac{1+(y^2-x^2)/2}{\int_0^y t^2 e^{-t^2/2} dt} \left[4R\frac{\nu\Theta(y)}{\mu\lambda(y)}(2+y^2) - \frac{1}{3\mu s^2}\int_y^\infty e^{(y^2-t^2)/2}\frac{d}{dt}(t^3\varphi)dt\right] dy \quad (45)$$

is more convenient than (34) and (31). We have here, besides R, other parameters of the problem. The arbitrary constants in (45) indicate indefiniteness in choosing the correction to the zeroth-order distribution-function approximation. In order to maintain the normalization $C_e = \mathcal{N}_e(2\pi mkT)^{-3/2}$ we demand

$$\int f(\mathbf{v}) d^3v = 0$$

and immediately obtain the condition

$$\int_0^\infty e^{-x^2/2}\psi(x)x^2 dx = 0 \ . \quad (46)$$

The calculation of arbitrary constants can be made in one of the following ways: (i) $\mathcal{A}_1 = 0$, $\mathcal{A}_2$ is the root of Eq. (46), (ii) using (46) and minimizing the maximum deviation from zero of the correction, or (iii) a least-squares treatment with the minimization of the integral

$$\int_0^\infty x^2 e^{-x^2}\psi^2(x) dx \ .$$

If λ, θ are the constants, we can rewrite (45) in the form

$$\psi(x)=\mathcal{A}_1+\mathcal{A}_2x^2+\left[\frac{\pi}{2}\right]^{1/2}4R\frac{\nu\theta}{\mu\lambda}I_1(x)-\left[\frac{\pi}{2}\right]^{1/2}I_2(x)/3\mu s^2\ ,$$

where the function $I_2(x)$ only depends on R and both of them can easily be calculated. The assumptions $\lambda=1$ and $\theta=\text{const}$ are not very important for the possibility of integration in (45), the more so as the exact calculation of the symmetrical correction to the distribution function is not usually of great interest.

If R is small, the value of $\psi(x)$ is not large either, and for $R>1$ we need only consider the asymptotic behavior of $\psi(x)$. It can be obtained with the help of the analysis of the integrals in (45) in the form

$$\psi(x)\to-\frac{R\nu\theta}{6\mu\lambda}x^6+\frac{x^4}{12\mu Rs^2}\ . \tag{47}$$

This relationship enables us to evaluate the correctness of the approximation that our method is based on.

According to the outlined scheme we obtain the solution of (38), the values of $I_1(x)$, $I_2(x)$, and the integral expression $W(R)$, given by (37), for the set of parameter $R\subset[10^{-6},10^4]$ values. The function $W(R)$, which plays the central role, is presented in Table I. We also show the values of the calculation errors. The point is that the solution of (38) is given by the set of function $\varphi(x)$ values in the homogeneous grid $x=nh$ knots and by the derivatives $\varphi'(x)$ of the same function. All these values are obtained as solutions of the linear algebraic equation system independently. Nevertheless, obviously, the relation

$$\varphi'(nh+h/2)\approx[\varphi(nh+h)-\varphi(nh)]/h$$

should hold. The values of the function

$$\delta=\left[\sum_{n=2}^{K}\left[\frac{\varphi_n'+\varphi_{n+1}'}{2}+\frac{\varphi_n-\varphi_{n+1}}{h}\right]^2\Bigg/\sum_{n=1}^{K}(\varphi_n')^2\right]^{1/2}$$

are also presented in Table I for all R. Here, $K=X/h$ is the complete number of knots on the interval $[0,X]$.

TEMPERATURE AND OTHER ELECTRON GAS CHARACTERISTICS

As a result of solving Eq. (38), all the terms of (35) are determined, and electron temperature can be calculated. If $\theta=\text{const}$, $\lambda=1$, and $T_i=T_n$, relationship (35), with the help of (37), becomes

$$1+8R\theta\frac{M_i}{M_n}-\frac{1}{3\mu s^2}W(R)=0\ , \tag{48}$$

where only μ and $R(T)$ depend on T. In the case $\theta\neq\text{const}$, we have in (48) the mean value $\bar{\theta}$, which can be found from (35) without any difficulties. Now we rewrite (48) in the form

TABLE I. Values $W(R)$, $RW(R)$, $\delta(R)$ (top to bottom, respectively) as functions of $R=a\times10^b$.

a \ b	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4
1	27.929	27.944	27.938	27.876	27.287	22.71	9.4612	1.6477	1.9213×10^{-1}	1.9865×10^{-2}	1.9981×10^{-3}	1.9997×10^{-4}
					0.027 29	0.2271	0.946 12	1.6477	1.9213	1.9865	1.9981	1.9997
	0.0088	0.0088	0.0089	0.0107	0.0268	0.005 64	0.007 44	0.0187	0.0554	0.179	0.278	0.669
1.4					27.035	21.206	7.6604	1.2221	1.3845×10^{-1}			
					0.037 85	0.296 88	1.0725	1.7109	1.9396			
					0.0261	0.004 94	0.008 42	0.0217	0.0656			
2	27.928	27.943	27.93	27.81	26.667	19.338	6.0032	0.883 81	9.7633×10^{-2}	9.9619×10^{-3}		
					0.053 33	0.386 76	1.2006	1.7676	1.9527	1.9924		
	0.0088	0.0088	0.009	0.0135	0.0225	0.048	0.009 64	0.0255	0.0786	0.245		
3					26.081	16.937	4.4516	0.606 84	6.5506×10^{-2}			
					0.078 24	0.508 11	1.3355	1.8205	1.9652			
					0.0165	0.0051	0.0113	0.0308	0.0968			
4					25.526	15.124	3.5563	0.462 88	4.9304×10^{-2}			
					0.1021	0.604 96	1.4225	1.8515	1.9722			
					0.0125	0.005 49	0.0127	0.0353	0.112			
5	27.944	27.941	27.91	27.611	24.998	13.696	2.9695	0.374 45	3.9532×10^{-2}	3.9929×10^{-3}		
					0.124 99	0.6848	1.4848	1.8723	1.9766	1.9965		
	0.0088	0.0088	0.0096	0.0216	0.01	0.005 87	0.0139	0.0393	0.126	0.32		
7.5					23.788	11.154	2.1141	0.2538	2.6441×10^{-2}			
					0.178 41	0.836 55	1.5856	1.9035	1.9831			
					0.0069	0.006 72	0.0165	0.048	0.155			

$$R_E=\left[1+8\frac{M_i}{M_n}\theta R\right](1-\sqrt{R_i/R}\,)/4RW(R) \quad (49)$$

using the new parameters independent of T,

$$R_i=R(T_i),\quad R_E=\frac{M_i}{12m}\frac{(eEl_0)^2}{4\pi e^4L_eN_el_0}\ .$$

The function $R_E(R)$ is given in Table I, but the inverse relation $R(R_E)$ and hence the relation $T(E)$ are not single valued at all points of their domains owing to nonmonotonicity of $R_E(R)$.

We derive from (49) the relationship

$$\frac{R_E'}{R_E}=-\frac{1}{R}\left[\frac{(RW)'}{W}-\frac{8M_i\theta R/M_n}{1+8M_i\theta R/M_n}-\frac{1}{2(\sqrt{R/R_i}-1)}\right],$$

which shows that $R_E'<0$ if $R_i<R_i^0$, where

$$R_i^0=R\left\{1+\frac{1}{2}\left[\frac{(RW)'}{W}-8\frac{M_i}{M_n}\theta R\Big/\left[1+8\frac{M_n}{M_i}\theta R\right]\right]\right\}^{-2}. \quad (50)$$

Table II, which is constructed using condition

$$\frac{M_i}{M_n}\theta=1 \quad (51)$$

with the help of the function $W(R)$ values, makes it possible to calculate the curve $R_i^0(R)$ (see Fig. 1). Figure 2 presents a few curves $R_E(R)$, built according to formula (49) under condition (51) for some different values of R. The positions of points R_1 and R_2 are determined by corresponding points on Fig. 1. The electron temperature dependence on field intensity $T(E)$ and the function $R(R_E)$, which is related to $T(E)$ directly, cannot be found as the result of the symmetrical transposition of Fig. 2, because of the above-mentioned single-value problem. The $T(E)$ or $R(R_E)$ behavior is described by a hys-

TABLE II. The functions $R^{3/4}W(R)=\text{const}\times\sigma$, $U(R)=(1/W)[d(RW)/dR]$, $V(R)=8R/(1+8R)$, and the function $R_i^0=R[1+1/2(U-V)]^{-2}$ when $U\geq V$. The equality $U=V$ holds at the point $R\approx 9.03\times10^{-2}$.

R	$R^{3/4}W(R)$	$U(R)$	$V(R)$	$R_i^0(R)$
8.75	1.1343	0.031 93	0.9859	
6.25	1.2417	0.039 73	0.9804	
4.5	1.2935	0.049 63	0.973	
3.5	1.3687	0.057 96	0.9655	
2.5	1.4818	0.070 98	0.9524	
1.7	1.5676	0.089 72	0.9315	
1.2	1.6452	0.110 18	0.9057	
0.875	1.7017	0.132 11	0.875	
0.625	1.7867	0.1586	0.833	
0.45	1.7927	0.1909	0.783	
0.35	1.822	0.2174	0.737	
0.25	1.8482	0.2579	0.667	
0.17	1.8087	0.3127	0.576	
0.12	1.7454	0.3689	0.49	
0.0875	1.6583	0.4252	0.4118	5.960×10^{-5}
0.0625	1.5531	0.4885	0.3333	3.507×10^{-3}
0.045	1.4079	0.5541	0.2647	6.048×10^{-3}
0.035	1.2972	0.6042	0.2188	6.632×10^{-3}
0.025	1.1403	0.6691	0.1667	6.280×10^{-3}
0.017	0.954 41	0.739	0.1197	5.204×10^{-3}
0.012	0.796 12	0.7945	0.0876	4.117×10^{-3}
0.008 75	0.665 14	0.8377	0.0654	3.224×10^{-3}
0.006 25	0.542 22	0.876	0.0476	2.431×10^{-3}
0.0045	0.438 91	0.9061	0.0347	1.817×10^{-3}
0.0035	0.3713	0.9247	0.0272	1.444×10^{-3}
0.0025	0.294 87	0.9445	0.0196	1.053×10^{-3}
0.0017	0.2248	0.9609	0.0134	7.284×10^{-4}
0.0012	0.175 12	0.972	0.0095	5.197×10^{-4}
0.000 75	0.1244	0.9825	0.006	3.280×10^{-4}
0.000 35	0.070 91	0.9916	0.0028	1.544×10^{-4}
0.000 15	0.037 74	0.9964	0.0012	6.645×10^{-5}

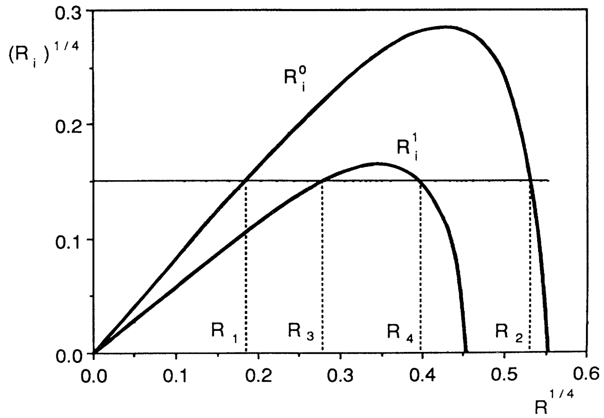

FIG. 1. Values of $(R_i^0)^{1/4}$ and $(R_i^1)^{1/4}$ [see (50) and (54)] vs the variable $R^{1/4}$. Coordinates of maxima: $R_i^0=6.64\times10^{-3}$ when $R=3.42\times10^{-2}$; $R_i^1=7.41\times10^{-4}$ when $R=1.46\times10^{-2}$. $R_i^0(0.1)\approx R_i^1(\frac{1}{24})=0$. If $R_i=5\times10^{-4}$: $R_1\approx1.2\times10^{-3}$, $R_2\approx0.08$, $R_3\approx0.0059$, $R_4\approx0.0253$.

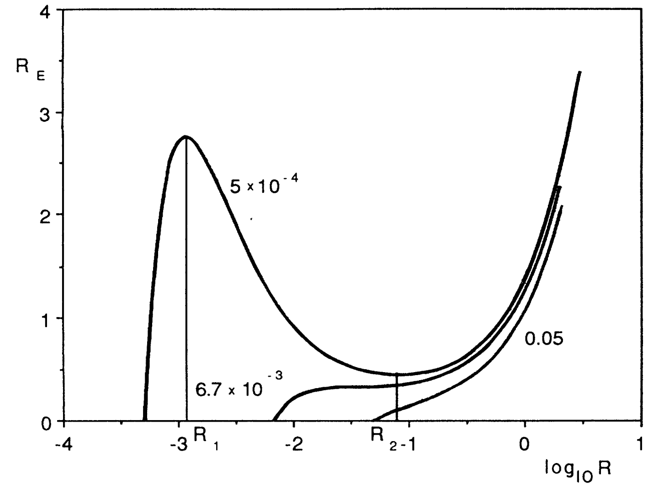

FIG. 2. Family of curves $R_E(\log_{10}R)$ for various values R_i. The curves tend towards the asymptotic value $R_E=R$.

teresis curve,[7] which can be built with the help of Fig. 2. Increasing intensity causes the abrupt change of the electron temperature from point A to point B (Fig. 3), while decreasing intensity causes the downturn between C and D. Our calculation is more accurate and consistent than the calculations in Ref. 3 and others; the entire dependence $T(E)$ is obtained using a single method.

The increase of θ means the approximate treatment of weakly inelastic e-n collisions, which are accompanied by the excitation of low-energy levels in molecules such as rotation and oscillation. The analysis of (48) shows that the growth of θ raises the curve $R_i^0(R)$. It leads to an expansion of the $T(E)$ inversion interval between points R_1 and R_2 of Fig. 1.

Now we consider the dependence of the specific absorption capacity on the intensity. By virtue of (36) it is almost independent of the electron concentration (see also Ref. 6). It is very convenient here to write $P(E)$ in the form

$$P(E)=\text{const}(E,T)\times R_E R^{3/4} W(R)\ . \tag{52}$$

Temporarily ignoring the hysteresis phenomena given in Fig. 3, we obtain from (49) and (52)

$$\text{sgn}\left[\frac{dP}{dE}\right]=\text{sgn}\left[\frac{dR_E}{dR}\right]\text{sgn}\left[\frac{d}{dR}\left\{R^{-1/4}\left[1+8R\theta\frac{M_i}{M_n}\right]\left[1-\left[\frac{R_i}{R}\right]^{1/2}\right]\right\}\right]. \tag{53}$$

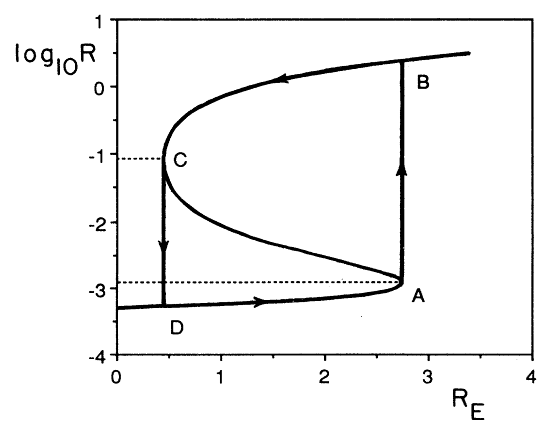

FIG. 3. Values of $\log_{10}R$ as a function of R_E when $R_i=5\times10^{-4}$. $R\rightarrow R_E$ for large R.

The second factor in (53) is negative at the interval pointed by the roots of the equation

$$R_i=R\left[1-24R\theta\frac{M_i}{M_n}\right]^2\left[3-8R\theta\frac{M_i}{M_n}\right]^{-2}\ . \tag{54}$$

Basing on (54), we build the dependence $R_i^1(R)$ at Fig. 1 and show the location of these roots R_3,R_4 for some particular $R_i=R_{i1}$. It is very important that the interval $[R_3,R_4]$ (see Figs. 2 and 3) is located inside $[R_1,R_2]$. Therefore, the electron-gas state under heating changes from point R_1 to point $R_B>R_2$ (Fig. 3) and from R_2 to $R_D<R_1$ under cooling. Changing the second term sign in (53) does not affect the sign of dR/dE. So $P(E)$ depends monotonically on E with the abrupt change at $E=E(R_1)$. The locations of the points $R_1=R_A$ and R_B as functions of the variable R are presented in Fig. 4. The graphs of $R_E(R)$ and the value

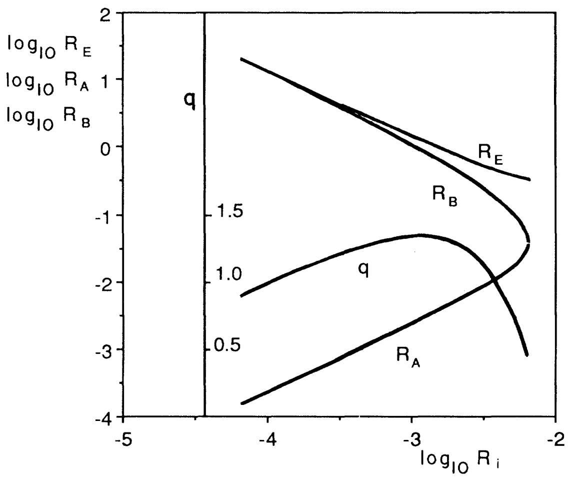

FIG. 4. The locations of the points R_A and R_B, and $q(R_i)$ as functions of variable R_i.

$$q(R_i)=R_B^{3/4}W(R_B)-R_A^{3/4}W(R_A) ,$$

which characterizes the capacity step of discontinuity, also are shown in Fig. 4.

According to Fig. 1, $R_A \approx 9R/4$, hence $T=3T/2$ in conformity with Ref. 7. If E decreases, $P(E)$ decreases monotonically too, but the downturn takes place in the point $R_C=R_2$ (see Figs. 1 and 3), therefore the dependence $P(E)$ shown in Fig. 5 has the hysteresis with the same corner points A,B,C,D as the $T(E)$ curve at Fig. 3. The downturn is not calculated, because of its sign definiteness in virtue of the monotonic growth of the function $R^{3/4}RW(R)$ (Fig. 6) at the interval $0<R<0.25$ and the location of $R_C=R_2<0.1$ inside this interval according to Fig. 1.

We complete this section by writing down the formula of electron conductivity of plasma

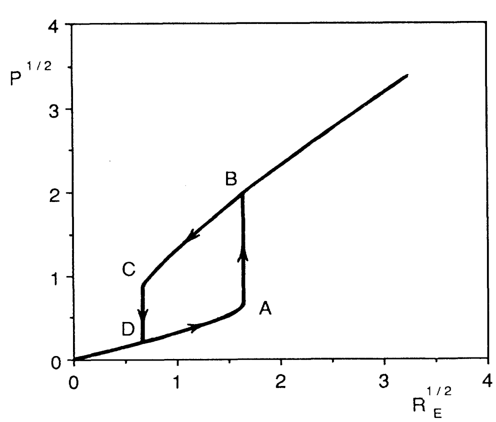

FIG. 5. The absorption capacity vs $\sqrt{R_E}$ (i.e., the electrical field intensity) when $R_i=5\times 10^{-4}$.

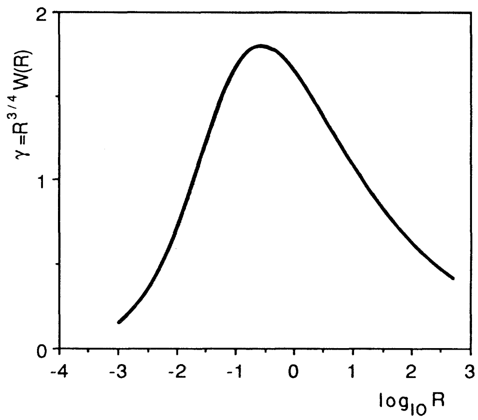

FIG. 6. The electron conductivity of the plasma as a function of R (i.e., the electron temperature). The maximum $\gamma=1.81$ is located at the point $R\approx 0.25$.

$$\sigma=\frac{P(E)}{E^2}=\frac{e}{3(m^2L_e)^{1/4}}\left[\frac{N_e l_0}{\pi}\right]^{3/4}R^{3/4}W(R) . \tag{55}$$

So Fig. 6 and the first column of Table I represent the value of conductivity up to a constant factor. The conductivity takes a maximum value at the point $R\approx 0.25$, which corresponds to different values of E, depending on the temperature of plasma heavy components R_i.

Let us analyze (55). According to Fig. 6 the conductivity is almost independent of electron temperature T and directly proportional to $(N_e/N)^{3/4}$, when $0.05<R<2$. If $R>2$, by virtue of Table I, $1.76<RW(R)<2$, and the conductivity is proportional to $N_e/N\sqrt{T}$. For $R<0.01$, Table I shows that $W(R)\approx$const, therefore the conductivity is almost independent of N_e and N and increases as $T^{3/2}$, in accordance with the well-known Spitzer formula.

APPROXIMATION OF THE FUNCTION $\varphi(x)$

We obtained many curves describing the function $\varphi(x)$ that takes up the central place in our calculation. Let us remark first of all that the usual approximation neglecting e-e collisions explicitly (their role is supposed to cause the electron distribution function to be Maxwell's) annuls the item in parentheses. The solution of (38) in this case can be obtained immediately,

$$\varphi^{\rm I}(x)=x^3/(1+Rx^4) \tag{56}$$

or $\varphi^{\rm I}(x)=x^3[1+Rx^4/\lambda(x)]^{-1}$ if $\lambda\equiv 1$. The calculation shows that each solution of Eq. (38), similarly to function (56), has only one maximum, and almost immediately after the maximum value, it then tends to its asymptotic behavior (42) and does not change its sign. The latter fact is obvious as a physical one: $\varphi(x)$ describes the electron flow, the direction of which depends on the field-intensity vector direction only and does not depend on the electron energy.

We consider the function $\varphi^{\rm I}(x)$ and

TABLE III. The functions $W(R)$, $W_{\rm I}(R)$, $W_{\rm II}(R)$, $W_{\rm IV}(R)$ (top to bottom), which were obtained with the help of Eq. (37) and φ, $\varphi^{\rm I}$, $\varphi^{\rm II}$, $\varphi^{\rm IV}$, respectively. $R=a\times10^{b}$.

a \ b	−6	−5	−4	−3	−2	−1	0	1 (×0.1)	2 (×0.01)	3 (×0.001)
1	27.944	27.938	27.876	27.287	22.71	9.4612	1.6477	1.9213	1.9865	1.9981
	47.996	47.962	47.622	44.672	30.519	10.083	1.6637	1.926	1.9875	1.9982
	27.998	27.987	27.871	26.8	20.534	8.1964	1.5467	1.8914	1.9808	
	27.979	27.972	27.901	27.222	22.25	9.2186	1.6418	1.9319	1.9873	1.9995
2	27.943	27.93	27.81	26.667	19.338	6.0032	0.88381	0.97633	0.99619	
	47.992	47.923	47.256	42.05	23.846	6.2142	0.88933	0.978	0.99642	0.99943
				25.781	16.96	5.3095	0.84628	0.96694		
	27.978	27.964	27.823	26.519	18.813	5.894	0.88313	0.97726	0.99635	0.99943
5	27.941	27.91	27.611	24.998	13.696	2.9695	0.37445	0.39532	0.39929	
	47.981	47.81	46.222	36.404	15.33	3.0172	0.3758	0.3957	0.3993	0.39989
	27.993	27.935	27.376	23.359	11.783	2.7198	0.36526	0.39337		
	27.976	27.94	27.593	24.679	13.287	2.944	0.37471	0.39558	0.39929	0.39989

$$\varphi^{\rm II}(x)=x^3/(Rx^4+\tfrac{12}{7}),\quad \varphi^{\rm III}(x)=x^3/[D(R)+Rx^4]\,,$$
$$\varphi^{\rm VI}(x)=x^3[Rx^4+(6x^4+304)/(x^4+304)]^{-1} \qquad (57)$$

as the models for the solution of (38); $\varphi^{\rm III}(x)$ is obtained from (42). In Tables III and IV, we present the values of $W(R)$, the locations and the values of the maxima, and the values of φ at the point $x=10$, where it must reach the asymptotic region if $R>0.01$. The tables were calculated with the help of the numerical solution and the functions (56) and (57), which simulate it.

Table III shows that $W(R)$, calculated with the help of $\varphi^{\rm I}(x)$, has rather large errors, to 70% if R is small. The function $\varphi^{\rm I}(x)$ itself can be six times bigger than the "correct" solution $\varphi(x)$ ($R<0.001, x>6$). So the explicit consideration of e-e collisions is important for describing the fast electron behavior. The model function $\varphi^{\rm IV}(x)$ simulates $\varphi(x)$ quite well according to Tables III and IV and may be used for approximations. We hope the function

$$x^3\left[\frac{Rx^4}{\lambda(x)}+\frac{6x^4+304}{x^4+304}\right]^{-1},$$

obtained in the same way as $\varphi^{\rm IV}(x)$ for $l(v)\neq$const, will be the satisfactory model of the solution to Eq. (38), but it needs an examination.

CONCLUSION

In conclusion, we consider the conditions of our method applicability and hence the reliability of reported

TABLE IV. The coordinates of maxima x_κ, $\varphi^\kappa(x_\kappa)$, and the values $\varphi^\kappa(10)$ of the function $\varphi(x)\equiv\varphi^0(x)$, which was obtained as the solution of Eq. (38), and the modeling functions $\varphi^{\rm I}(x)$, $\varphi^{\rm II}(x)$, $\varphi^{\rm III}(x)$, $\varphi^{\rm IV}(x)$ (up to down consecutively when k grows) for various values of R.

R	10^{-7}	10^{-6}	10^{-5}	10^{-4}	10^{-3}	10^{-2}	10^{-1}	10^0	10^1	10^2	10^3	10^4	10^5
x_κ					9.709	5.617	2.795	1.405	0.7603	0.4266	0.2371	0.135	0.08036
					7.4	4.162	2.34	1.316	0.7401	0.4162	0.234	0.1316	0.07401
					8.468	4.76	2.678	1.506	0.847	0.476	0.268	0.1506	0.08468
					10.59	5.421	2.981	1.672	0.94	0.5286	0.2972	0.1671	0.09399
					11.48	5.835	2.29	1.31	0.74	0.418	0.234	0.132	0.0738
$\varphi^\kappa(x_\kappa)$					75.42	14.8	2.936	0.5275	9.471×10^{-2}	1.736×10^{-2}	3.093×10^{-3}	5.599×10^{-4}	1.007×10^{-4}
					101.3	18.02	3.205	0.5699	10.13×10^{-2}	1.802×10^{-2}	3.205×10^{-3}	5.699×10^{-4}	1.013×10^{-4}
					88.56	15.75	2.801	0.498	10.89×10^{-2}	1.570×10^{-2}	2.800×10^{-3}	4.980×10^{-4}	8.856×10^{-5}
					70.8	13.84	2.516	0.4486	7.979×10^{-2}	1.419×10^{-2}	2.523×10^{-3}	4.487×10^{-4}	7.979×10^{-5}
					64.98	12.00	2.883	0.563	10.12×10^{-2}	1.802×10^{-2}	3.205×10^{-3}	5.699×10^{-4}	1.013×10^{-4}
$\varphi^\kappa(10)$	175	175	172	156	73.5	10.00	1.01	0.101	1.01×10^{-2}	1.01×10^{-3}	1.01×10^{-4}	1.01×10^{-5}	
	999	990	909	500	90.91	9.901	0.999	0.09999	1.00×10^{-2}	1.00×10^{-3}	1.00×10^{-4}	1.00×10^{-5}	
	583	580	551	368	85.37	9.832	0.9983	0.09998	1.00×10^{-2}	1.00×10^{-3}	1.00×10^{-4}	1.00×10^{-5}	
	167	166	165	150	70.44	9.72	0.9974	0.09997	1.00×10^{-2}	1.00×10^{-3}	1.00×10^{-4}	1.00×10^{-5}	
	171	171	168	146	63.08	9.447	0.9942	0.09994	1.00×10^{-2}	1.00×10^{-3}	1.00×10^{-4}	1.00×10^{-5}	

results. The method is based on the condition of plasma neutrality, the strong mutual correlations in charged-particle motion, and the absence of plasma-boundary influence. These are the usual conditions for the plasma problems:[7,12]

$$r_D/r>1,\quad l/r_D>1,\quad r_D \ll \Lambda ,$$

where $r_D=[4\pi N_e(e^2/k)(1/T_e+1/T_i)]^{-1/2}$ is the Debye radius, $r\approx N_e^{-1/3}$ is the mean distance between plasma charged particles, and Λ is the typical linear size of the plasma. These conditions are almost independent of the electron temperature T_e and point out the upper and the lower restrictions for the charged-particle concentration N_e.

The functions $f_0(v)$ and $f_1(v)$ should be small in comparison with the Maxwellian

$$\Phi(v)=\exp(-mv^2/2kT)$$

at least into the interval of electron velocities, where the main contribution in the collision integral is formed. It means, according to (3), (21), (28),

$$\mu\psi(x)\ll 1,\quad \frac{x}{s}\varphi(x)\ll 1 \quad \text{for } x\subset[0,5] . \tag{58}$$

The second inequality (58) with the help of replacing $\varphi(x)$ by the approximation $\varphi^{\mathrm{IV}}(x)$ gives

$$s \gg (R+0.01)^{-1} . \tag{59}$$

If we consider small R, when $R\sim R_i$, (59) leads to the inequality

$$E<0.1e/r_D^2 ,$$

which is close to the usual condition of a weak field for the electrons whose distribution function is almost undisturbed. The upper limits of the field intensity and the electron temperature are much more interesting. To obtain these limits we continue to study (58). If x is small, $\psi(x)$ is rather large, but the factor μ is very small, therefore the evaluation of $\psi(x)$ for large x with the help of its asymptotic behavior (47) is sufficient. The symmetrical correction $\psi(x)$ rapidly increases in this domain because the distribution function has to approach Druyvesteyn's function.[12,17] We require that (59) and

$$Rm\theta x_m^6/6M_n<1,\quad x_m^4/12\mu Rs^2<1 \tag{60}$$

hold for $x_m\sim 5$. The first inequality (60) gives immediately the maximum value of R:

$$R<n/\theta , \tag{61}$$

where n is the atomic number of the plasma neutral-particle component. Equation (61) means, for example, that 4, 15, and 40 are the maximum values of R, if we want to our method to be correct for helium, neon, and argon, respectively.

When R is not small, we have by virtue of (49) and Table I the relations

$$R\approx R_E \Longrightarrow kT\approx eEl\sqrt{M_n/12m\theta} , \tag{62}$$

which ensure the correctness of (59) almost always. The second inequality (60) can be written in the form

$$Rs^2 \gg 50 . \tag{63}$$

Equations (63), (61), and (62) restrict the electrical field intensity and the electron temperature from above. Choosing between them the stronger inequality (63) we obtain these limits with the help of (62):

$$E<E_{\max}=e\sqrt{N_e/l\theta} ,$$
$$kT<kT_{\max}=\frac{e^2}{2\theta}\sqrt{NlM_n/3m} .$$

The electron temperature can be calculated in a stronger electrical field by our method with another zeroth approximation for the velocity distribution function.

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