PHYSICAL REVIEW A VOLUME 43, NUMBER 8 15 APRIL 1991

Level spacings for harmonic-oscillator systems

Akhilesh Pandey
School of Physical Sciences, Jawaharlal Nehru University, New Delhi 110 067, India

Ramakrishna Ramaswamy*
Division of Theoretical Studies, Institute for Molecular Science, Myodaiji, Okazaki 444, Japan
(Received 14 August 1990)

From the viewpoint of eigenvalue level statistics, harmonic-oscillator systems are unusual. Although integrable, these systems are nongeneric, and a spacing distribution does not exist even as the number of levels $N\rightarrow\infty$. The origins of this pathological behavior are explored using methods of number theory and ergodic analysis. However, such nongenericity is extremely fragile, and the smallest nonlinearity asymptotically restores generic behavior. These results are of relevance to the study of molecular spectra, as well as to the quasienergy spectra of integrable quantum maps.

I. INTRODUCTION

In the search for possible quantum parallels to complex classical dynamical behavior, attention has increasingly been focused on the statistical properties of quantum eigenvalue spectra. From the study of the spectral fluctuations of bound systems with two or more degrees of freedom, the following picture emerges:[1,2] when the underlying classical system has mainly chaotic motion, the eigenvalue spacing distribution falls into one of three random-matrix universality classes,[3,4] depending on the symmetries of the Hamiltonian.[5-7] When the underlying classical system is integrable, a different universality class obtains. By using an elegant semiclassical analysis, Berry and Tabor[8] showed that so long as the constant energy surfaces in the space of action variables was curved and convex from above, then, in integrable systems, the nearest-neighbor eigenvalue spacings were Poisson distributed.

These connections—between the nature of the classical dynamics on the one hand, and the resulting stationary-state quantum-mechanical properties on the other—have been the subject of considerable study. A host of applications to model systems which, classically, are chaotic,[7,9-11] integrable,[8,12,13] or mixed[14] have been made with considerable success. These ideas, and in particular the methods of random-matrix theory, have found several applications[3,15-19] in the analysis of complex nuclear, atomic, and molecular systems.

Within the class of integrable systems, however, there is one important exception.[8,20] The simplest integrable systems, namely, uncoupled harmonic oscillators, are nongeneric. The energy contours in action space have no curvature, and this results in correlations among the levels. The Berry-Tabor (BT) result[8] of a Poisson distribution for the level spacings does not apply to harmonic-oscillator systems, and such *nongeneric* behavior is the focus of the present work.

One result in this paper is that for finite degrees of freedom, there is no stationary distribution for the nearest-neighbor level spacings. That this might be the case was already indicated by BT, who studied examples in two dimensions. They also noted that in contrast to generic integrable systems that showed level clustering, harmonic oscillators with irrationally related frequencies exhibited a level "repulsion." While we confirm that this behavior is only artificially similar to the level repulsion characteristic of chaotic or complex systems, we also elucidate the mainly number-theoretic origins of this phenomenon.

By confining attention to segments of eigenlevel spectra, the energy eigenvalues can be generated through iterative mappings.[20] In addition, such eigenlevel maps arise naturally in the quasienergy spectrum of integrable quantum maps which obtain for periodically forced quantum systems.[21] These mappings can then be studied using methods of the ergodic theory of dynamical systems.[22] We examine the level autocorrelation functions, and show that this function does not decay *only* for harmonic-oscillator systems. The specific case of two degrees of freedom can be analyzed completely. This is discussed in Sec. II, where eigenlevel maps for component spectra are introduced. We show there that regardless of the ratio of the two frequencies, there are at most only three distinct spacings in any segment of the energy spectrum.[20] We then consider the question of the asymptotic spacing distribution which is obtained by superposing different segments: even then there is no limiting distribution.

Our results have relevance to the analysis of typical eigenvalue spectra. A harmonic-oscillator model is standard for a variety of systems, ranging from the normal-mode description of polyatomic molecules[23] to the phonon spectrum of solids.[24] This is most pertinent in the low-energy regime that is experimentally also the most easily accessible. The lack of existence of a spacing distribution indicates the need for care in the analysis of such data.

In Sec. III of this paper, the ergodic analysis of energy-level maps is presented. We explore the differences between harmonic and nonharmonic systems, and also address the question of how a transition from the nongeneric case to the generic occurs. We find that this transition is rapid and discontinuous in the limit of asymptotic energy, similar to what is encountered in the analogous transition between the several universality classes [Gaussian orthogonal ensemble (GOE), Gaussian unitary ensemble (GUE), Gaussian symmetric ensemble (GSE)] of random-matrix theory.[18] Higher-dimensional systems are briefly dealt with, in Sec. IV, using mainly numerical methods. This is followed, in Sec. V, by a summary.

II. TWO-DIMENSIONAL HARMONIC OSCILLATORS (TDHO's)

In this section the case of $d=2$ degrees of freedom is analyzed. The analysis of the spacing distribution is reduced first to a geometric problem, the distribution of arc lengths generated by an irrational rigid rotation of a circle. We also give a concise proof that this distribution is singular and that there can be at most three distinct arc-lengths.

For simplicity, the quantal energy levels of the two-dimensional harmonic oscillator can be written as $E_{mn}=m+\alpha n$, where $\alpha \leq 1$ is the ratio of frequencies of the two oscillators, m and n are the quantum numbers, and energy is being measured in units of the larger quantum.

A particular choice of m gives the mth component spectrum, which is uniformly spaced with spacing α and starts at $E=m$. Consider the unit segment of the TDHO spectrum between $E=M$ and $E=M+1$ for arbitrary integer M. This segment contains $N+1$ levels with average spacing $1/N$, where

$$N=-[-(M+1)/\alpha] \overset{\text{large } M}{\rightarrow} M/\alpha \ . \tag{2.1}$$

$[x]$ and $\{x\}$ denote the integer and fractional part of x, respectively. These levels are located at energies

$$E=M+x_j \ , \tag{2.2a}$$

with

$$x_j=\{j\alpha\}, \quad j=0,1,\ldots,N-1 \ , \tag{2.2b}$$

and the final level at $E=M+1$. Thus the levels in each unit segment (except for $j=N$) are generated by a finite orbit of 0 under the rigid rotation,

$$x \rightarrow x+\alpha (\text{mod } 1) \ . \tag{2.2c}$$

By the *ordered spectrum* we mean the sequence of x_j's arranged in increasing order. These will be denoted by X_j (with $X_N \equiv x_N = 1$).

A. Preliminaries

We first fix notation and recall some definitions.[25] The (arbitrary) frequency ratio α, which will henceforth be denoted α_1 for notational consistency, and which we take to be ≤ 1, is written as a simple continued fraction in terms of positive integers a_k

$$\begin{aligned}\alpha_1&=1/\{a_1+1/[a_2+1/(a_3+\cdots)]\}\\ &=[a_1,a_2,\ldots,a_k,\ldots]\\ &=[a_1,a_2,\ldots,1/\alpha_k] \ ,\end{aligned} \tag{2.3}$$

with

$$\alpha_k=[a_k,a_{k+1},\ldots] \ . \tag{2.4}$$

The a sequence is finite for rational α_1, infinite for irrational and eventually periodic for quadratic irrational. An important operation in this context is the Gauss-shift transformation T defined by[22]

$$\alpha_{k+1}=T\alpha_k=T^k\alpha_1 \ , \tag{2.5}$$

$$T[a_k,a_{k+1},\ldots]=[a_{k+1},a_{k+2},\ldots] \ , \tag{2.6}$$

which defines the Gauss sequence $\alpha_1,\alpha_2,\ldots$. Finite products of the α's play a crucial role in the theory of the spacings,

$$\Delta_i=\prod_{j=1}^{i}\alpha_j \ . \tag{2.7}$$

Note that truncating the infinite continued fraction at a_k gives the so-called kth convegent,

$$p_k/q_k=[a_1,a_2,\ldots,a_k] \tag{2.8}$$

($p_0=0$, $q_0=1$). The convergents form an alternating sequence of rational approximations to α_1, with errors given by

$$\alpha_1-p_k/q_k=(-1)^k\Delta_{k+1}/q_k \ . \tag{2.9}$$

B. The three spacings

For rational $\alpha_1=p/q$ it is clear that the spectrum generated by the map Eq. (2.2) is uniformly spaced if $N=q$; there is then only one distinct spacing which is $1/q$. If $N>q$ then there are degeneracies as well, so that there are two distinct spacings, 0 and $1/q$. For $N<q$ it turns out that there can be *at most* three distinct spacings. This result, surprisingly, also holds for *arbitrary* α and *arbitrary* N. We consider this general case.

The eigenlevel map given in Eq. (2.2) is graphically represented as in Fig. 1, from which it is clear that for $j\leq a_1$ there are only two distinct spacings [see Fig. 1(a)] Δ_1 and $(1-j\Delta_1)$. For $j=a_1$, the second spacing takes its minimum possible value, $1-j\Delta_1\equiv 1-q_1\Delta_1=\Delta_2$. Note here that

$$q_1\Delta_1+\Delta_2=1 \ . \tag{2.10}$$

At the next iteration a third spacing, namely $\Delta_1-\Delta_2$, is created, with $x_{a_1+1}\in[0,\Delta_1]$ [see Fig. 1(b)]. Further iterates $x_{a_1+j}, j<q_2$, fall in the intervals of length Δ_1 created by the first a_1 iterations of the map. Consequently, it is enough to restrict attention to the q_1th composition of the map on the first interval $[0,\Delta_1]$, which mirrors what happens in all other intervals of length Δ_1,

$$\bar{x}_{k+1}=\bar{x}_k-\Delta_2, \quad 0\leq k\leq a_2, \quad \bar{x}_0=\Delta_1 \ . \tag{2.11}$$

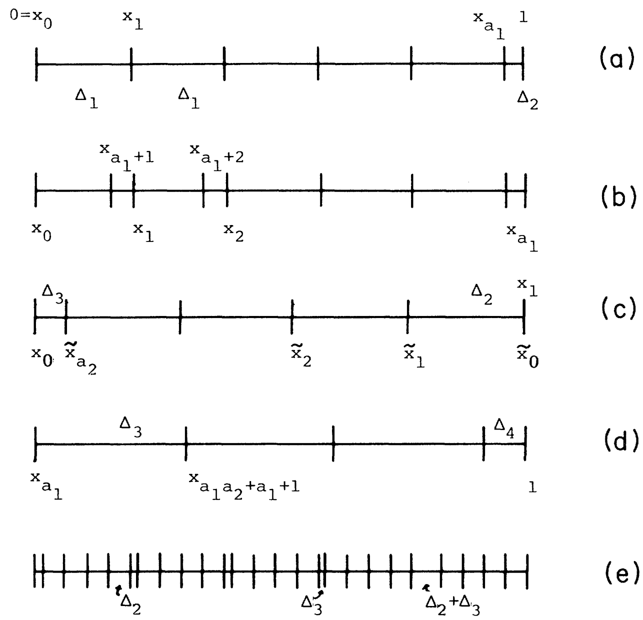

FIG. 1. Graphical construction illustrating the existence of at most three spacings. The irrational ratio of frequencies is $\alpha_1=[5,4,3,2,1,1,1,\ldots]$, and (a)–(d) show the successive creation of spacings Δ_1, Δ_2, Δ_3, and Δ_4. (e) shows a typical partition of the interval [0,1] with only three spacings.

Upon rescaling by a factor $-\Delta_1$ and shifting by 1, this becomes another rigid rotation, however, now by the angle α_2,

$$\tilde{x}_{k+1}=\tilde{x}_k+\alpha_2,\quad 0\le k\le a_2,\quad \tilde{x}_0=0\ . \tag{2.12}$$

The only spacings that appear are Δ_2, $\Delta_1-k\Delta_2$, and $\Delta_1-(k-1)\Delta_2$ for $j<q_1+q_2$ iterations of the original map, Eqs. (2.2b) and (2.2c). For $k=a_2$, there are again only two spacings since $\Delta_1-a_2\Delta_2=\Delta_3$ [Fig. 1(c)], and further

$$q_2\Delta_2+q_1\Delta_3=1\ . \tag{2.13}$$

The next iterate creates a new spacing, $\Delta_2-\Delta_3$ with $x_{q_1+q_2}\in[x_{q_1},1]$, the first interval of length Δ_2. Subsequent iterates fall only in the intervals of length Δ_2, for which it suffices to look at the map [following rescaling and shifting as above; see Fig. 1(d)]

$$\tilde{x}_{k+1}=\tilde{x}_k+\alpha_3,\quad 0\le k\le a_3,\quad \tilde{x}_0=0\ . \tag{2.14}$$

That there can never be more than three spacings [Fig. 1(e), for example] derives from the fact that after $t_i=q_i+q_{i-1}-1$ iterations ($i=1,2,\ldots$), there are only two spacings, Δ_i and Δ_{i+1}, with the unit interval partitioned as

$$\Delta_i q_i+\Delta_{i+1}q_{i-1}=1\ . \tag{2.15}$$

Each interval of length Δ_i is first created at the left or right extreme of [0,1] according as i is odd or even, reflecting the fact that successive convergents of α_1 are alternately less than or greater than α_1. The three spacings that will occur for arbitrary number of iterates N can be deduced easily. For $t_i\le N\le t_{i+1}$, the spacings are[20]

$$\begin{aligned} s_1&=\Delta_{i+1}\ ,\\ s_2&=\Delta_i-k\Delta_{i+1},\quad k<a_i\\ s_3&=s_1+s_2\ . \end{aligned} \tag{2.16}$$

The frequency of each spacing can be enumerated from the unique (for integer k,z) decomposition

$$N=q_{i-1}+kq_i+z\ , \tag{2.17}$$

with $1\le k\le a_{i+1}$ and $0\le z\le q_i$. It is easy to show that the respective weights for the three spacings are

$$\begin{aligned} w_1&=N-q_i\ ,\\ w_2&=z\ ,\\ w_3&=q_i-z\ . \end{aligned} \tag{2.18}$$

C. The limiting distribution

In each unit segment of the energy spectrum, the distribution (normalized to unit average spacing) is

$$P_N(s)=\frac{1}{N}\sum_{i=1}^{3}w_i\delta(s-Ns_i)\ , \tag{2.19}$$

with w and s given as in Eqs. (2.18) and (2.16) above. In the case of rational $\alpha=p/q$ and $N>q$ degeneracies dominate the spectrum so that

$$P_N(s)\overset{N\to\infty}{\to}\delta(s)\ , \tag{2.20}$$

which emphasizes the significance of degeneracies, but does not give the correct average spectrum. A further averaging,

$$P(s;N_{\min},N_{\max})=\frac{\sum_{N_{\min}}^{N_{\max}}NP_N(s)}{\sum_{N_{\min}}^{N_{\max}}N}\ , \tag{2.21}$$

yields the same result for $N_{\max}\to\infty$.

For irrational α there are no degeneracies since none of the s_i's in Eq. (2.16) can be zero. It is possible, however, that for some α and some resulting N, the probability of observing two levels arbitrarily close to one another remains nonzero. However, as the weight of these cases can be made arbitrarily small in the average, the irrational TDHO system will generically[8,20] show "level repulsion."

As $N\to\infty$, however, $P_N(s)$ does not approach a unique limit, but changes continually with $N_{\max}$. Thus there is no stationary form even as the energy increases. (If α is a quadratic irrational with period 1, then a spacing distribution can be defined in some sense; see Ref. 20.)

In discussing the $N\to\infty$ limit we need to know whether α_k and $C_k=q_{k-1}/q_k$ have well-defined limits as

$k \to \infty$. For quadratic irrationals this is indeed the case. However, for almost all α's, a Gaussian theorem states that the α_k and C_k settle on the joint distribution[22,26]

$$f(x,y)=\frac{1}{\ln 2}\frac{1}{(xy+1)^2}, \quad 0 \le x,y \le 1 \tag{2.22}$$

from which one obtains

$$\mathrm{Prob}(a_k=p)=\frac{\ln[(p+1)^2/p(p+2)]}{\ln 2} . \tag{2.23}$$

Thus for a typical irrational number, any integer p is encountered infinitely often in the continued-fraction expansion. In the large-k limit,[22,26]

$$-\lim_{k\to\infty}\frac{\ln\Delta_k}{k}=\lim_{k\to\infty}\frac{\ln q_k}{k}=\frac{\pi^2}{12\ln 2} . \tag{2.24}$$

Now consider the piecewise continuous function

$$\Delta(E)=S , \tag{2.25}$$

where S represents the smallest (unnormalized) spacing in the energy spectrum $[0,E]$. $\Delta(E)$ takes values $\Delta_1,\Delta_2,\Delta_3,\ldots$ at energies $E_i \approx p_i$, the numerators of the successive convergents. As a consequence, for large stretches of energy, the smallest spacing remains constant. It can easily be shown, using Eqs. (2.23) and (2.24) above, that

$$\Delta(E)\sim\frac{1}{E} \tag{2.26}$$

for two degrees of freedom. In the generic integrable system, however, the smallest spacing decreases much faster with energy.

III. ERGODIC ANALYSIS OF LEVEL MAPS

The analysis of unit segments of component spectra leads, as discussed above, to the study of the distribution of $\{j\alpha\}$, $j=0,1,\ldots$ in the case of the TDHO system. Such analysis can be extended to a much wider class of integrable systems, so long as one of the degrees of freedom remains harmonic. Then eigenvalues in a unit segment of the component spectrum can be generated through a map similar to Eq. (2.2),

$$x_k=\{f(k)\} , \tag{3.1}$$

where $f(y)$ is the function appropriate to the problem. If f is a polynomial of degree ≥ 1,

$$f(y)=b_0y^r+b_1y^{r-1}+\cdots+b_r , \tag{3.2a}$$

or exponential

$$f(y)=b\sigma^y , \tag{3.2b}$$

then[22] the sequences x_k are uniformly distributed on the interval [0,1] if at least one of the coefficients b_j above is irrational, and for $\sigma > 1$ and almost all b in the exponential case.

To give a specific example in two dimensions, consider a particle moving in a potential which is harmonic in one direction and consists of rigid walls in the other. The expression for the quantum eigenvalues, $E_{nm}=n+\alpha m^2$, leads to the map

$$x_j=\{j^2\alpha\} ,$$

or (3.3)

$$x_{j+1}=x_j+(2j+1)\alpha(\mathrm{mod}\,1)$$

in unit intervals of the (scaled) energy.

In higher dimensions $d>2$ similar maps can be defined (see Sec. IV below) where analogous results show that, in the typical case, the corresponding sequences of eigenvalues are uniformly distributed on $d-1$ dimensional tori.[22]

The above cases cover a wide variety of integrable systems.

In $d=2$ dimensions, although the sequence of eigenvalues x_k that obtain from the rotation, Eq. (2.2), is ergodic on the interval, we have shown above that a limiting spacing distribution does not exist. Ergodicity alone is therefore not sufficient that such a distribution should exist, although it would seem a necessary condition.

In order to highlight the differences between generic and nongeneric systems, we now compute the two-level correlation function for a typical nonlinear map, specifically the example given above in Eq. (3.3). Note that the latter system satisfies the requirement that the energy surfaces are curved in action space, and thus the eigenvalue statistics are expected to be Poissonian. It is convenient to consider the two-level density correlation function

$$R_2(r)=\frac{1}{N^2}\sum_{j\ne k}\delta(r/N+x_j-x_k) \tag{3.4}$$

$$=\frac{1}{N^2}\sum_{k>0}(N-k)[G_k(r)+G_k(-r)] , \tag{3.5}$$

where we take $|r| \ll N$ and

$$G_k(r)=\frac{1}{N-k}\sum_n\delta(r/N-x_{n+k}+x_n) . \tag{3.6}$$

For the TDHO, $x_{n+k}=\{x_n+x_k\}$, so for large N (using the property that the rigid rotation is ergodic),

$$\begin{aligned}G_k(r)&=\int_0^1 dt\,\delta\left[\frac{r}{N}+t-\{t+x_k\}\right]\\&=\left[1-\frac{|r|}{N}\right]\left[\delta\left[x_k-\frac{r}{N}\right]+\delta\left[1-x_k+\frac{r}{N}\right]\right]\\&\to\delta\left[\frac{r}{N}-x_k\right]+\delta\left[\frac{r}{N}+1-x_k\right] ,\end{aligned} \tag{3.7}$$

from which it follows that

$$R_2(r)=\frac{1}{N^2}\sum_k(N-k)\left[\delta\left[\frac{|r|}{N}-x_k\right]+\delta\left[\frac{|r|}{N}-1+x_k\right]\right] . \tag{3.8}$$

It is tempting to argue here that as k and x_k are independently distributed, $R_2(r)\to 1$ for large N. This is indeed true for large (but still much smaller than N) values of $|r|$, but for finite r, the ergodic result does not hold. The fluctuations in $R_2(r)$, and in its integral, remain large and consequently the limit does not exist. This can be seen above for small values of $|r|$, for which $R_2(r)\approx P_N(|r|)$, but numerical experiments support this for larger values of $|r|$ as well.

In contrast, the map specified in Eq. (3.3), with $x_{n+k}=\{x_n+2nk\alpha+x_k\}$, is not only ergodic, but in addition, both $\{n\alpha\}$ and $\{n^2\alpha\}$ are independently distributed. We therefore obtain for large N and $k>0$

$$\begin{aligned} G_k(R) &= \int dt_1 \int dt_2 \delta(r/N-(t_1+t_2-x_k)) \\ &= 1-|r|/N \\ &\to 1 , \end{aligned} \tag{3.9}$$

so that

$$R_2(r)=\frac{2}{N^2}\sum_k (N-k)\to 1 , \tag{3.10}$$

implying Poisson statistics. Note that the function $G_k(r)$ is essentially a measure of the correlation between x_k and x_{n+k}. The correlation coefficient can be shown to be $1-6x_k(1-x_k)$ and δ_{k0} for the maps (2.2) and (3.3), respectively. For the harmonic system which is nongeneric, correlations survive—indeed they oscillate about zero—for arbitrary k, whereas in the latter, generic case, they die down immediately. (Note that for the map $x_j=\{2^j\alpha\}$, the correlation coefficient dies as 2^{-k}, but even for this case, the BT results apply.) This contrasting behavior of correlation coefficients is, we believe, the main difference between generic and nongeneric integrable systems.

Consider, finally, a nonlinear perturbation of the rotation (2.2),

$$x_j=\{\alpha j+\beta j^2\} , \tag{3.11}$$

with α and β (the perturbation parameter) mutually incommensurable. This, for example, describes the quasienergy spectrum corresponding to the unitary operator $\exp[i(\alpha J_z+\beta J_z^2)]$ for fixed total angular momentum J (see also Ref. 27). Since $\alpha j+\beta j^2$ is a convex function of j, one should expect Poisson statistics for the spacings for any nonzero value of β and *sufficiently large N*. In particular, there should be a transition from nongeneric behavior for small N to the generic for large N.

To quantify this assertion, we proceed as follows. Consider small β; then the ordered spectrum can be written as $X_k(\alpha,\beta)\approx X_k(\alpha,0)+\beta Y_k^2$, where $0<Y_k<N$ is an integer such that $X_k=x_{Y_k}$. By inspection of the spacings $X_{k+1}-X_k$ one can see that the degeneracy in the spectrum of spacings is completely broken for any nonzero β. (If the perturbation were linear, then degeneracies will survive.) As β increases, the levels will cross and the spacing distribution will make a transition to the Poisson with the crossover value of the perturbation parameter determined by the condition that the level shift due to perturbation is of the order of the spacing itself. In this case, $\beta_{\text{crossover}}\approx N^{-3}$.

In Fig. 2 we show the results of numerical studies of the spacing distribution for the map (3.11) with fixed α and $N=1000$, and β varying. As argued above, the

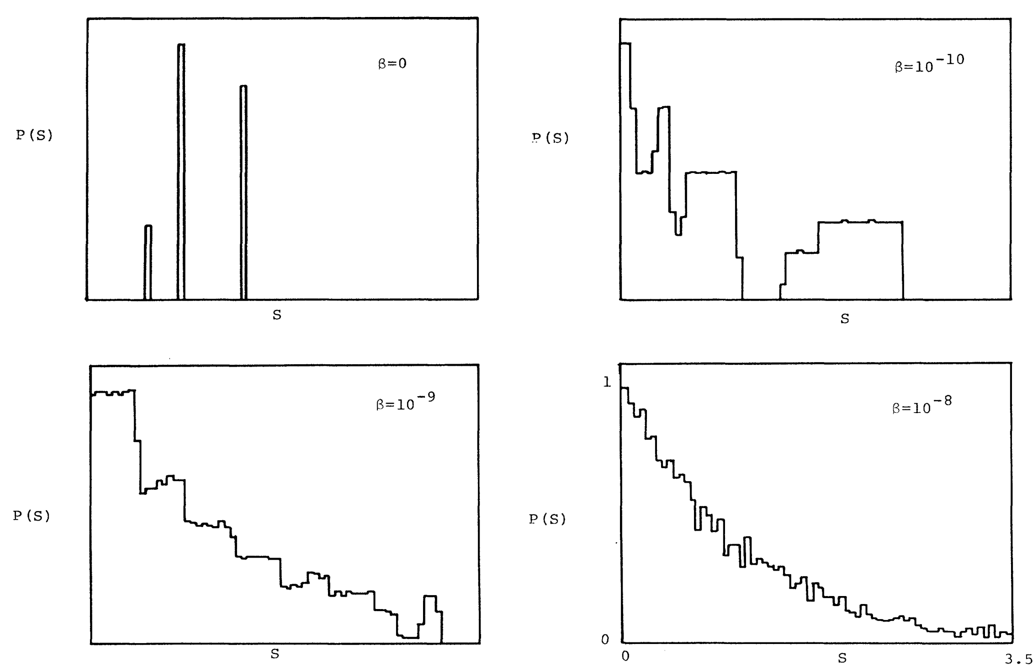

FIG. 2. Illustration of the crossover behavior when the levels are given by Eq. (3.11). In all cases, α is the golden mean, $N=1000$, and the value of β is indicated in the panels. The spectrum has been unfolded such that the mean spacing is unity.

crossover occurs for $\beta \approx 10^{-8}$. The transition to the generic case is extremely rapid, and discontinuous for $N \to \infty$ as a function of β, but smooth as a function of βN^3.

This fragility of the nongenericity of harmonic-oscillator systems is similar to that in random-matrix theories where discontinuous changes are encountered when a symmetry of the system is broken.[18]

IV. HIGHER-DIMENSIONAL HARMONIC SYSTEMS

Extension of the above ideas to the study of spacing distributions in higher-dimensional harmonic systems, where the energy eigenvalues are given by

$$E = n + \sum_{i=2}^{d} \omega_i m_i \ , \tag{4.1}$$

yield ergodic maps on $(d-1)$-dimensional tori. We restrict attention to the case when all frequencies ω_i are irrational and mutually irrational as well, since for rational ratios, degeneracies will ultimately dominate the spectrum.

Classical systems of an infinite number of uncoupled harmonic oscillators are routinely used to model stochastic phenomena, most notably as a source of random noise.[28] One might therefore speculate that although the TDHO (or other low-dimensional harmonic oscillator) does not have a limiting spacing distribution, for a typical distribution of oscillator frequencies, ω, a spacing distribution might exist in the $d \to \infty$ limit.

It may also be mentioned that the harmonic level spacing problem is *dual* to the generalized quasicrystal problem.[29] Recall the quasicrystal construction:[30] Lattice points of a d-dimensional hypercubic lattice contained between two parallel $(d-1)$-dimensional hyperplanes which intersect the axes with irrational slope are projected onto one of the planes. This gives a lattice with quasiperiodic ordering in $(d-1)$ dimensions. We are, however, interested in the energy, which for these harmonic systems is measured as a distance along the direction $(1111\ldots11)$. By fixing the segment $[M, M+1]$ of interest, the two hyperplanes are fixed, and the sequence of levels, $\{x\}$ are the projections of the lattice points on the diagonal $(1111\ldots11)$, measured from the hyperplane determined by $E = M$.

For $d=2$, both projections have identical properties, and the three-spacing result quoted in Sec. II B is valid both for the resulting quasiperiodic one-dimensional (1D) lattice, as well as for the level spacings. When the slope is the golden mean, $\alpha = (\sqrt{5}+1)/2$, this gives the well-known Fibonacci construction of a one-dimensional quasiperiodic lattice with only two lattice spacings. Our result in Sec. II B shows that with a judicious choice for the irrational α and appropriate N, one can generate lattices with only two spacings in any desired sequence; of course, in general, there can be at most three lattice spacings.

For $d=3$, the eigenvalue map is

$$x_{jk} = \omega_1 j + \omega_2 k (\text{mod } 1) \ \text{ if } [\omega_1 j + \omega_2 k] < M+1 \ . \tag{4.2}$$

Since the levels arise from the intermeshing of two different subcomponents, the spacings "numerology" depends strongly on the number-theoretic properties of ω_1 and ω_2. Unlike the three-spacing result for TDHO's, it turns out now that *any* number of different spacings N_s can occur. Furthermore, N_s has a complicated dependence on M as well as on the ratio ω_1/ω_2. In general, though, there continues to be a high degree of degeneracy as for TDHO's, so that N_s is much less than the number of levels in the interval $[M, M+1]$, and the distribution is

$$P(s) = \sum_{i=1}^{N_s} w_i \delta(s - s_i) \ \ (N_s > 3) \ . \tag{4.3}$$

In numerical experiments, we have noted that regardless of the exact value of N_s the spacings themselves have a three-dimensional basis: each individual spacing can be expressed as a linear combination of three "fundamental" spacings.

It is not clear whether further analysis along the lines of Sec. II B is possible. Pairs of mutually irrational irrational numbers do not have convenient (or unique) simultaneous best rational approximations, except for particular cases.[31] This is one impediment to the development of a complete number-theoretic treatment of the spacings degeneracy in the $d=3$ case.

As the number of degrees of freedom increases, the combinatorics gets even more complicated since the number of frequency ratios that must be taken into account increases. For small d, though, the spacings distribution tends to remain, as in Eq. (4.3), a sum of δ functions so that there is no limiting distribution as $N \to \infty$. The question of whether a distribution exists even as $d \to \infty$ remains an open one, although for typical irrational frequency ratios, it seems unlikely. Recall that for $d=2$, the smallest spacing scaled as $1/E$ with energy [Eq. (2.26)]. In d dimensions, this result generalizes to $\Delta(E) \sim 1/E^{d-1}$. Since the average spacing also scales in the same way, the normalized smallest spacing is still ~ 1.

V. SUMMARY

In this paper we have explored the pathology of harmonic-oscillator systems in the context of level spacing statistics. In the case of two degrees of freedom, it is possible to see the origins of such behavior very clearly in a variety of ways: the lack of curvature for constant energy surfaces in action space, the "at most three distinct spacings" result in any unit interval of energy, or the persistance of correlations between energy levels as typified by the ergodic analysis.

In higher dimensions, the pathological behavior remains, but the analysis is not as transparent as for the two-dimensional case. It seems, however, that the conclusions are not very different from that in two dimensions. The nongenericity of such systems is delicately poised, inasmuch as the addition of the smallest nonlinear term suffices to guarantee that the spacing distribution and other spectral correlations revert to the Poisson, and the system regains all the features of generic integrable systems. We have derived here simple estimates for when and how the transition from the nongeneric to the generic occurs.

ACKNOWLEDGMENTS

We thank D. Dhar and S. Thomae for discussions and helpful correspondence. In addition, R.R. would like to thank Hiroki Nakamura and the Institute for Molecular Science for hospitality.

---

*On leave from School of Physical Sciences, Jawaharlal Nehru University, New Delhi 110 067, India.

[1] M. V. Berry, Proc. R. Soc. London, Ser. A **413**, 183 (1988).
[2] B. Eckhardt, Phys. Rep. **163**, 205 (1988).
[3] T. A. Brody, J. Flores, P. A. Mello, J. B. French, A. Pandey, and S. S. M. Wong, Rev. Mod. Phys. **53**, 385 (1981).
[4] M. L. Mehta, *Random Matrices and the Statistical Theory of Energy Levels* (Academic, New York, 1967).
[5] F. J. Dyson, J. Math. Phys. **3**, 140 (1962).
[6] M. Robnik and M. V. Berry, J. Phys. A **17**, 669 (1986).
[7] R. Scharf, B. Dietz, F. Haake, and M. V. Berry, Europhys. Lett. **5**, 383 (1988).
[8] M. V. Berry and M. Tabor, Proc. R. Soc. London, Ser. A **356**, 375 (1977).
[9] O. Bohigas, M. J. Giannoni, and C. Schmit, Phys. Rev. Lett. **52**, 1 (1984).
[10] M. V. Berry and M. Robnik, J. Phys. A **19**, 649 (1986).
[11] T. H. Seligman and J. J. M. Verbaarschot, Phys. Lett. **108A**, 183 (1985).
[12] G. Casati, B. V. Chirikov, and I. Guarneri, Phys. Rev. Lett. **54**, 1350 (1985).
[13] M. Feingold, Phys. Rev. Lett. **55**, 2626 (1985).
[14] M. V. Berry and M. Robnik, J. Phys. A **17**, 2413 (1984).
[15] O. Bohigas, R. U. Haq, and A. Pandey, Phys. Rev. Lett. **54**, 1645 (1985).
[16] N. Rosenzweig and C. E. Porter, Phys. Rev. **120**, 1698 (1960).
[17] T. Zimmermann, H. Köppel, L. S. Cederbaum, G. Persch, and W. Demtröder, Phys. Rev. Lett. **61**, 3 (1988).
[18] J. B. French, V. K. B. Kota, A. Pandey, and S. Tomsovic, Ann. Phys. (N.Y.) **181**, 198 (1988).
[19] S. Sinha and R. Ramaswamy, J. Phys. B **22**, 2985 (1989).
[20] A. Pandey, O. Bohigas, and M. J. Giannoni, J. Phys. A **22**, 4083 (1989).
[21] M. V. Berry, N. L. Balasz, M. Tabor, and A. Voros, Ann. Phys. (N.Y.) **122**, 26 (1979).
[22] Ya. G. Sinai, *Introduction to Ergodic Theory* (Princeton University Press, Princeton, 1976); I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinai, *Ergodic Theory* (Springer-Verlag, New York, 1982).
[23] E. B. Wilson, J. C. Decius, and P. C. Cross, *Molecular Vibrations* (McGraw-Hill, New York, 1955).
[24] J. Ziman, *Principles of the Theory of Solids* (Cambridge University Press, Cambridge, 1972).
[25] A. Ya. Khinchin, *Continued Fractions* (University of Chicago Press, Chicago, 1964).
[26] P. Lévy, *Théorie de l'Addition des Variables Aléatoires* (Gauthier-Villars, Paris, 1954), Chap. IX.
[27] G. Casati, I. Guarneri, and F. M. Izrailev, Phys. Lett. A **124**, 263 (1987); Ya. G. Sinai, Physica A **163**, 197 (1990).
[28] G. W. Ford, M. Kac, and P. Mazur, J. Math. Phys. **6**, 504 (1965).
[29] P. J. Steinhardt and S. Ostlund, *The Physics of Quasicrystals* (World Scientific, Singapore, 1987); see also B. Sutherland, in *Disorder and Nonlinearity,* edited by A. R. Bishop (Springer-Verlag, Berlin, 1989).
[30] A. Katz and M. Duneau, J. Phys. (Paris) **47**, 181 (1986).
[31] S. Kim and S. Ostlund, Phys. Rev. A **34**, 3426 (1986).