Class of exactly solvable master equations describing coupled nonlinear oscillators

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Through use of the notation of thermofield dynamics, an exact solution of a class of master equations describing coupled nonlinear oscillators is presented.

I. INTRODUCTION

In a recent work[1] we highlighted the usefulness of the thermofield dynamics[2-4] notation for master equations that appear in studies of quantum optics. This notation enables one to transcribe master equations as Schrödinger-like equations, thereby making them amenable to algebraic techniques such as those used for solving the Schrödinger equation for quadratic parametric processes in quantum optics and quantum acoustics.[5,6] In a subsequent work[7] we applied these methods to the master equation of a nonlinear oscillator described by the Hamiltonian

$$H = \hbar\omega(a^\dagger a) + \hbar\chi(a^\dagger a)^2 \ .$$

Various aspects of this nonlinear oscillator with or without damping have been investigated by a number of authors.[8-21] In particular, an exact solution of the master equation for this problem was obtained by Daniel and Milburn[16] and by Peřinova and Lukš[17] by solving the corresponding Fokker-Planck equation for the Q function. In our work we showed that this master equation can be solved rather elegantly using the notation of thermofield dynamics, and in a way that hardly requires more effort than is necessary for the linear case. An exact solution of this problem along similar lines has also been independently given by Peřinova and Lukš.[22] The aim of the present work is to generalize our earlier work on the master equation for a single nonlinear oscillator to that describing coupled nonlinear oscillators.

This work is organized as follows. In Sec. II we give a brief summary of thermofield dynamics and discuss, in the context of master equations, its connection with the operators introduced by Davies.[23-25] In Sec. III we use the thermofield-dynamics notation to give an exact solution for a class of master equations describing coupled nonlinear oscillators. Finally, in Sec. IV we consider some special cases that have been discussed in the context of wave propagation in a nonlinear medium and which fall into the class of master equations discussed in Sec. III.

II. BRIEF SUMMARY OF THERMOFIELD DYNAMICS

In thermofield dynamics[2-4] one associates, with a density operator ρ acting on a Hilbert space $\mathcal{H}$, a state vector $|\rho^\alpha\rangle$, $0 \le \alpha \le 1$, in the extended Hilbert space $\mathcal{H}\otimes\mathcal{H}^*$, so that

$$\langle A\rangle = \mathrm{Tr}\, A\rho = \langle \rho^{1-\alpha}| A |\rho^\alpha\rangle \ . \tag{1}$$

The state $|\rho^\alpha\rangle$ is given by

$$|\rho^\alpha\rangle = \rho^\alpha |I\rangle \ , \tag{2}$$

where

$$|I\rangle = \sum_N |N,N\rangle \ , \tag{3}$$

where $|N\rangle$ constitutes any complete orthonormal set in $\mathcal{H}$. (We use the notations and conventions of Ref. 1.) The relation (3) is simply the counterpart of the resolution of the identity

$$I = \sum_N |N\rangle\langle N| \ , \tag{4}$$

in terms of a complete orthonormal set $|N\rangle$ in $\mathcal{H}$.

In dealing with bosonic systems it is natural to use, for $|N\rangle$, the number states $|n\rangle$ and to introduce creation and annihilation operators a, $\tilde{a}$, $a^\dagger$, and $\tilde{a}^\dagger$ as follows:

$$a|n,m\rangle = \sqrt{n}\,|n-1,m\rangle \ , \tag{5a}$$

$$\tilde{a}|n,m\rangle = \sqrt{m}\,|n,m-1\rangle \ , \tag{5b}$$

$$a^\dagger|n,m\rangle = \sqrt{m+1}|n+1,m\rangle \ , \tag{5c}$$

$$\tilde{a}^\dagger|n,m\rangle = \sqrt{m+1}|n,m+1\rangle \ . \tag{5d}$$

The operators a and $a^\dagger$ commute with $\tilde{a}$ and $\tilde{a}^\dagger$. The relations (5) are the analogs of the following relations in the usual notation:

$$a|n\rangle\langle m| = \sqrt{n}\,|n-1\rangle\langle m| \ , \tag{6a}$$

$$|n\rangle\langle m|a^\dagger = \sqrt{m}\,|n\rangle\langle m-1| \ , \tag{6b}$$

$$a^\dagger|n\rangle\langle m| = \sqrt{n+1}|n+1\rangle\langle m| \ , \tag{6c}$$

$$|n\rangle\langle m|a = \sqrt{m+1}|n\rangle\langle m+1| \ . \tag{6d}$$

In other words, $\tilde{a}$ and $\tilde{a}^\dagger$, respectively, simulate the action of $a^\dagger$ and a on $|n\rangle\langle m|$ from the right.

From the expression for $|I\rangle$ in terms of the number states,

$$|I\rangle=\sum_n |n,n\rangle \ , \tag{7}$$

it follows that

$$a|I\rangle=\tilde{a}^{\dagger}|I\rangle \ , \tag{8a}$$

$$a^{\dagger}|I\rangle=\tilde{a}|I\rangle \ . \tag{8b}$$

Given the evolution equation for ρ^{α}, the relations (8) enable one to transcribe it into a Schrödinger-like equation for $|\rho^{\alpha}\rangle$ associated with the density operator ρ. For dissipative systems this is only possible for $\alpha=1$.

A. An illustration

Consider, for example, the master equation for a linear oscillator

$$\frac{\partial}{\partial t}\rho=-i\omega[a^{\dagger}a,\rho]+\tfrac{1}{2}\gamma(\bar{n}+1)(2a\rho a^{\dagger}-a^{\dagger}a\rho-\rho a^{\dagger}a) +\tfrac{1}{2}\gamma\bar{n}(2a^{\dagger}\rho a-aa^{\dagger}\rho-\rho aa^{\dagger}) \ . \tag{9}$$

Applying $|I\rangle$ on (9) from the right and using (8), the master equation (9), in the thermofield-dynamics notation goes over to the following equation for the state $|\rho\rangle$:

$$\frac{\partial}{\partial t}|\rho\rangle=-i\hat{H}|\rho\rangle \ , \tag{10}$$

where

$$-i\hat{H}=-i\omega(a^{\dagger}a-\tilde{a}^{\dagger}\tilde{a})+\tfrac{1}{2}\gamma(\bar{n}+1)(2a\tilde{a}-a^{\dagger}a-\tilde{a}^{\dagger}\tilde{a}) +\tfrac{1}{2}\gamma\bar{n}(2a^{\dagger}\tilde{a}^{\dagger}-aa^{\dagger}-\tilde{a}\tilde{a}^{\dagger}) \ . \tag{11}$$

Introducing the operators

$$K_{+}=a^{\dagger}\tilde{a}^{\dagger}, \quad K_{-}=a\tilde{a}, \quad K_3=\tfrac{1}{2}(a^{\dagger}a+\tilde{a}^{\dagger}\tilde{a}+1) \ , \quad K_0=(a^{\dagger}a-\tilde{a}^{\dagger}\tilde{a}) \ , \tag{12}$$

we may rewrite (11) as

$$-i\hat{H}=-i\omega K_0+\gamma(\bar{n}+1)K_{-}+\gamma\bar{n}K_{+} -\gamma(2\bar{n}+1)K_3+\tfrac{1}{2}\gamma \ . \tag{13}$$

The operators K_{+}, K_{-}, and K_3 generate the su(1,1) algebra

$$[K_{-},K_{-}]=2K_3, \quad [K_3,K_{\pm}]=\pm K_{\pm} \ , \tag{14}$$

K_0 is a Casimir operator. Use of the disentangling theorem for su(1,1) (Ref. 26),

$$\exp(\gamma_{+}K_{+}+\gamma_3K_3+\gamma_{-}K_{-}) =\exp(\Gamma_{+}K_{+})\exp[(\ln\Gamma_3)K_3]\exp(\Gamma_{-}K_{-}) \ , \tag{15}$$

where

$$\Gamma_{\pm}=\frac{2\gamma_{\pm}\sinh\phi}{2\phi\cosh\phi-\gamma_3\sinh\phi} \ , \tag{16}$$

$$\Gamma_3=\left[\frac{2\phi}{2\phi\cosh\phi-\gamma_3\sinh\phi}\right]^2 \ , \tag{17}$$

$$\phi^2=\left[\frac{\gamma_3^2}{4}\right]-\gamma_{+}\gamma_{-} \ , \tag{18}$$

and of the fact that K_{+}, K_{-}, and K_3 have simple actions on $|n,m\rangle$, enabling one to solve (11) and hence (9) purely algebraically, as was shown in Refs. 1 and 7.

B. Connection with the operators introduced by Davies

In the specific context of the master equation for a linear oscillator (9) with $\bar{n}=0$, Davies[23] introduced an algebraic technique for its solution. Extending his work slightly, one may introduce the following operators

$$\mathcal{K}_{+}\rho=a^{\dagger}\rho a, \quad \mathcal{K}_{-}\rho=a\rho a^{\dagger} \ , \quad \mathcal{K}_3\rho=\tfrac{1}{2}(a^{\dagger}a\rho+\rho aa^{\dagger}), \quad \mathcal{K}_0\rho=[a^{\dagger}a,\rho] \ . \tag{19}$$

In terms of these, the master equation (9) may be written as

$$\frac{\partial}{\partial t}\rho=-i\mathcal{L}\rho \ , \tag{20}$$

where

$$\mathcal{L}=-i\omega\mathcal{K}_0+\gamma(\bar{n}+1)\mathcal{K}_{-}+\gamma\bar{n}\mathcal{K}_{+}+\gamma(2\bar{n}+1)\mathcal{K}_3+\tfrac{1}{2}\gamma \ . \tag{21}$$

These operators are clearly similar to the operators K_{+}, K_{-}, K_3, and K_0 introduced above and obey the same algebra, as can easily be verified. Thus, from an algebraic point of view, the two techniques are the same. However, by introducing tide operators to replace the action, from the right, of a and $a^{\dagger}$ on ρ, the thermofield-dynamics notation enables one to carry out certain algebraic manipulations that, at first sight, may not be apparent in the method of Davies. This may be clearly seen from our treatment of the master equation of the nonlinear oscillator.[7]

III. EXACT SOLUTION OF A CLASS OF MASTER EQUATIONS DESCRIBING COUPLED NONLINEAR OSCILLATORS

Generalizing our previous work on the master equation for a single nonlinear oscillator, we consider the following master equation:

$$\frac{\partial}{\partial t}\rho=-i\sum_{i=1}^{N}\omega_i[a_i^{\dagger}a_i,\rho]-i\sum_{i,j=1}^{N}\chi_{ij}[a_i^{\dagger}a_ia_j^{\dagger}a_j,\rho] +\tfrac{1}{2}\sum_{i=1}^{N}\gamma_i(\bar{n}_i+1)(2a_i\rho a_i^{\dagger}-2a_i^{\dagger}a_i\rho-\rho a_i^{\dagger}a_i) +\tfrac{1}{2}\sum_{i=1}^{N}\gamma_i\bar{n}_i(2a_i\rho a_i^{\dagger}-a_i^{\dagger}a_i\rho-\rho a_i^{\dagger}a_i) \ . \tag{22}$$

In the thermofield-dynamics notation, the master equation (22) goes over

$$\frac{\partial}{\partial t}|\rho\rangle=-i\hat{H}|\rho\rangle \ , \tag{23}$$

where

$$-i\hat{H}=-i\sum_{i=1}^{N}\omega_i(a_i^\dagger a_i-\tilde{a}_i^\dagger\tilde{a})-i\sum_{i,j=1}^{N}\chi_{ij}(a_i^\dagger a_i a_j^\dagger a_j-\tilde{a}_i^\dagger\tilde{a}_i\tilde{a}_j^\dagger a_j)+\tfrac{1}{2}\sum_{i=1}^{N}\gamma_i(\bar{n}_i+1)(2a_i\tilde{a}_i-a_i^\dagger a_i-\tilde{a}_i^\dagger\tilde{a}_i)+\tfrac{1}{2}\sum_{i=1}^{N}\gamma_i\bar{n}_i(2a_i^\dagger\tilde{a}_i^\dagger-a_ia_i^\dagger-\tilde{a}_i\tilde{a}_i^\dagger)\ . \tag{24}$$

Rewriting the second term on the right-hand side (rhs) of (24) as

$$\sum_{i,j=1}^{N}\chi_{ij}(a_i^\dagger a_i a_j^\dagger a_j-\tilde{a}_i^\dagger\tilde{a}_i\tilde{a}_j^\dagger\tilde{a}_j)=\sum_{i,j=1}^{N}\chi_{ij}(a_i^\dagger a_i+\tilde{a}_i^\dagger\tilde{a}_i)(a_j^\dagger a_j-\tilde{a}_j^\dagger\tilde{a}_j)\ , \tag{25}$$

we find that $-i\hat{H}$ may be expressed in terms of $K_\pm^i$, K_3^i, and K_0^i as follows:

$$-i\hat{H}=\sum_{i=1}^{N}\left[-i\left(\omega_i-\sum_{j=1}^{N}\chi_{ij}\right)K_0^i+\tfrac{1}{2}\gamma_i+\gamma_i(\bar{n}_i+1)K_-^i+\gamma_i\bar{n}_iK_+^i-\left(\gamma_i(2\bar{n}_i+1)+\sum_{j=1}^{N}2i\chi_{ij}K_0^j\right)K_3^i\right]\ . \tag{26}$$

From (23) and (25) we have

$$|\rho(t)=\exp(-i\hat{H}t)|\rho(0)\rangle=\prod_{i=1}^{N}\exp(\gamma_0^iK_0^i+\tfrac{1}{2}\gamma_it)\exp(\gamma_+^iK_+^i+\gamma_-^iK_-^i+\gamma_3^iK_3^i)|\rho(0)\rangle\ , \tag{27}$$

where

$$\gamma_0^i=-i\left(\omega_i-\sum_{j=1}^{N}\chi_{ij}\right)t \tag{28a}$$

$$\gamma_+^i=\gamma_i\bar{n}_it\ , \tag{28b}$$

$$\gamma_-^i=\gamma_i(\bar{n}_i+1)t\ , \tag{28c}$$

$$\gamma_3^i=-\left(\gamma_i(2\bar{n}_i+1)+\sum_{j=1}^{N}2i\chi_{ij}K_0^j\right)t\ . \tag{28d}$$

Since K_0^i's are Casimir operators, the disentangling theorem (15) may still be used to write (27) as

$$|\rho(t)\rangle=\prod_{i=1}^{N}\exp(\gamma_0^iK_0^i+\tfrac{1}{2}\gamma_it)\exp(\Gamma_+^iK_+^i)\times\exp[(\ln\Gamma_3^i)K_3^i]\exp(\Gamma_-^iK_-^i)|\rho(0)\rangle\ , \tag{29}$$

where $\Gamma_\pm^i$ and Γ_3^i may be calculated from $\gamma_\pm^i$ and γ_3^i given by (28) using (16)–(18).

Consider now an arbitrary initial condition for $\rho(0)$

$$\rho(0)=\sum_{\mathbf{m},\mathbf{n}=0}^{\infty}\rho_{\mathbf{m},\mathbf{n}}(0)|\mathbf{m}\rangle\langle\mathbf{n}|\ , \tag{30}$$

which for $|\rho(0)\rangle$ corresponds to

$$|\rho(0)\rangle=\sum_{\mathbf{m},\mathbf{n}=0}^{\infty}\rho_{\mathbf{m},\mathbf{n}}(0)|\mathbf{m},\mathbf{n}\rangle\ . \tag{31}$$

Substituting this in the rhs of (29) and successively applying the operators on $|\mathbf{m},\mathbf{n}\rangle$, we obtain, in exactly the same manner as for a single oscillator, the following expression for $\rho_{\mathbf{m},\mathbf{n}}(t)$:

$$\rho_{\mathbf{m},\mathbf{n}}(t)=\sum_{\mathbf{q}=0}^{\min(\mathbf{m},\mathbf{n})}\sum_{\mathbf{p}=0}^{\infty}\prod_{i=1}^{N}\left\{\exp[\gamma_0^i(m_i-n_i)+\tfrac{1}{2}\gamma_it]\left[\binom{m_i+p_i-q_i}{p_i}\binom{n_i+p_i-q_i}{p_i}\binom{m_i}{q_i}\binom{n_i}{q_i}\right]^{1/2}\times[\Gamma_+^i(m-n)]^{q_i}[\Gamma_3^i(m-n)]^{(m_i+n_i-2q_i+1)/2}[\Gamma_-^i(m-n)]^{p_i}\right\}\rho_{\mathbf{m}+\mathbf{p}-\mathbf{q},\mathbf{n}+\mathbf{p}-\mathbf{q}}(0)\ . \tag{32}$$

For the Q function

$$Q(\boldsymbol{\alpha},t)=\langle\boldsymbol{\alpha}|\rho(t)|\boldsymbol{\alpha}\rangle\ ,$$

we then obtain

$$Q(\boldsymbol{\alpha},t)=\sum_{\mathbf{m},\mathbf{n}=0}^{\infty}\prod_{i=1}^{N}\left\{\exp[\gamma_0^i(m_i-n_i)+\tfrac{1}{2}\gamma_it]\frac{(\alpha_i^*)^{m_i}}{m_i!}\frac{(\alpha_i)^{n_i}}{n_i!}\exp[(\Gamma_+^i-1)|\alpha_i|^2](\Gamma_3^i)^{(m_i+n_i+1)/2}\right\}g_{\mathbf{m},\mathbf{n}}(0)\ , \tag{33}$$

where

$$g_{\mathbf{m},\mathbf{n}}(0)=\sum_{\mathbf{p}=0}^{\infty}\prod_{i=1}^{N}\frac{(\Gamma_-^i)^{p_i}}{p_i!}\sqrt{(m_i+p_i)!}\sqrt{(n_i+p_i)!}\rho_{\mathbf{m}+\mathbf{p},\mathbf{n}+\mathbf{p}}(0)\ . \tag{34}$$

In particular, for the initial condition

$$\rho(0)=|\boldsymbol{\alpha}_0\rangle\langle\boldsymbol{\alpha}_0| \ , \tag{35}$$

one finds that

$$g_{\mathbf{m},\mathbf{n}}(0)=\prod_{i=1}^{N}\{(\alpha_{0i})^{m_i}(\alpha_{0i}^{*})^{n_i}\exp[(\Gamma_{-}^{i}-1)|\alpha_{0i}|^2]\} \ . \tag{36}$$

which when substituted in (33) yields the expression for the corresponding Q function.

IV. SOME SPECIAL CASES

Two Hamiltonians involving two modes have been discussed in the literature in the context of propagation of electromagnetic waves through a nonlinear medium. The one due to Agarwal and Puri[27] is given by

$$H=p(a^{\dagger 2}+b^{\dagger 2})(a^2+b^2)+\frac{q}{2}:(a^{\dagger}a+b^{\dagger}b)^2: \ , \tag{37}$$

and that due to Tombesi and Mecozzi[13] is given by

$$H=\lambda(a^{\dagger}b+ab^{\dagger})+\frac{\varepsilon}{2}(a^{\dagger}b+ab^{\dagger})^2 \ . \tag{38}$$

If dissipation is included via the usual master equation techniques, then for the case

$$\gamma_a=\gamma_b, \quad \bar{n}_a=\bar{n}_b \ , \tag{39}$$

by defining

$$c=\frac{a+ib}{\sqrt{2}}, \quad d=\frac{a-ib}{\sqrt{2}}, \quad \tilde{c}=\frac{\tilde{a}-i\tilde{b}}{\sqrt{2}}, \quad \tilde{d}=\frac{\tilde{a}+i\tilde{b}}{\sqrt{2}} \tag{40}$$

and

$$c=\frac{a+b}{\sqrt{2}}, \quad c=\frac{a+b}{\sqrt{2}}, \quad \tilde{c}=\frac{\tilde{a}+\tilde{b}}{\sqrt{2}}, \quad \tilde{d}=\frac{\tilde{a}-\tilde{b}}{\sqrt{2}} \ , \tag{41}$$

the two master equations, in the thermofield-dynamics notation, can be cast into the form discussed earlier and may therefore be solved exactly.

V. CONCLUDING REMARKS

We have presented a class of exactly solvable master equations describing coupled nonlinear oscillators. The solution was obtained by first transcribing the master equation as Schrödinger-like equation in the thermofield-dynamics notation and then using purely algebraic techniques. This method of solution for the problem at hand turns out to be much simpler and much more elegant than attempting to solve the Fokker-Planck equation for the P function or the Q function, which in this case turns out to be multivariate Fokker-Planck equation with nonlinear drift and nonconstant diffusion coefficients.

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