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Polarization dynamics and interactions of solitons in a birefringent optical fiber

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Dynamics of vector solitons are studied within the framework of the general model of coupled nonlinear Schrödinger equations. The analysis is based upon the perturbation theory for the case when the system is close to the exactly integrable Manakov form. Evolution of the soliton's polarization, coupled by the linear-birefringence terms (which take account of the difference in the group velocity between two linearly polarized modes) to the positional degree of freedom of the soliton, is studied. A four-dimensional dynamical system for the two coupled degrees of freedom integrates to a two-dimensional conservative system. Depending on the value of an arbitrary integration constant, there are four different types of the phase portrait of the latter system. For each value of the polarization angle, there exist two stationary vector solitons, at least one of them being stable. Generic trajectories on the two-dimensional phase plane correspond to oscillations of the polarization coupled to oscillations of the position of the soliton. A generalized model including the polarization-rotating linear coupling is also analyzed. Next, interaction of two slightly overlapping vector solitons is considered, and it is demonstrated that a stable bound state is possible. A stable periodic chain of the slightly overlapping solitons is also found. Finally, radiative decay of a vector soliton is investigated for the case when it has a large component in one subsystem and a small component in another subsystem.

I. INTRODUCTION

The present paper is devoted to the study of soliton dynamics within the framework of a model describing propagation of optical solitons in a nonlinear birefringent fiber. This subject attracts great current interest (see, e.g., Refs. 1–16) owing to both its fundamental meaning and potential applications in optical logic devices (e.g., soliton switches[12]). The analysis to be developed in the present work will be based on the system of coupled nonlinear Schrödinger (NS) equations governing propagation of envelopes of electromagnetic waves in the fiber. In a general case, this system may be written in the following dimensionless form:[10]

$$iu_z+icu_\tau-\omega u+u_{\tau\tau}+2(|u|^2+|v|^2)u+2\beta|v|^2u+2\beta' v^2u^*=0\ , \qquad (1.1a)$$

$$iv_z-icv_\tau+\omega v+v_{\tau\tau}+2(|v|^2+|u|^2)v+2\beta|u|^2v+2\beta' u^2v^*=0\ , \qquad (1.1b)$$

where $u(z,t)$ and $v(z,t)$ are envelopes of the waves with mutually orthogonal linear polarizations, z is the propagation distance, and $\tau=t-z/v_{\rm gr}$, where t is time, and $V_{\rm gr}$ is a mean group velocity. The cross-phase-modulation coefficients β and β' are usually assumed to be related as follows:

$$|\beta|=|\beta'| \qquad (1.2)$$

(note that the sign of β is invariant, while the sign of β' is not, as it can be changed by the substitution $u,v\rightarrow iu,v$ or $u,v\rightarrow u,iv$). At last, the linear birefringence coefficients c and ω take account of the difference in, respectively, group and phase velocities between the two linear polarizations. In what follows, it will be convenient to remove the group-velocity terms by means of the obvious transformation

$$u(z,\tau)\equiv U(z,\tau)\exp\left[i\left[-\frac{c}{2}\tau+\frac{c^2}{4}z\right]\right]\ , \qquad (1.3a)$$

$$v(z,\tau)\equiv V(z,\tau)\exp\left[i\left[\frac{c}{2}\tau+\frac{c^2}{4}z\right]\right]\ . \qquad (1.3b)$$

The corresponding coupled NS equations take the form

$$iU_z-\omega U+U_{\tau\tau}+2(|U|^2+|V|^2)U+2\beta|V|^2U+2\beta' V^2U^*\exp(2ic\tau)=0\ , \qquad (1.4a)$$

$$iV_z+\omega V+V_{\tau\tau}+2(|V|^2+|U|^2)V+2\beta|U|^2V+2\beta' U^2V^*\exp(-2ic\tau)=0\ . \qquad (1.4b)$$

Equations (1.4) can be derived from the Lagrangian density

$$\mathcal{L}=\tfrac{1}{2}iU^*U_z-\tfrac{1}{2}iUU_z^*-|U_\tau|^2+\tfrac{1}{2}iV^*V_z-\tfrac{1}{2}iVV_z^*-|V_\tau|^2-\omega|U|^2+\omega|V|^2+|U|^4+|V|^4+2(1+\beta)|U|^2|V|^2+\beta' V^2(U^*)^2e^{2ic\tau}+\beta'(V^*)^2U^2e^{-2ic\tau}\ . \qquad (1.5)$$

The Lagrangian representation of Eqs. (1.4) will play an important role in the analysis developed below.

In the case $\beta=\beta'=0$ (and $\omega=0$), the coupled NS equations (1.4) are amenable to application of the inverse scattering transform.[17] In this case Eqs. (1.4) possess a family of the exact soliton solutions:

$$U_{\rm sol}=2i\eta(\cos\theta)\mathrm{sech}[2\eta(\tau-\tau^{(0)})]\times\exp[i(\phi-\tfrac{1}{2}\psi'+\tfrac{1}{2}V\tau)]\ , \quad (1.6a)$$

$$V_{\rm sol}=2i\eta(\sin\theta)\mathrm{sech}[2\eta(\tau-\tau^{0})]\times\exp[i(\phi+\tfrac{1}{2}\psi'+\tfrac{1}{2}V\tau)]\ , \quad (1.6b)$$

where η and V are arbitrary amplitude and inverse velocity of the soliton, the arbitrary parameter θ ($0\le\theta\le\pi/2$) is a polarization angle, and ψ' is an arbitrary soliton's internal phase. The common phase ϕ and the soliton's "coordinate" $\tau^{(0)}$ evolve according to the equations

$$\dot{\phi}=4\eta^2-\tfrac{1}{4}V^2\ , \quad (1.7a)$$

$$\dot{\tau}^{(0)}=V\ , \quad (1.7b)$$

the dot standing for differentiation in z. In the general case, when both U and V components of the soliton (1.6) are different from zero, it will be called a vector soliton.[14] The particular case $\sin(2\theta)=0$, when only one component is present, will be referred to as a simple soliton.

To study the dynamics of the solitons in the nonintegrable case $\beta,\beta'\neq0$, it is natural to develop a perturbative analysis, assuming that β and β' are small parameters.[13] An important feature of the nonintegrable case is the fact that the soliton's polarization (the angle θ) may vary in z. To investigate the polarization dynamics of the vector soliton by means of the perturbation theory, let us presume that in the case of small β and β' the soliton's wave form remains close to that given by Eqs. (1.6), while the parameters θ, $\dot{\tau}^{(0)}$, and ψ' become slowly varying functions of z. Then one should derive evolution equations for θ, $\dot{\tau}^{(0)}$, and ψ', and investigate the corresponding phase space. For the particular case $c=0$ [see Eqs. (1.1)], this has been recently done in Ref. 13. In this particular case, the parameter $\dot{\tau}^{(0)}$ does not suffer the slow evolution, so that the phase space becomes two dimensional (see below). It has been demonstrated in Ref. 13 that two qualitatively different situations are possible in the case $c=0$: Either both simple solitons ($\theta=0$ and $\theta=\pi/2$) are stable and no vector solitons exist, or one simple soliton becomes unstable, and simultaneously there appears a stable vector soliton (the instability of a simple soliton has been first revealed in numerical simulations performed in Ref. 1). In the latter situation, the vector soliton has a uniquely determined value of θ [in the cw approximation corresponding to $u=u(z)$, $v=v(z)$, i.e., $u_\tau=v_\tau\equiv0$, a vector state, if any, also has a uniquely determined polarization, see, e.g., the papers quoted in Ref. 18].

In Sec. II of the present paper, the dynamics of the vector soliton are investigated in the general case ($c\neq0$). It is demonstrated that the parameter c couples the "positional" degree of freedom $\tau^{(0)}$ to the slow variables θ and ψ'. Thus the original dynamical system for the two coupled degrees of freedom (positional and polarizational ones) is four dimensional. However, it can be transformed into a three-dimensional system, which can be further integrated once. Thus we arrive at a two-dimensional conservative dynamical system with two independent parameters, one being a function of the ratio η/c (recall η is the soliton's amplitude), the other being an arbitrary integration constant. Investigation of the stationary points of the two-dimensional dynamical system demonstrates that there are two different stationary solitons for every value of the polarization angle θ. This is a drastic difference from the aforementioned particular case $c=0$, where a vector soliton exists at a single value of θ. At different values of the two parameters, there are four different types of a phase portrait of the two-dimensional dynamical system. Two of them resemble the phase portraits revealed for the particular case $c=0$ in Ref. 13 (no vector soliton and two stable simple solitons, or one stable vector soliton and two simple solitons, one of which is stable). Two other types are new: Two stable simple solitons and two vector solitons, one of which is stable; or two unstable simple solitons and two stable vector ones. Generic dynamical trajectories on the phase plane correspond to oscillations of the polarization of the soliton coupled to oscillations of the position of its center.

In Sec. III, interaction between two slightly overlapping vector solitons is considered (in the particular case $c=0$). It is demonstrated that their bound state with a nonzero "binding energy" is possible, and, under a certain condition [when ω is small in comparison with β, see Eqs. (1.1)], the bound state is stable against small perturbations. In Sec. IV, a periodic array of vector solitons with an arbitrary (but large) spacing is considered for the same particular case $c=0$. It is demonstrated that the array may be stable, a corresponding stability condition being less restrictive than that for the two-soliton bound state.

The analysis developed in Secs. II–IV is based upon the Hamilton canonical equations of motion for slowly varying soliton parameters, which can be derived starting from the Lagrangian (1.5). This analysis corresponds to the so-called adiabatic approximation that neglects a disturbance of the soliton's wave form (1.6). Beyond the framework of this approximation, an adiabatically stable vector soliton may be subject to radiative decay; i.e., it may emit quasilinear waves (radiation) at the expense of losing its energy. In Sec. V, the radiative decay is investigated for the simplest case when the vector soliton has a large U component and a small V component, the latter one being described by the linearized equation (1.4b). A law of the radiative decay of the soliton's amplitude η [see Eqs. (1.6)] is found in an explicit form by means of the perturbation theory based upon the inverse scattering transform.[19] It is also demonstrated that, in the general case ($c\neq0$), the emission intensity is asymmetric. Owing to this circumstance, the radiative decay of the vector soliton is accompanied by its recoil-induced acceleration.

In the concluding section, Sec. VI, the one-soliton dynamics are analyzed within the framework of a generalized model incorporating the optical activity effect (i.e., rotation of the polarization induced, e.g., by a homogeneous twist of the fiber). The optical activity is accounted for by additional linear-coupling terms in Eqs. (1.1). In this case, the general four-dimensional dynamical system for the polarizational and positional degrees of freedom θ and $\tau^{(0)}$ can be again integrated to a two-dimensional conservative system. Possible phase portraits of that system are investigated in some detail. A general inference is that it does not admit dynamical trajectories corresponding to a permanent rotation of the polarization, i.e., going from $\theta=-\infty$ to $+\infty$. Only limited oscillations of the polarization prove to be possible.

II. THE POLARIZATION DYNAMICS OF A VECTOR SOLITON

To derive evolution equations for the slowly varying quantities θ, $\tau^{(0)}$, and ψ' in the case when the parameters β and β' in Eqs. (1.4) are small, let us insert the unperturbed solitonic wave form (1.6), with z-dependent ϕ, ψ', θ, and $\tau^{(0)}$, into the Lagrangian density (1.5). Next, one should calculate the full Lagrangian $L\equiv\int_{-\infty}^{+\infty}\mathcal{L}\,d\tau$ and define, with regard to Eq. (1.7b), the generalized momenta p_ψ and p_τ conjugate to the independent generalized coordinates ψ' and $\tau^{(0)}$:

$$p_\psi\equiv\frac{\partial L}{\partial\dot{\psi}'}=2\eta\cos(2\theta)\ , \tag{2.1a}$$

$$p_\tau\equiv\frac{\partial L}{\partial\dot{\tau}^{(0)}}=2\eta\dot{\tau}^{(0)}\ . \tag{2.1b}$$

Proceeding from Lagrangian to the Hamiltonian description, one may introduce the density of the perturbation Hamiltonian:

$$\mathcal{H}_{\text{pert}}=-\omega|V|^2+\omega|U|^2-2\beta|U|^2|V|^2 -\beta'V^2(U^*)^2e^{2ic\tau}-\beta'(V^*)^2U^2e^{-2ic\tau} \tag{2.2}$$

corresponding to Eqs. (1.4) [the terms $\sim\pm\omega$ in Eqs. (1.4) are also treated as a perturbation]. The next step is to define the full perturbation Hamiltonian

$$H_{\text{pert}}\equiv\int_{-\infty}^{+\infty}\mathcal{H}\,d\tau=4\eta\omega\cos(2\theta)-\tfrac{2}{3}\eta^3\sin^2(2\theta)\{8\beta+\beta'\pi(c/\eta)[(c/\eta)^2+4][\sinh(\pi c/2\eta)]^{-1}\cos(2\psi)\}\ , \tag{2.3}$$

where

$$\psi\equiv\psi'+c\tau^{(0)}\ . \tag{2.4}$$

The canonical equations of motion

$$\dot{p}_\psi=-\frac{\partial H_{\text{pert}}}{\partial\psi'},\quad \dot{\psi}'=\frac{\partial H_{\text{pert}}}{\partial p_\psi}\ , \tag{2.5a}$$

$$\dot{p}_\tau=-\frac{\partial H_{\text{pert}}}{\partial\tau^{(0)}}\ , \tag{2.5b}$$

yield, on inserting Eqs. (2.1), (2.3), and (2.4), equations of motion which can be written in the following form:

$$\frac{d\theta}{d\zeta}=\sin(2\theta)\sin(2\psi)\ , \tag{2.6}$$

$$\frac{d^2\psi}{d\zeta^2}=-4[B+\cos(2\psi)]\sin^2(2\theta)\sin(2\psi) -4\cos(2\theta)\sin(2\psi)\frac{d\psi}{d\zeta}\ , \tag{2.7}$$

$$\frac{d\psi'}{d\zeta}=K'+2[B'+\cos(2\psi)]\cos(2\theta)\ , \tag{2.8}$$

where

$$\zeta\equiv\tfrac{1}{3}\beta'\eta^2\pi(c/\eta)[(c/\eta)^2+4][\sinh(\pi c/2\eta)]^{-1}z\ , \tag{2.9}$$

$$B'\equiv8(\beta/\beta')\sinh(\pi c/2\eta)[(c/\eta)^2+4]^{-1}(\pi c/\eta)^{-1}\ , \tag{2.10a}$$

$$B\equiv8(\beta/\beta'+3c^2/16\beta'\eta^2)\sinh(\pi c/2\eta) \times\{\pi(c/\eta)[(c/\eta)^2+4]\}^{-1}\ ,$$

$$K'\equiv6(\omega/\beta')\sinh(\pi c/2\eta)\{\pi c\eta[(c/\eta)^2+4]\}^{-1}\ . \tag{2.10b}$$

In the particular case $c=0$, when, according to Eq. (2.4), $\psi\equiv\psi'$, Eqs. (2.6) and (2.8) coincide with equations deduced in Ref. 13, while Eq. (2.7) becomes a corollary of those two equations.

Obviously, Eq. (2.8) separates from the third-order system of Eqs. (2.6) and (2.7). The latter equation can be simplified if one substitutes the product $\sin(2\theta)\times\sin(2\psi)$ in the first term on its right-hand side by $d\theta/d\zeta$ [see Eq. (2.6)]. After this substitution, Eq. (2.7) can be integrated once to yield the following equation:

$$\frac{d\psi}{d\zeta}=K+2[B+\cos(2\psi)]\cos(2\theta) \tag{2.11}$$

[cf. Eq. (2.8)], where K is an arbitrary constant of integration. Thus Eqs. (2.6) and (2.11) constitute the eventual two-dimensional two-parametric (B and K) dynamical system governing evolution of polarization of the vector soliton.

Let us investigate a phase portrait of the system (2.6) and (2.11). First of all, the conservation of the Hamiltonian gives rise to the following integral of motion of this system:

$$K\cos(2\theta)-[B+\cos(2\psi)]\sin^2(2\theta)=C\ , \tag{2.12}$$

C being an arbitrary constant. Setting $C=\pm K$ in Eq. (2.12), one finds the solutions $\theta=0$ and $\pi/2$ corresponding to the simple solitons. The vector solitons correspond to the following stationary points of the system of equations (2.6) and (2.11):

$$\sin(2\psi)=0\ ,\qquad \cos(2\theta)=-(K/2)[B+\cos(2\psi)]^{-1}\ . \tag{2.13}$$

The expression (2.13) comprises two different vector solitons with $\cos(2\psi)=\pm 1$, i.e., with $\psi=0$ and $\pi/2$.

Since K is an arbitrary constant, Eq. (2.13) means that there is a pair of vector solitons for each value of the polarization angle θ: One gets the solitons corresponding to a given θ setting $K=-2(B\pm 1)\cos(2\theta)$, where $\pm 1 \equiv \cos(2\psi)$. This is a drastic difference from the particular case $c=0$ considered in Ref. 13 (and also from the cw case corresponding to $u_\tau = v_\tau \equiv 0$), where stationary solitons were possible only for three special values of θ: $\theta=0$, $\theta=\pi/2$, and $\cos(2\theta)=-\frac{1}{2}K'\text{sgn}(\beta/\beta')$, the parameter K' being defined by Eq. (2.10b). The reason for this difference is that in the particular case $c=0$ the positional degree of freedom is decoupled from the polarizational one. Formally, this is reflected by the fact that the two-dimensional dynamical system for the variables θ and ψ' [see Eqs. (2.6) and (2.8)] derived in Ref. 13 contains the single parameter K' instead of the two ones K and B in the system of Eqs. (2.6) and (2.11).

It is straightforward to investigate the stability of the stationary solutions (2.13) against infinitesimal disturbances within the framework of the linearized system of Eqs. (2.6) and (2.11). The stability condition takes the form

$$\cos(2\psi)[B+\cos(2\psi)]\geq 0\ . \tag{2.14}$$

According to Eq. (2.14), both vector solitons [corresponding to $\cos(2\psi)=+1$ and to $\cos(2\psi)=-1$] are stable in the case $|B|<1$. In the opposite case $|B|>1$, only the soliton with $\cos(2\psi)=\text{sgn}B$ is stable.

Subsequent investigation of the phase plane (ψ,θ) based on Eqs. (2.6), (2.11), and (2.12) brings us to the following conclusions. In the case

$$|K|>2(|B|+1) \tag{2.15}$$

the vector solitons (2.13) do not exist, and both simple solitons ($\theta=0$ and $\pi/2$) are stable. In this case the phase plane (ψ,θ) is trivial (Fig. 1). As a matter of fact, this situation (with $|B|=1$) has been already considered in Ref. 13.

At $|K|=2(|B|+1)$ there appears the vector soliton (2.13) with $\cos(2\psi)=\text{sgn}B$. In the range

$$2(|B|-1)<|K|<2(|B|+1) \tag{2.16}$$

the phase plane takes the form shown in Fig. 2: One of the two simple solitons (corresponding to $\theta=0$ in Fig. 2) is unstable, and the vector soliton is stable. This situation (with $|B|=1$) has also been considered previously in Ref. 13.

At $|K|=2(|B|-1)$ there appears the second vector soliton (2.13) with $\cos(2\psi)=-\text{sgn}B$. This soliton is unstable. Simultaneously, the stability of the simple soliton (the one with $\theta=0$ in Fig. 2) is retrieved. In the range

$$|K|<2(|B|-1) \tag{2.17}$$

we have two stable simple solitons, the stable vector soliton, and the unstable one (Fig. 3). This case was absent in Ref. 13, as it was presumed there that $|B|=1$ ($c=0$). The separatrix connecting the saddles S' (depicted by the bold line in Fig. 3) corresponds to

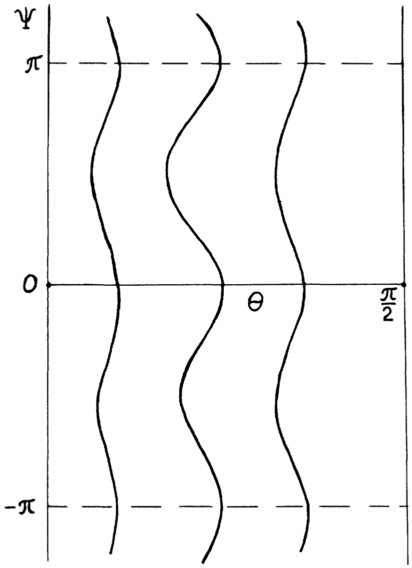

FIG. 1. The phase plane of the dynamical system of Eqs. (2.6) and (2.11) in the case (2.15). The stable simple soliton corresponds to the lines $\theta=0$ and $\pi/2$.

$$C=-(\text{sgn}B)\tfrac{1}{4}[K^2+(|B|-1)^2]/(|B|-1) \tag{2.18}$$

in Eq. (2.12). It is noteworthy that the separatrix is very "sensitive" to additional perturbations explicitly depen-

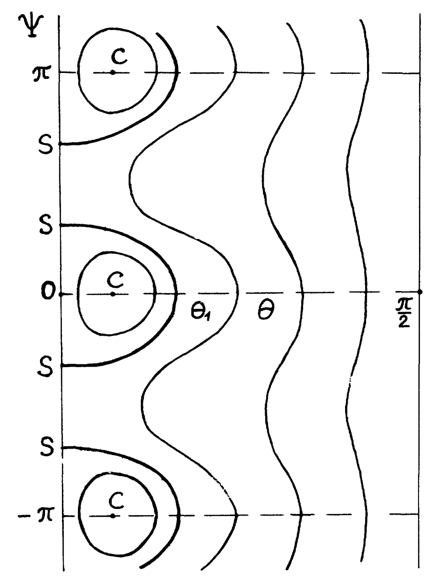

FIG. 2. The phase plane of Eqs. (2.6) and (2.11) in the case (2.16). For definiteness, it is assumed $KB<0$. The unstable simple soliton corresponds to the saddles S with the coordinates $\theta=0$, $\cos(2\psi)=-(B+\frac{1}{2}K)$, and the stable vector soliton corresponds to the center C with the coordinates $\sin(2\psi)=0$, $\cos(2\theta)=\frac{1}{2}|K|/(|B|+1)$, see Eqs. (2.13). The coordinate of the point θ_1 is given by Eq. (2.20).

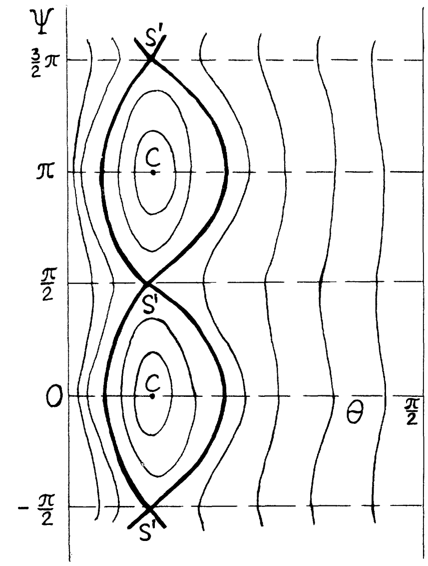

FIG. 3. The phase plane of Eqs. (2.6) and (2.11) in the case (2.18). The stable vector soliton corresponds to the center C with the same coordinates as in Fig. 2, and the unstable vector soliton is the saddle S' with the coordinates $\sin(2\psi)=0$, $\cos(2\theta)=\frac{1}{2}|K|/(|B|-1)$.

dent on the evolutional variable ζ [i.e., as a matter of fact, on the coordinate z, see Eq. (2.9)]: A small ζ-periodic or random perturbation can generate a narrow stochastic layer in a vicinity of the separatrix (see, e.g., Ref. 20). A perturbation of this type may naturally arise if the birefringent fiber is periodically or randomly inhomogeneous. Thus one may expect appearance of a vector soliton with chaotically varying polarization in the regime (2.17), provided a weak spatial inhomogeneity is taken into account.[18]

It was implied above that $|B|$ takes values $|B|>1$. In the case $\beta<0$, the values $|B|<1$ are possible too according to Eq. (2.10). Let us consider the phase plane of the system of Eqs. (2.6) and (2.11) with $|B|<1$. Using Eq. (2.12) one can readily find that in the case $|K|>2(|B|+1)$ the phase plane takes the form shown in Fig. 1. At $2(1-|B|)<|K|<2(|B|+1)$ the situation is the same as in Fig. 2. In the case

$$|K|<2(1-|B|)\ , \tag{2.19}$$

which is only possible if $|B|<1$, one reveals a qualitatively new situation illustrated by Fig. 4: Both simple solitons are unstable (they correspond to the saddles S and $\tilde{S}$), and there are two stable vector solitons (the center C and $\tilde{C}$), the one $\tilde{C}$ appearing at $|K|=2(1-|B|)$. The separatrices connecting pairs of the saddles S or $\tilde{S}$ are boundaries between confined and free dynamical trajectories. The separatrices intersect the θ axis ($\psi=0$) at $\theta=\theta_1$ and at $\theta=\theta_2$, where

$$\cos^2\theta_1=\tfrac{1}{2}|K|/(1+|B|)\ ,\qquad \sin^2\theta_2=\tfrac{1}{2}|K|/(1-|B|) \tag{2.20}$$

(the same expression for θ_1 pertains to Fig. 2). As follows from Eqs. (2.20), the situation shown in Fig. 4(a) ($\theta_1<\theta_2$) takes place if

$$|K|>1-B^2\ , \tag{2.21}$$

and in the opposite case the separatrices are located as shown in Fig. 4(b). Note that the inequality (2.19) is a

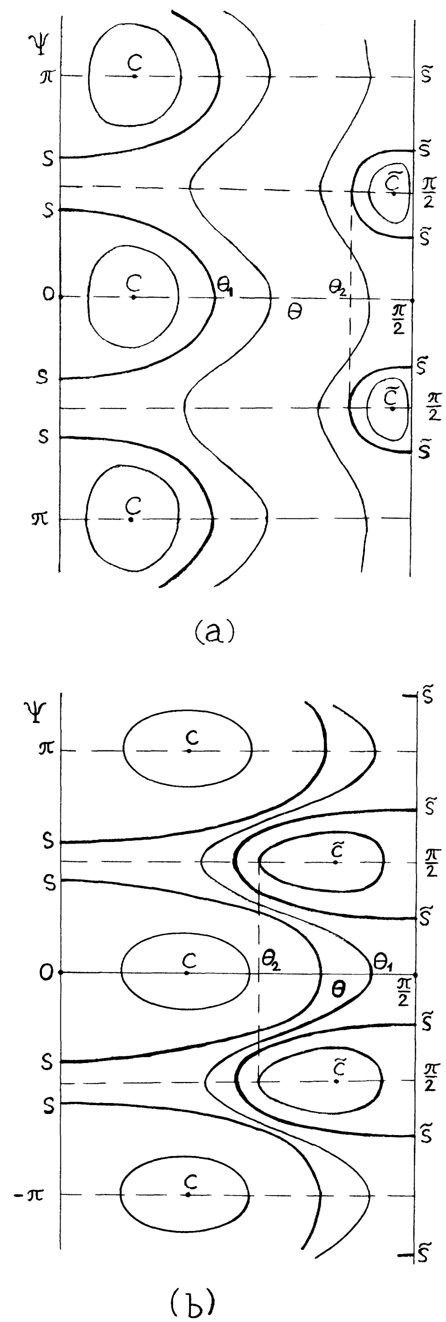

FIG. 4. The phase plane of Eqs. (2.6) and (2.11) in the case (2.19): (a) $\theta_1<\theta_2$, i.e., $1-B^2<|K|<2(1-|B|)$ [see Eqs. (2.19) and (2.21)]; (b) $\theta_2<\theta_1$, i.e., $|K|<1-B^2$.

corollary of the one $|K| < 1 - B^2$.

To conclude this section, let us discuss the meaning of generic (oscillatory) dynamical trajectories on the phase planes (Figs. 1–4). As follows from the definition of the variable ψ [given by Eq. (2.4)] and the additional evolution equation (2.8) for the phase ψ', the soliton's "coordinate" $\tau^{(0)}$ evolves according to the following equation:

$$\frac{d}{d\zeta}\tau^{(0)} = c^{-1}[(K-K') + 2(B-B')\cos(2\theta)] \ . \tag{2.22}$$

According to Eq. (2.22), oscillations of the polarization angle θ are accompanied by oscillations of the position of the soliton. Besides, the soliton moves with some mean velocity. However, the mean velocity is, as a matter of fact, an additional arbitrary parameter of the solution, since the underlying equations (1.1) are invariant with respect to the Galilean transformations. At last, in the particular case $c=0$ considered in Ref. 13, when the polarizational and positional degrees of freedom are decoupled, the oscillations of the soliton's position are absent.

III. A BOUND STATE OF TWO VECTOR SOLITONS

In this section we will consider interaction between two vector solitons with different (z-dependent) parameters ϕ_j, ψ_j', θ_j, and $\tau_j^{(0)}$ ($j=1,2$) [see Eqs. (1.3)], and with close amplitudes η_j. The analysis will be confined to the simplest particular case $c=0$.

The soliton-soliton interaction will be analyzed under the fundamental assumption that overlapping between the solitons is weak, i.e., $e^{-2\eta_{1,2}L} \ll 1$, where $L \equiv |\tau_1^{(0)} - \tau_2^{(0)}|$ is proportional to the distance between the centers of the solitons, see Eqs. (1.6). Straightforward analysis following the lines of that developed previously for slightly overlapping solitons in the single NS equation[21] makes it possible to find an effective potential of the soliton-soliton interaction:

$$W_{\text{eff}}(L) \approx -128\eta^3 e^{-2\eta L}\{\cos\theta_1\cos\theta_2\cos[\phi - \tfrac{1}{2}(\psi_1' - \psi_2')] + \sin\theta_1\sin\theta_2\cos[\phi + \tfrac{1}{2}(\psi_1' - \psi_2')]\} \ , \tag{3.1}$$

where $\phi \equiv \phi_1 - \phi_2$, and it is implied that η_1 and η_2 take a common value η. In other words, the full Hamiltonian of the two-soliton system is

$$H = H_0 + \sum_{j=1}^{2} H_{\text{pert}}^{(j)} + W_{\text{eff}} \ , \tag{3.2}$$

where

$$H_0 = -\tfrac{16}{3}(\eta_1^3 + \eta_2^3) + \eta_1(\dot{\tau}_1^{(0)})^2 + \eta_2(\dot{\tau}_2^{(0)})^2 \tag{3.3}$$

is the unperturbed Hamiltonian, and $H_{\text{pert}}^{(j)}$ are the perturbation-induced terms (2.3) for both solitons. The definition of the generalized momenta $(p_\psi)_j$ conjugate to the generalized coordinates ψ_j' retains the form of Eq. (2.1a),

$$(p_\psi)_j = 2\eta\cos(2\theta_j) \tag{3.4}$$

[here, as well as in Eq. (3.1), a difference between η_1 and η_2 may be neglected]. In the analysis developed in the preceding section for the single soliton, we did not deal with an equation of motion for the variable ϕ, as it was not coupled to Eqs. (2.5). This time, equations for the phases ϕ_j must be taken into account. Using the underlying Lagrangian density (1.5), it is easy to arrive at the well-known expression for the generalized momenta conjugate to ϕ_j:

$$(p_\phi)_j \equiv \frac{\partial L}{\partial \phi_j} = -4\eta_j \ . \tag{3.5}$$

At last, the generalized coordinates $\tau_j^{(0)}$ are conjugate to the momenta [cf. Eq. (2.1b)]

$$(p_\tau)_j = 2\eta_j \dot{\tau}_j^{(0)} \ . \tag{3.6}$$

Having the full Hamiltonian given by Eqs. (3.1), (3.2), and (3.3) and the definition of the momenta conjugate to the independent generalized coordinates ψ_j', ϕ_j, and $\tau_j^{(0)}$ [see Eqs. (3.4)–(3.6)], one can immediately write the canonical Hamilton's equations of motion. In a general case, those equations have a rather cumbersome form. However, it is easy to realize that there is the reduction

$$\theta_1 = -\theta_2 \equiv \theta, \quad \psi_1' = \psi_2' \equiv \psi \tag{3.7}$$

compatible with the equations of motion. In what follows we will concentrate on this particular case. Insertion of Eq. (3.7) into Eq. (3.1) yields the simplified form of the interaction Hamiltonian:

$$W_{\text{eff}}(L) = -128\eta^3 e^{-2\eta L}(\cos\phi)\cos(2\theta) \ . \tag{3.8}$$

Choosing the quantities ψ, ϕ, and $L \equiv \tau_1^{(0)} - \tau_2^{(0)}$ as new independent generalized coordinates, and redefining the corresponding conjugate momenta, one can bring the canonical equations of motion into the following eventual form:

$$\frac{d\theta}{d\zeta} = \sin(2\theta)\sin(2\psi) \ , \tag{3.9a}$$

$$\frac{d\psi}{d\zeta} = K' + \kappa(\cos\phi)e^{-l} + 2[B + \cos(2\psi)]\cos(2\theta) \ , \tag{3.9b}$$

$$\frac{d^2 l}{d\zeta^2} = \tfrac{1}{18}\kappa^2 e^{-l}(\cos\phi)\cos(2\theta) \ , \tag{3.9c}$$

$$\frac{d^2\phi}{d\zeta^2} = \tfrac{1}{18}\kappa^2 e^{-l}(\sin\phi)\cos(2\theta) \ , \tag{3.9d}$$

where the quantities ζ, K', and B are defined by Eqs. (2.9) and (2.10) ($B=\pm 1$ and $K' = 3\omega/4\beta'\eta^2$ in the considered particular case $c=0$), $l \equiv 2\eta L$, and

$$\kappa \equiv 36/\beta' \ . \tag{3.10}$$

Note that κ is a large quantity as, from the very beginning, the coefficient β' was assumed to be small.

The dynamical system (3.9) has a family of stationary solutions

$$\cos(2\theta)=0\ ,\quad \sin(2\psi)=0\ ,\quad e^{-l}=K'/\kappa\cos\phi \tag{3.11}$$

with an arbitrary relative phase ϕ (recall $\phi\equiv\phi_1-\phi_2$). Evidently, the stationary solution (3.11) describes a bound state of two vector solitons. A very important issue is stability of the bound state against small perturbations. In the present work we will confine ourselves to the perturbations that do not violate the fundamental reduction (3.7).

Linearizing Eqs. (3.9) on the background of the stationary solution (3.11), one can obtain a sixth-degree equation for the instability growth rate γ. That equation has two zero roots, and other four of them are determined by the equation

$$(\gamma^2)^2\cos^2\phi+8[B+\cos(2\psi)]\cos(2\psi)(\cos^2\phi)\gamma^2 + \tfrac{2}{9}\kappa^2 K' e^{-l}\cos(2\psi)\cos\phi\ . \tag{3.12}$$

To provide the stability of the stationary solution (3.11) ($\mathrm{Re}\gamma\le 0$), one should demand that Eq. (3.12), regarded as a square equation for γ^2, must have only real negative roots. It is easy to find that this requirement amounts to the system of three inequalities:

$$\cos(2\psi)[B+\cos(2\psi)]\ge 0\ , \tag{3.13a}$$

$$K'(\cos\phi)\cos(2\psi)\ge 0\ , \tag{3.13b}$$

$$72[B+\cos(2\psi)]^2\ge|\kappa|(K')^2/\cos^2\phi\ . \tag{3.14}$$

Note that the stability condition (3.13a) for the bound state coincides with that (2.10′) for the single vector soliton.

Let us recall that in the stationary states $\cos(2\psi)=\pm 1$ [see Eq. (3.11)], and in the case under consideration $B=\pm 1$. Thus the stability condition (3.14) cannot be satisfied by the stationary state with $\cos(2\psi)=-B$. As for the state with $\cos(2\psi)=+B$, it was stressed above that the parameter κ was a large quantity $\sim(\beta')^{-1}$, see Eq. (3.10). Thus, in a generic case ($|\omega|\sim|\beta|\equiv|\beta'|$, $c\lesssim\eta$) the right-hand side of Eq. (3.14) is large, while the left-hand side is not. To satisfy the inequality (3.14), we should assume that the parameter K' is small, so that

$$|\kappa|(K')^2/\cos^2\phi\lesssim 1\ . \tag{3.15}$$

Inserting the definitions of κ and K', given by Eqs. (3.10) and (2.10b), into the inequality (3.15) (and assuming, as above, $c\lesssim\eta$), one arrives at the following condition necessary for the stability of the bound state of two vector solitons:

$$\omega^2\lesssim|\beta|^3\cos^2\phi\ . \tag{3.16}$$

It is important to note that the underlying assumption $e^{-l}\ll 1$ means that the expression $K'/\kappa\cos\phi$ must be small, see Eq. (3.11). This requirement amounts to the following inequality:

$$\omega^2\gg\beta^4\cos^2\phi\ . \tag{3.17}$$

Evidently, the inequalities (3.16) and (3.17) are compatible due to our underlying assumption that β is a small parameter.

It is relevant to analyze a possibility of coexistence of the stable two-soliton bound state with stable one-soliton states. The smallness of the parameter K', necessary for the stability of the bound state, means that we deal with the case (2.16), when one simple soliton and one vector soliton are stable (Fig. 2) (recall that in the case $c=0$ K' plays the role of K). According to Eq. (2.13), the smallness of K' implies as well that the polarization angle θ of the free vector soliton is close to $\pi/4$, i.e., the free solitons are close in form to the bound ones whose polarization is exactly $\pi/4$, see Eq. (3.11). To see which vector-soliton state, free or bound, is "more stable" in the case when both are stable against the infinitesimal perturbations, it is natural to calculate the "binding energy" of the bound state, i.e., a difference between the sum of the Hamiltonians of the free solitons and the full Hamiltonian of their bound state. As is well known, the single NS equation has exact solutions (sometimes called breathers) describing two-soliton bound states, but their binding energy is exactly equal to zero.[22] To define the binding energy, we will deal with the Hamiltonian of the dimensionless system (2.6) and (2.8), which is nothing but the quantity C determined by Eq. (2.12), with K replaced by K' ($B'\equiv B$ in the case $c=0$). It is important that a coefficient relating the dimensionless Hamiltonian C to the original one H defined by Eq. (2.3) be positive. According to Eqs. (2.9) and (2.10b) this implies $\beta'>0$. As was mentioned in the Introduction, β' can always be chosen positive, which will be assumed in what follows. The full Hamiltonian (3.2) contains also the unperturbed term (3.3) which is the same for the solitons in the free and bound states, provided their amplitudes η are equal [it is implied $\dot{\tau}_j^{(0)}=0$], and the interaction potential W_{eff}, which is equal to zero in the stationary bound state according to Eqs. (3.8) and (3.11). So, it is sufficient to compare the values of the dimensionless Hamiltonian (2.12) for the free and bound states.

Insertion of Eqs. (3.11) into Eq. (2.12) yields

$$C=C_b\equiv=[B+\cos(2\psi)] \tag{3.18}$$

for the bound state, and insertion of Eqs. (2.13) (with K replaced by K') yields

$$C=C_f\equiv-\tfrac{1}{4}\{(K')^2+4[B+\cos(2\psi)]^2\}\times[B+\cos(2\psi)]^{-1} \tag{3.19}$$

for the two free solitons. At last, the "binding energy" is

$$E_b\equiv C_f-C_b=-\tfrac{1}{8}(K')^2B\ , \tag{3.20}$$

where it has been taken into account that we consider the case $|B|=1$, and in the stationary stable state $\cos(2\psi)=\mathrm{sgn}B$ according to Eqs. (2.14) and (3.13a).

The full Hamiltonian is an integral of motion of Eqs. (1.1) (in the optical-fiber theory, it has the physical meaning of the conserved full modulation). If the binding energy is negative, i.e., $B>0$ according to Eq. (3.20), the bound state may, in principle, decay into free vector solitons, shedding off the excessive energy in the form of radiation (quasilinear waves), whose Hamiltonian is strictly positive (of course, this possibility does not imply an instability of the bound state against infinitesimal disturbances). The decay is not possible if the binding energy is positive, i.e., $B<0$. According to the definition of B [see Eq. (2.10a)], and to the fact that β' has been chosen positive, this means that the bound state is absolutely stable against the decay provided $\beta<0$. Recall that the sign of the cross-phase-modulation coefficient β in Eqs. (1.1), unlike that of β', is invariant.

IV. PERIODIC CHAINS OF VECTOR SOLITONS

Generalizing the analysis of the bound states of vector solitons, it is natural to consider periodic chains of solitons (the attention is again confined to the simplest case $c=0$). We assume that in an equilibrium state the chain's spacing L_0, i.e., the time delay between adjacent solitons, is constant:

$$\tau_n^{(0)}=L_0 n\ , \tag{4.1}$$

$\tau_n^{(0)}$ being the coordinate of the nth soliton [see Eq. (1.6)]. Next it is necessary to specify the phase shifts $\phi_n-\phi_{n-1}$ between adjacent solitons in the equilibrium state [the phases ϕ_n are defined according to Eq. (1.6)]. Two different types of the chain are possible: (i)

$$\phi_n=\phi_0 n$$

[cf. Eq. (4.1)], and (ii)

$$\phi_n=\tfrac{1}{2}(-1)^n\phi_0\ ,$$

ϕ_0 being a constant.

The spacing L_0 in both cases (i) and (ii), as well as the phase shift ϕ_0 in the case (i), are arbitrary parameters, i.e., a soliton chain must exist at any values of these parameters. In what follows, only rarefield chains will be considered:

$$l_0\equiv 2\eta L_0\gg 1\ . \tag{4.2}$$

In the case (ii), ϕ_0 is not arbitrary; this parameter will be found below from a stationary solution of equations of motion.

Proceeding to analysis of dynamics of the soliton chains, we will confine our attention to their internal oscillations of the simplest type: The variable parameter $\psi(z)$ is the same for all the solitons, and the variable spacings

$$l(z)\equiv 2\eta(\tau_{2n}^{(0)}-\tau_{2n-1}^{(0)}) \tag{4.3}$$

are the same for all n, while the double spacings remain constant:

$$\tau_{2n}^{(0)}-\tau_{2n-2}^{(0)}=\tau_{2n+1}^{(0)}-\tau_{2n-1}^{(0)}\equiv 2l_0\ . \tag{4.4}$$

At last, following Eq. (3.7), we set

$$\theta_n=(-1)^n\theta(t)\ . \tag{4.5}$$

As for the phase shifts, in the case (i) it is assumed that [cf. Eq. (4.3)]

$$\phi(t)\equiv\phi_{2n}-\phi_{2n-1} \tag{4.6}$$

are the same for all n, and the double phase shifts remain constant:

$$\phi_{2n}-\phi_{2n-2}=\phi_{2n+1}-\phi_{2n-1}\equiv 2\phi_0\ , \tag{4.7}$$

cf. Eq. (4.4). In the case (ii), the variable phase shift is defined as follows:

$$\phi_{2n}-\phi_{2n-1}=\phi_{2n}-\phi_{2n+1}\equiv\phi(t)\ . \tag{4.8}$$

Let us choose the variables $\psi(t)$, $\phi(t)$, and $l(t)$ as independent generalized coordinates. Using the expressions for the full one-soliton Hamiltonian $H_1\equiv H_0+H_{\text{pert}}$ [where H_0 and H_{pert} are defined, respectively, by Eqs. (3.3) and (2.3)] and for the potential (3.8) of the soliton-soliton interaction, one can derive dynamical equations which take the following final form: In the case (i),

$$\frac{d\theta}{d\zeta}=\sin(2\theta)\sin(2\psi)\ , \tag{4.9a}$$

$$\frac{d\psi}{d\zeta}=K'-\kappa[e^{-l}\cos\phi+e^{-(2l_0-l)}\cos(2\phi_0-\phi)]+2[B+\cos(2\psi)]\cos(2\theta)\ , \tag{4.9b}$$

$$\frac{d^2l}{d\zeta^2}=\tfrac{1}{18}\kappa^2[e^{-l}\cos\phi-e^{-(2l_0-l)}\cos(2\phi_0-\phi)]\cos(2\theta)\ , \tag{4.9c}$$

$$\frac{d^2\phi}{d\zeta^2}=\tfrac{1}{18}\kappa^2[e^{-l}\sin\phi-e^{-(2l_0-l)}\sin(2\phi_0-\phi)]\cos(2\theta)\ , \tag{4.9d}$$

and in the case (ii),

$$\frac{d\theta}{d\zeta}=\sin(2\theta)\sin(2\psi)\ , \tag{4.10a}$$

$$\frac{d\psi}{d\zeta}=K'-\kappa(e^{-l}+e^{-(2l_0-l)})\cos\phi+2[B+\cos(2\psi)]\cos(2\theta)\ , \tag{4.10b}$$

$$\frac{d^2l}{d\zeta^2}=\tfrac{1}{18}\kappa^2(e^{-l}-e^{-(2l_0-l)})\cos\phi\cos(2\theta)\ , \tag{4.10c}$$

$$\frac{d^2\phi}{d\zeta^2}=\tfrac{1}{18}\kappa^2(e^{-l}+e^{-(2l_0-l)})\sin\phi\cos(2\theta)\ . \tag{4.10d}$$

In Eqs. (4.9) and (4.10), the quantities K', B, κ, and ζ are the same as in Eqs. (3.9).

A general stationary solution to Eqs. (4.9) is

$$\sin(2\theta)\sin(2\psi)=0\ , \tag{4.11a}$$

$$K'-2\kappa e^{-l_0}\cos\phi_0+2[B+\cos(2\psi)]\cos(2\theta)=0\ . \tag{4.11b}$$

Equations (4.10) have two stationary solutions:

$$\sin(2\psi)=0\ ,$$
$$\cos(2\theta)=0\ , \tag{4.12}$$
$$\cos\phi_0=(K'/2\kappa)e^{l_0}\ ,$$

and

$$\sin(2\psi)=0\ ,$$
$$\sin\phi_0=0\ , \tag{4.13}$$
$$\cos(2\theta)=\tfrac{1}{2}(2\kappa e^{-l_0}\cos\phi_0-K')/[B+\cos(2\psi)]\ .$$

Recall that the parameters l_0 and ϕ_0 in the solution (4.11), as well as l_0 (but *not* ϕ_0) in the solutions (4.12) and (4.13), are arbitrary. Note the similarity between the solution (4.12) and the solution (3.11) of Eqs. (3.9) for the two-soliton state.

Investigation of the stability of the solutions (4.11), (4.12), and (4.13) within the framework of the linearized equations (4.9) and (4.10) yields the following results. For the solutions (4.11) and (4.13), the linearized equations for infinitesimal disturbances of the variables l and ϕ detach from those for θ and ψ, and straightforward analysis of the subsystem for l and ϕ demonstrates that it gives rise to an instability. Thus the solutions (4.11) and (4.13) are unstable.

For the solution (4.12), only the linearized equation for the disturbance of l is detached from the other equations, and it yields two zero roots for the instability growth rate γ (recall that two zero roots have also been obtained when investigating the stability of the two-soliton bound state in the preceding section). The other four roots are determined by the equation

$$(\gamma^2)^2\cos^2\phi_0+8[B+\cos(2\psi)]\cos(2\psi)\cos^2\phi_0\gamma^2 + \tfrac{4}{9}\kappa^2K'e^{-l_0}\cos(2\psi)\sin^2\phi_0=0 \tag{4.14}$$

similar to Eq. (3.12). The stability conditions ensuing from Eq. (4.14) (i.e., that γ^2 must be real negative) seem similar to Eqs. (3.13):

$$[B+\cos(2\psi)]\cos(2\psi)\geq 0\ , \tag{4.15a}$$
$$K'\cos(2\psi)\geq 0\ , \tag{4.15b}$$
$$72[B+\cos(2\psi)]^2\geq|\kappa|(K')^2\tan^2\phi_0\ . \tag{4.15c}$$

Despite the similarity, the stability conditions (4.15) are easier to satisfy. Indeed, let us recall that the difficulty with Eq. (3.15) was in the fact that the coefficient $|\kappa|(K')^2$ was large. In Eq. (4.15c), this can be compensated for the smallness of $\tan^2\phi_0$.

To interpret the solution (4.12) and its stability conditions, let us note its difference from the similar solution (3.11) for the two-soliton bound state. The latter solution determines the equilibrium "distance" l in terms of the arbitrary phase parameter ϕ. The former solution determines the phase difference ϕ_0 in terms of the given chain's spacing l_0. This solution exist in the range

$$e^{l_0}\leq 2|\kappa/K'| \tag{4.16}$$

[recall that κ is a large parameter $\sim\beta^{-1}$ according to its definition (3.10), so that the inequality (4.16) does not contradict the underlying assumption (4.2)]. If one follows the evolution of the solution (4.12) with the growth of the parameter e^{-l_0}, one notes that at $e^{l_0}=2|\kappa/K'|$, when the solution appears, $\sin\phi_0=0$, so that the nascent solution is stable according to Eq. (4.15c). Moreover, according to Eq. (4.12), at this point there appear, as a matter of fact, two different solutions with

$$\sin\phi_0=\pm[1-(K'/2\kappa)^2e^{2l_0}]^{1/2}\ , \tag{4.17}$$

and *both* are stable. The corresponding bifurcation diagram is shown schematically in Fig. 5. The unstable branch with $\sin\phi_0\equiv 0$ (depicted by the dashed line in Fig. 5), existing both at $e^{l_0}>2|\kappa/K'|$ and at $e^{l_0}<2|\kappa/K'|$, is nothing but the solution (4.13).

Thus, according to Eqs. (4.15c) and (4.16), the stable stationary solution (4.12) exists in the interval

$$(K')^2\equiv\epsilon_1\leq\epsilon\equiv 4\kappa^2e^{-2l_0} \leq (K')^2+72|\kappa|^{-1}(|B|+1)^2\equiv\epsilon_2\ . \tag{4.18}$$

In Eq. (4.18) it has been taken into account that Eq. (4.15a), coinciding with Eqs. (2.14) and (3.13a), dictates that one should choose $\cos(2\psi)=\mathrm{sgn}B$ [recall $\cos(2\psi)=\pm 1$ according to Eq. (4.12)]. In the generic case, when K' is not especially small as in the case (3.16), the interval (4.18) is narrow because κ is large.

As follows from Eq. (4.14), the instability setting in at $\epsilon=\epsilon_2$ is oscillatory since the corresponding instability growth rate γ is complex. Thus it is natural to expect that at $\epsilon=\epsilon_2$ there appears some ζ-periodic solution, the corresponding frequency being $\omega_0\approx 2\sqrt{2}$ according to Eq. (4.14). In principle, one can study that solution by means of expanding in powers of $(\epsilon-\epsilon_2)$, but this issue will not be considered here.

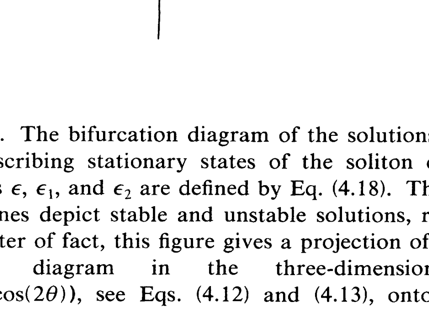

FIG. 5. The bifurcation diagram of the solutions (4.12) and (4.13) describing stationary states of the soliton chain. The quantities ϵ, ϵ_1, and ϵ_2 are defined by Eq. (4.18). The solid and dashed lines depict stable and unstable solutions, respectively. As a matter of fact, this figure gives a projection of the full bifurcation diagram in the three-dimensional space $(\epsilon,\sin\phi_0,\cos(2\theta))$, see Eqs. (4.12) and (4.13), onto the plane $(\epsilon,\sin\phi_0)$.

V. RADIATIVE DECAY OF A VECTOR SOLITON

In this section we will deal with a vector soliton which consists of a large U component coupled to a small component in the V subsystem. Let us rewrite Eqs. (1.4) in terms of the new wave fields $\tilde{U}\equiv Ue^{i\omega z}$ and $\tilde{V}\equiv Ve^{-i\omega z}$:

$$i\tilde{U}_z+\tilde{U}_{\tau\tau}+2(|\tilde{U}|^2+|\tilde{V}|^2)\tilde{U}+2\beta|\tilde{V}|^2\tilde{U}+2\beta'\tilde{V}^2\tilde{U}^*e^{2ic\tau+4i\omega z}=0\ , \quad (5.1a)$$

$$i\tilde{V}_z+\tilde{V}_{\tau\tau}+2(|\tilde{V}|^2+|\tilde{U}|^2)\tilde{V}+2\beta|\tilde{U}|^2\tilde{V}+2\beta'\tilde{U}^2V^*e^{-2ic\tau-4i\omega z}=0\ . \quad (5.1b)$$

In the zeroth approximation, the $\tilde{U}$ component of the soliton is taken in the form [cf. Eqs. (1.6) and (1.7)]

$$\tilde{U}^{(0)}_{\rm sol}=2i\eta\,{\rm sech}(2\eta\tau)e^{4i\eta^2 z}\ . \quad (5.2)$$

For the $\tilde{V}$ subsystem we employ the linearized equation (5.1b),

$$i\tilde{V}_z+\tilde{V}_{\tau\tau}+8(1+\beta)\eta^2{\rm sech}^2(2\eta\tau)\tilde{V}=8\beta'\eta^2{\rm sech}^2(2\eta\tau)e^{-2ic\tau-4i\omega z+8i\eta^2 z}\tilde{V}^*\ . \quad (5.3)$$

The right-hand side of Eq. (5.3) will be treated as a perturbation. We will deal with a weakly localized solution to Eq. (5.3), which in the zeroth approximation ($\beta'=0$) has the following form:

$$\tilde{V}^{(0)}(\tau,z)\approx b\,[{\rm sech}(2\eta\tau)]^{(2/3)\beta}\tanh(2\eta\tau)e^{(16i/9)\eta^2\beta^2 z}\ , \quad (5.4)$$

where it is implied that β is small, and b is an arbitrary amplitude.

To take account of the perturbation in the right-hand side of Eq. (5.3), let us employ an equation of balance for the energy (number of quanta) of the $\tilde{V}$ subsystem,

$$N_V\equiv\int_{-\infty}^{+\infty}d\tau|\tilde{V}(\tau)|^2\ , \quad (5.5)$$

which is an integral of motion in the case $\beta'=0$. According to Eq. (5.3),

$$\frac{d}{dz}N_V=16\beta'\eta^2\int_{-\infty}^{+\infty}d\tau\,{\rm sech}^2(2\eta\tau)\times{\rm Im}\{e^{-2ic\tau-4i\omega z+8i\eta^2 z}\times[\tilde{V}^*(\tau,z)]^2\}\ . \quad (5.6)$$

Assuming that in the presence of the perturbation the amplitude b in Eq. (5.4) becomes a slowly varying function of z, we insert Eq. (5.4) into Eqs. (5.6) and (5.5) to obtain

$$b^2(z)\approx b_0^2\left[1-\left[\frac{4\pi}{9}\right](\beta')^2(c/\eta)\frac{2\eta^2-c^2}{2\eta^2-\omega}\cos[4(2\eta^2-\omega)z]/\sinh\left[\frac{\pi c}{2\eta}\right]\right]\ , \quad (5.7)$$

where b_0^2 is a constant. As is seen from Eq. (5.7), under the action of the perturbation the amplitude of the $\tilde{V}$ component performs small oscillations unless the amplitude η is close to the resonant value

$$\eta^2_{\rm res}=\omega/2\ . \quad (5.8)$$

So far, the last term in Eq. (5.1a) was ignored. Let us now take account of it as a small perturbation. Inserting the expressions (5.2) and (5.4) into this term, it is sufficient to take the expression (5.4) in the approximation $\beta=0$, i.e., $\tilde{V}^{(0)}=b\tanh(2\eta\tau)$. Thus we obtain the following effective perturbing term in Eq. (5.1a):

$$P=-4i\beta'\eta b^2{\rm sech}(2\eta\tau)\tanh^2(2\eta\tau)e^{2ic\tau+(4i\omega-8i\eta^2)z}. \quad (5.9)$$

As is well known, a small perturbing term may give rise to decay of the NS soliton into radiation.[23,18] Far from the soliton, the radiation is a superposition of the small-amplitude (quasilinear) waves

$$\tilde{U}_{\rm rad}=B(q)\exp[-i(q^2z+q\tau)]\ . \quad (5.10)$$

A perturbation-induced evolution equation for the radiation spectral amplitudes $B(q)$ can be deduced by means of the perturbation theory based on the inverse scattering transform:[23,19]

$$\frac{dB(q)}{dz}=-[(q/2)^2+\eta^2]^{-1}e^{-iq^2z}\int_{-\infty}^{+\infty}d\tau\,e^{-iq\tau}\{[q/2-i\eta\tanh(2\eta\tau)]^2P^*-\eta^2e^{-8i\eta^2z}{\rm sech}(2\eta\tau)P\}\ , \quad (5.11)$$

where P is the perturbing term (5.9). Inserting Eq. (5.9) into Eq. (5.11), it is straightforward to see that the first and second terms in the integrand give rise to generation of radiation at the wave numbers, respectively,

$$q=\pm q_1,\quad q_1\equiv2(2\eta^2-\omega)^{1/2} \quad (5.12)$$

(provided $\eta^2>\omega/2$) and

$$q=\pm q_2,\quad q_2\equiv2(\omega-4\eta^2)^{1/2}\ , \quad (5.13)$$

provided $\eta^2<\omega/4$.

The final aim of our analysis is to find a law of radiative decay of the soliton. This can be done by means of a balance equation for the energy (number of quanta) of the $\tilde{U}$ field,

$$N_u\equiv\int_{-\infty}^{+\infty}d\tau|\tilde{U}(\tau)|^2\ , \quad (5.14)$$

cf. Eq. (5.5). In terms of the spectral amplitudes $B(q)$, the energy of the radiation field can be expressed as follows:[24]

$$N_u^{(\rm rad)}=\frac{1}{2\pi}\int_{-\infty}^{+\infty}dq|B(q)|^2\ , \quad (5.15)$$

and the soliton's energy is[24]

$$N_u^{(\mathrm{sol})}=4\eta \ . \tag{5.16}$$

The emission intensity can be characterized by the rate of emission of energy[23]

$$\frac{dN_u}{dz}=\frac{1}{\pi}\int_{-\infty}^{+\infty}dq\,\mathrm{Re}\left[B^*(q)\frac{dB(q)}{dz}\right] . \tag{5.17}$$

Insertion of Eq. (5.11) into Eq. (5.17) and straightforward calculations following the lines of Ref. 23 yield an expression for dN_u/dz which has a rather cumbersome form in the general case. If $\eta^2 \gg \omega$ and $\eta^2 \gg c^2$, we may set $\omega=c=0$, and the expression (5.17) takes the form

$$\frac{dN_u^{(\mathrm{rad})}}{dz}=A(\beta')^2b^4\eta^{-1} , \tag{5.18}$$

where $A\approx 1.17$. In this case, the emission takes place at the wave numbers $q\approx \pm 2\sqrt{2}\eta$, see Eq. (5.12).

In the presence of the perturbing term (5.9), the total energy of the $\tilde{U}$ field, $N_u\equiv N_u^{(\mathrm{sol})}+N_u^{(\mathrm{rad})}$, is not an integral of motion (only N_u+N_v is a strictly conserved quantity). However, it is easy to see that, both in orders β' and $(\beta')^2$, the perturbing term gives rise to an expression for dN_u/dz which rapidly oscillates and results in no systematic change of the energy. Thus, neglecting those oscillations, we may find the law of the radiative decay of the soliton from the balance equation

$$\frac{d}{dz}N_u^{(\mathrm{sol})}=-\frac{d}{dz}N_u^{(\mathrm{rad})} . \tag{5.19}$$

Inserting Eqs. (5.15), (5.16), and (5.18) into Eq. (5.19), we obtain the evolution equation for the soliton's amplitude:

$$\frac{d\eta}{dz}=-\tfrac{1}{4}A(\beta')^2b^4\eta^{-1} . \tag{5.20}$$

The solution of Eq. (5.20) is evident:

$$\eta^2(z)=\eta_0^2-\tfrac{1}{2}A(\beta')^2b^4z , \tag{5.21}$$

η_0 being an initial value of the amplitude at $z=0$. According to Eq. (5.21), before the complete decay the soliton travels the finite distance $Z=(2/A)\eta_0^2(\beta'b^2)^{-2}$. However, Eqs. (5.20) and (5.21) become invalid when the amplitude is very small. Indeed, the analysis based on the balance equation (5.19) implied that the decay rate $\eta^{-1}|d\eta/dz|$ was much smaller than the soliton's frequency $4\eta^2$, i.e., $|d\eta/dz| \ll \eta^3$. As follows from Eq. (5.21), this assumption requires $\eta^2 \gg (\beta')^2b^2$. As a matter of fact, the analysis developed above needs a stronger assumption $\eta^2\gg b^2$, to guarantee that the soliton's amplitude in the $\tilde{U}$ subsystem is much greater than in the $\tilde{V}$ subsystem.

To derive Eqs. (5.20) and (5.21), we have set $\omega=c=0$. If ω is not zero, Eqs. (5.12) and (5.13) demonstrate that the soliton with the amplitude lying in the interval $\omega/4<\eta^2<\omega/2$ [provided $\omega>0$, cf. Eq. (5.8)] does not decay in the lowest order of the perturbation theory. However, the radiative decay in this interval will take place in higher orders.[23] If $\omega<0$, the decay takes place at all values of η. The analysis based on the general equation (4.11) demonstrates that at $\eta\to 0$ (and $\omega<0$) the energy emission rate (4.17) becomes exponentially small $\sim\exp(-\pi\sqrt{-\omega}/\eta)$, and, accordingly, at $z\to\infty$ the decay becomes very slow:

$$\eta(z)\approx \pi\sqrt{-\omega}/\ln(z/z_0) . \tag{5.22}$$

[the expression (5.22) satisfies the condition $|\dot{\eta}|\ll\eta^3$].

At last, it is worth noting that, if $c\neq 0$, the emission becomes asymmetric: As follows from Eq. (5.11), in this case the intensities of emission corresponding to $q=+q_{1,2}$ and $q=-q_{1,2}$ [see Eqs. (5.12) and (5.13)] are different. The asymmetry gives rise to a radiative recoil force which accelerates the soliton. The joint analysis of the balance of energy and momentum makes it possible to investigate this effect in detail. For instance, in the particular case $\omega=-c^2$ the asymptotic law of motion of the decaying soliton takes the form $d\tau^{(0)}/dz=4c$, where the soliton's coordinate $\tau^{(0)}$ is defined according to Eqs. (1.6).

VI. EFFECTS OF THE OPTICAL ACTIVITY

It seems interesting to extend the analysis of the soliton's dynamics developed in the present paper to a more general model incorporating the polarization-rotating effect (optical activity), which may be induced, e.g., by a homogeneous twist of the birefringent fiber. The optical activity is described by the additional linear-coupling terms λv and λu in Eqs. (1.1a) and (1.1b), respectively (λ is an optical-activity coefficient, which will be assumed to be of the order β, see below). After the transformation [Eqs. (1.3)], the additional coupling terms give rise to the additional term

$$\Delta\mathcal{H}=-U^*Ve^{ic\tau}-UV^*e^{-ic\tau} \tag{6.1}$$

in the Hamiltonian density (2.2). On inserting the unperturbed solitonic waveform (1.6), the term (6.1) generates the one

$$\Delta H=-[\pi\lambda c/\sinh(\pi c/2\eta)]\sin(2\theta)\cos\Psi \tag{6.2}$$

in the perturbation Hamiltonian (2.3). Investigation of the generalized model including the additional perturbing term (6.2) demonstrates that, as well as in the case $\lambda=0$ (Sec. II), the four-dimensional dynamical system of the Hamilton's equations of motion for the polarizational and positional degrees of freedom θ and $\tau^{(0)}$ can be integrated to the following two-dimensional conservative system:

$$\frac{d\theta}{d\zeta}=\sin(2\theta)\sin(2\Psi)+\mu\sin\Psi , \tag{6.3a}$$

$$\frac{d\Psi}{d\zeta}=K+2[B+\cos(2\Psi)]\cos(2\theta)+2\mu\cot(2\theta)\cos\Psi , \tag{6.3b}$$

cf. Eqs. (2.6) and (2.11). Here

$$\mu\equiv\tfrac{3}{4}(\lambda/\beta'\eta^2)[(c/\eta)^2+4]^{-1} \tag{6.4}$$

[cf. Eqs. (2.10)], and the quantities ζ, Ψ, K, and B are the same as in Sec. II. The system of Eqs. (6.3) conserves the integral of motion

$$C = K\cos(2\theta) - [B + \cos(2\psi)]\sin^2(2\theta) - 2\mu\sin(2\theta)\cos\psi ,$$

which is the dimensionless Hamiltonian of the system [cf. Eq. (2.12)]. Of basic interest is the case when the new governing parameter μ is of order 1, i.e., as follows from Eq. (6.4), $\lambda \sim \beta$.

Evidently, Eqs. (6.3) do not admit stationary points corresponding to the simple solitons. As for the stationary vector solitons, it is easy to see that Eqs. (6.3) give rise to the following stationary points:

$$\cos\psi = -\tfrac{1}{2}\mu/\sin(2\theta) , \tag{6.5a}$$

$$\cos(2\theta) = -(K/2)(B-1)^{-1} \tag{6.5b}$$

[cf. Eq. (2.13)], and

$$\sin\psi = 0 , \tag{6.6a}$$

the corresponding value of θ being determined by the equation

$$(B+1)\cos(2\theta) + \mu\cos\psi\cot(2\theta) = -K/2 , \tag{6.6b}$$

where $\cos\psi = \pm 1$ according to Eq. (6.6a). A straightforward qualitative (graphical) investigation of Eq. (6.6b) demonstrates that it may have four or two roots in the interval $-\pi/2 < \theta < \pi/2$. In particular, at small μ two roots are close to the ones

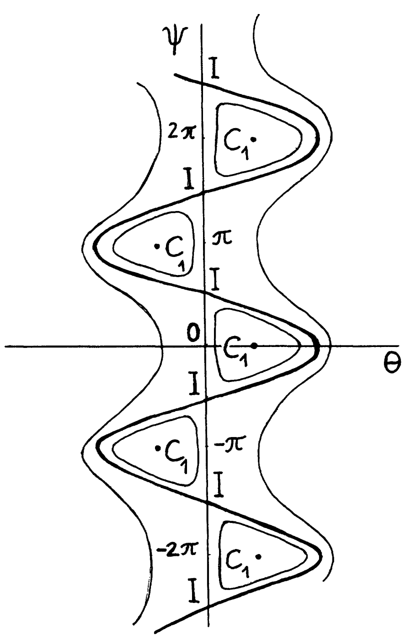

FIG. 6. The phase portrait of the dynamical system (6.3) with small μ near the vertical line $\theta = 0$ in the case (6.9). C_1 are the centers with the coordinates given by Eqs. (6.6a) and (6.8a); I are the intersection points determined by Eqs. (6.10) and (6.12).

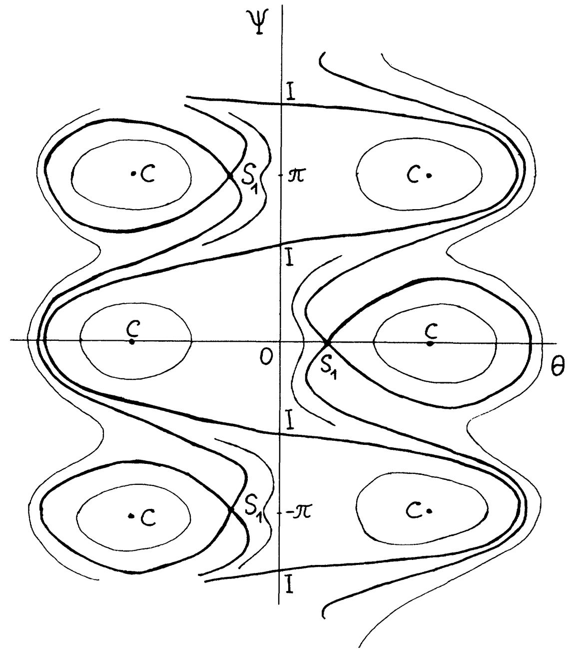

FIG. 7. The modification of the phase plane of Figs. 2 and 4 at small μ in the case opposite to that of (6.9). S_1 are the saddles with the coordinates (6.6a) and (6.8a).

$$\theta = \pm\tfrac{1}{2}\cos^{-1}[(K/2)(B+1)^{-1}] \tag{6.7}$$

which exist at $\mu = 0$ [see Eq. (2.13)], and two other roots are

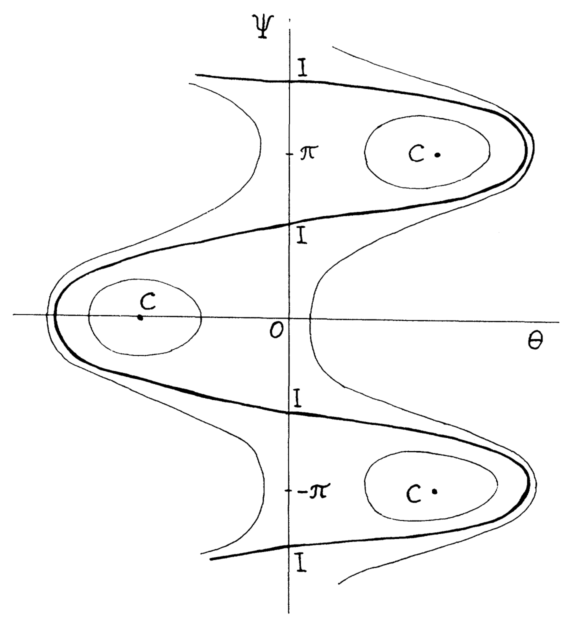

FIG. 8. The phase portrait after the annihilation of the saddles S_1 with the mate centers C.

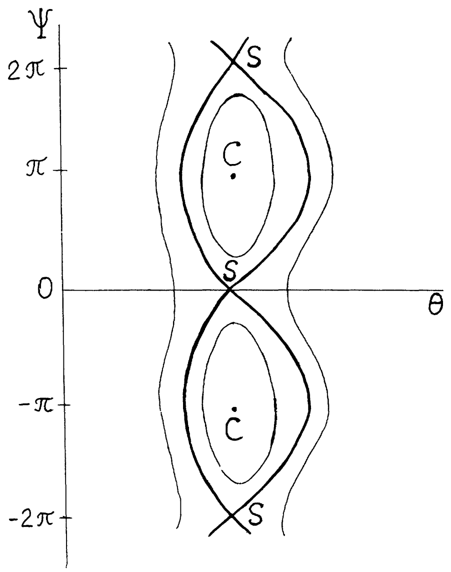

FIG. 9. The chain of the separatrix loops of Fig. 3 after the "period doubling" at the boundary of the parametric region (6.13).

$$\theta_1 \approx -\mu(\cos\psi)(K+2B+2)^{-1} \tag{6.8a}$$

and

$$\theta_2 \approx \pi/2 - \mu(\cos\psi)(K-2B-2)^{-1} \tag{6.8b}$$

(recall $\cos\psi = \pm 1$). In the limit $\mu \to 0$, the vector solitons (6.8) go over into the simple ones.

At μ sufficiently large, Eq. (6.6b) may lose two roots. In this case, either root (6.7) merges with one of the roots (6.8), and both disappear. It follows from Eq. (6.6b) that the merger bifurcation takes place at

$$\sin^3(2\theta) = -\mu(\cos\psi)/(B+1) \ .$$

At small μ, it is also easy to investigate the stability of the stationary point determined by Eqs. (6.6a) and (6.6b). The linearized equations (6.3) yield the stability condition

$$(K+2B)^2 - 4 \geq 0 \ . \tag{6.9}$$

It is important to note that the condition (6.9) is opposite to that under which, in the case $\mu=0$, there exist the saddles S shown in Figs. 2 and 4. Quite analogously, the stability condition for the stationary point (6.8b),

$$(K-2B)^2 - 4 \geq 0$$

[cf. Eq. (6.9)], is just opposite to that providing the existence of the saddles $\tilde{S}$ (Fig. 4).

In the case $\mu=0$ the dynamical trajectories could not intersect the vertical lines $\sin(2\theta)=0$ (see Figs. 1–4). At $\mu \neq 0$, the intersection is possible at the points where $\cos\psi$ vanishes, i.e., at

$$\psi = \psi_n \equiv \frac{\pi}{2}(2n+1), \quad n = 0, \pm 1, \pm 2, \ldots \ . \tag{6.10}$$

Using Eqs. (6.3), one can readily find that, in a vicinity of the intersection point ($\theta=0$, $\psi=\psi_n$), the dynamical trajectories are governed by the asymptotic equation

$$\frac{d\tilde{\psi}}{d\theta} = (-1)^n M - \frac{\tilde{\psi}}{\theta} \ , \tag{6.11}$$

where $\tilde{\psi} \equiv \psi - \psi_n$, the index n is the same as in Eq. (6.10), and $M \equiv (K+2B-2)/\mu$. As follows from Eq. (6.11), the trajectory intersecting the axis $\theta=0$ has, at the intersection point, the slope

$$\left.\frac{d\psi}{d\theta}\right|_{\theta=0} = \tfrac{1}{2}(-1)^n M \ . \tag{6.12}$$

Using these results, one can construct a phase portrait of the dynamical system (6.3). At small μ, a modification of the phase planes investigated in Sec. II at $\mu=0$ is significant only at small values of $\sin(2\theta)$ (i.e., near the simple-soliton states). In the case when the condition (6.9) holds, or, according to what was said above, when the stationary point (6.8a) is stable (a center), and the saddles S (Figs. 2 and 4) do not exist, a vicinity of the line $\theta=0$ takes the form shown in Fig. 6.

In the opposite case, when the stationary point (6.8a) is unstable (a saddle), the phase plane of Fig. 2 or 4 is modified as shown in Fig. 7.

To conclude this section, let us briefly discuss possible bifurcations of the phase portraits at larger values of μ. As was mentioned above, one of the points (6.7) may disappear together with either of the ones (6.8). In terms of Fig. 7, this implies annihilation of the saddles S_1 with the mate centers C. Thus we arrive at the phase portrait of Fig. 8.

At last, it is easy to see that the stationary points (6.5) exist provided

$$\mu^2 \leq 4 - K^2/(1-B)^2 \ . \tag{6.13}$$

At the boundary of the parametric region (6.13), the periodic chain of the separatrix loops shown (for $\mu=0$) in Fig. 3 suffers a "period doubling" and takes the form shown in Fig. 9, or that differing from it by a vertical shift by $\Delta\psi=\pi$.

Thus the analysis of the phase plane of the system (6.3) demonstrates the absence of the polarization-rotating trajectories, i.e., the ones going from $\theta=-\infty$ to $+\infty$ (see Figs. 6–9); instead, we have only oscillations of the polarization.

A more detailed analysis of the model including the optical-activity effect, is planned to be presented elsewhere.[25]

Note added in proof. Figure 7, where the separatrices connect the nearest saddles S_1, represents the simplest variant of the phase plane (θ, ψ) in the case when μ is small and the inequality (6.9) does not hold. More complicated variants, when each nth saddle is connected by the separatrices with the $(n-N)$th and $(n+N)$th ones, are possible for any $N \geq 2$ as well.

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