Squeezing and its graphical representations in the anharmonic oscillator model

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(Received 14 June 1990; revised manuscript received 10 September 1990)

The problem of squeezing and its graphical representations in the anharmonic oscillator model is considered. Explicit formulas for squeezing, principal squeezing, and the quasiprobability distribution (QPD) function are given and illustrated graphically. Approximate analytical formulas for the variances, extremal variances, and QPD are obtained for the case of small nonlinearities and large numbers of photons. The possibility of almost perfect squeezing in the model is demonstrated and its graphical representations in the form of variance lemniscates and QPD contours are plotted. For large numbers of photons the crescent shape of the QPD contours is hardly visible and quite regular ellipses are obtained.

I. INTRODUCTION

Squeezing of quantum fluctuations in optical fields is a problem that nowadays attracts many physicists, both theorists and experimentalists. The number of publications on the subject is growing at a high rate (see, for example, two special issues[1] of the specialized optical journals). Numerous experiments[2-8] have successfully confirmed the possibility of producing squeezed states of light in various nonlinear optical processes. Nonlinearity of the process is crucial in producing such nonclassical states. One of the simplest though very instructive models (because it admits of exact solutions) that can be used to describe nonlinear interaction of light in a medium is the anharmonic oscillator model. Some years ago, Tanas and Kielich[9] have shown that intense light propagating through a nonlinear Kerr medium can squeeze its own fluctuations. They referred to this effect as self-squeezing, and have proved the possibility of as much as 98% squeezing in this process. The model used in Ref. 9 was in fact a two-mode version of the anharmonic oscillator which describes the propagation of elliptically polarized light in a nonlinear Kerr medium. The one-mode version of the self-squeezing effect applicable for circularly polarized light propagating in an isotropic Kerr medium has been considered by Tanaś,[10] in terms of an anharmonic oscillator with interaction Hamiltonian $\kappa a^{\dagger 2} a^2$, who has shown that the same amount of self-squeezing as in the two-mode case is attainable. This very simple, strictly solvable model of the anharmonic oscillator with interaction Hamiltonian $\kappa a^{\dagger 2} a^2$ or $\kappa (a^{\dagger} a)^2$ appears to be very attractive and many properties of the quantum states generated from the model have been discussed recently.[11-22] The squeezed states to which it leads are not minimum-uncertainty states and differ essentially from the two-photon coherent states of Yuen,[23] most often used as models for squeezed states.

Milburn[11] has discussed the evolution of the quasiprobability distribution function $Q(\alpha,\alpha^*,t)$ for the anharmonic oscillator showing periodic recurrences of its initial form and predicting 37% squeezing for a mean number of photons $|\alpha|^2=0.25$. Milburn and Holmes have shown that dissipation in the model rapidly destroys the quantum recurrence effects. Kitagawa and Yamamoto,[13] who also considered the quasiprobability distribution $Q(\alpha,\alpha^*,t)$ for the states thus obtained, refer to squeezing in this case as "crescent"-shaped squeezing (in contrast to "elliptical"-shaped squeezing) because of the crescent shape of the quasiprobability distribution contours. The "self-squeezing" of Tanaś and Kielich[9] and the squeezing that produces crescent contours of Kitagawa and Yamamoto[13] are but different terms for what is virtually the same mechanism of squeezing.

Some aspects of third-[24] and second-[25] harmonic generation by self-squeezed light have also been discussed, and the possibility of controlling the self-squeezing process by means of an external magnetic field has been suggested.[26]

Yurke and Stoler[27] have shown that the states produced in the anharmonic oscillator model become a superposition of a finite number of coherent states under a proper choice of the evolution time. Tombesi and Mecozzi[28] have obtained the superposition states for the two-mode case and arbitrary initial state of the field. We have recently shown[29] that superpositions with not only even but also odd numbers of components can be obtained, and that the maximum number of well distinguishable states is proportional to the field amplitude $|\alpha|$.

The anharmonic oscillator model has also been discussed by Peřinova and Lukš[17] from the point of view of photon statistics and squeezing. In particular, Lukš, Peřinova, and Peřina[30] introduced the concept of "principal squeezing" which is associated with the geometrical representation of the quadrature field variance as an ellipse.[31] Loudon[32] has recently proposed another geometrical representation of the quadrature field variance by Booth's elliptical lemniscate. The last two geometrical representations are interrelated but their relation to the contours of the Q function is not so simple as it would appear at a first glance, when one is apt to identify the shape of the Q-function contours with the phase-space picture of the quadrature field variance.

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In this paper we give explicit analytical formulas for the quadrature variances (standard squeezing), their minimum and maximum values (principal squeezing), and the Q function for the anharmonic oscillator model in two different ranges of parameters: (i) a small number of photons and strong nonlinearity (long time), and (ii) a large number of photons and weak nonlinearity (short time). In case (i), direct numerical evaluation of the exact analytical formulas is feasible and the results are well known. In case (ii), we use the saddle-point technique to evaluate the summations for a large number of photons to obtain approximate analytical formulas for the quadrature field variances and the Q function. These formulas are illustrated graphically using different representations of squeezing. This region (ii) is the most interesting from the point of view of the practical applicability of the model and, thus, the relatively simple analytical formulas that hold in this region are especially valuable. These formulas are the main result of our paper.

II. THE ANHARMONIC OSCILLATOR MODEL AND ITS EVOLUTION

We consider an anharmonic oscillator model described by the Hamiltonian

$$\begin{aligned} H &= H_0 + H_I \ , \\ H_0 &= \hbar\omega a^\dagger a = \hbar\omega\hat{n} \ , \\ H_I &= \tfrac{1}{2}\hbar\kappa a^{\dagger 2} a^2 = \tfrac{1}{2}\hbar\kappa\hat{n}(\hat{n}-1) \ , \end{aligned} \tag{1}$$

where a ($a^\dagger$) is the boson annihilation (creation) operator, and κ is the coupling (anharmonicity) constant, which is real and can be related to the nonlinear susceptibility $\chi^{(3)}$ of the medium[9] if the anharmonic oscillator model is used to describe propagation of laser light in a nonlinear Kerr medium. We take the nonlinear part H_I of the Hamiltonian (1) in normal order which makes the transition to classical fields quite transparent. Another version of the nonlinear part of the Hamiltonian (1) which is proportional to the square of the free part H_0 of the Hamiltonian (or the square of the number operator $\hat{n}=a^\dagger a$) is often in use. The difference between the two versions of the anharmonic oscillator seems to be trivial because they can be matched by changing the free oscillator frequency ω. When homodyne detection of squeezing is to be applied, however, this extra phase shift can be significant in the long-time limit.[33] There are also some consequences of this difference for the generation of discrete superpositions of coherent states in the model.[29] Here, we are not going to discuss such differences, and shall use the normally ordered version of the interaction Hamiltonian.

In the Heisenberg picture the equation of motion for the annihilation operator a reads

$$\dot{a} = -\frac{i}{\hbar}[a,H] = -i(\omega+\kappa a^\dagger a)a \ . \tag{2}$$

Since the number of photons $\hat{n}=a^\dagger a$ is a constant of motion [it commutes with the Hamiltonian (1)], Eq. (2) has the simple exponential solution

$$a(t) = \exp\{-it[\omega+\kappa a^\dagger(0)a(0)]\}a(0) \ . \tag{3}$$

The solution (3) is the exact operator solution describing the time evolution of the system. This solution can be directly applied in calculations of such field characteristics as variances, correlation functions, or higher statistical moments. The free evolution $e^{-i\omega t}$ in Eq. (3) can be factored out and, since the problems we are going to address in this paper do not involve free evolution, we shall drop it later on. This means that the operator $a(t)$ will represent only the slow part of the evolution which is due to the nonlinear part of the Hamiltonian.

In the case of light propagating through a nonlinear medium with refractive index η (instead of a field in a cavity), we can also replace the evolution time t by $-\eta z/c$ (on neglecting the dispersion of the medium), where z is the path traversed by the photons in the nonlinear medium. In effect, the solution (3) can be written as

$$a(\tau) = \exp[i\tau a^\dagger(0)a(0)]a(0) \ . \tag{4}$$

where

$$\tau = -\frac{\eta}{c}\kappa z \ . \tag{5}$$

In the Schrödinger picture, we are searching for the state evolution of the field propagating in a Kerr medium. This evolution can be described with the help of the evolution operator[13]

$$U(\tau) = \exp\left[\frac{i\tau}{2}a^{\dagger 2}a^2\right] = \exp\left[\frac{i\tau}{2}\hat{n}(\hat{n}-1)\right] \ , \tag{6}$$

with τ given by Eq. (5).

The state of the field outgoing from the medium is thus given by

$$|\psi(\tau)\rangle = U(\tau)|\psi_0\rangle \ , \tag{7}$$

where $|\psi_0\rangle$ is the state of the field incoming into the medium.

A convenient representation of the state of the field is the quasiprobability distribution (QPD) in the complex α plane defined as[11]

$$\begin{aligned} Q(\alpha,\alpha^*,\tau) &= \mathrm{Tr}[\rho(\tau)|\alpha\rangle\langle\alpha|] \\ &= \langle\alpha|\rho(\tau)|\alpha\rangle \\ &= \langle\alpha|\psi(\tau)\rangle\langle\psi(\tau)|\alpha\rangle \ , \end{aligned} \tag{8}$$

where $|\alpha\rangle$ is a coherent state. This function satisfies the relations

$$\int Q(\alpha,\alpha^*,\tau)\frac{1}{\pi}d^2\alpha = 1 \tag{9}$$

and

$$0 \le Q(\alpha,\alpha^*,\tau) \le 1 \ . \tag{10}$$

Some properties of this function for the anharmonic oscillator both in classical and quantum description of the oscillator have been discussed by Milburn[11] as well as Milburn and Holmes.[12]

If the state of the incoming beam is a coherent state $|\alpha_0\rangle$, the resulting state of the outgoing beam is given, according to (7), by

$$|\psi(\tau)\rangle=|\alpha_0,\tau\rangle=\exp(-\tfrac{1}{2}|\alpha_0|^2)\sum_{n=0}^{\infty}\frac{\alpha_0^n}{\sqrt{n!}}\exp(i\theta_n)|n\rangle\ , \tag{11}$$

where the phase θ_n is given by

$$\theta_n=\frac{\tau}{2}n(n-1)\ . \tag{12}$$

Because of the presence of the additional phases θ_n, the resulting state of the field is a generalized coherent state[34] which can become, under certain conditions,[35] a discrete superposition of coherent states. Some properties of such superpositions have been discussed recently.[27–29]

Inserting the state (11) into Eq. (8) one obtains for the QPD the expression

$$Q(\alpha,\alpha^*,\tau)=\exp(-|\alpha|^2-|\alpha_0|^2)\times\left|\sum_{n=0}^{\infty}\frac{(\alpha^*\alpha_0)^n}{n!}\exp(i\theta_n)\right|^2\ , \tag{13}$$

where θ_n is given by Eq. (12). For $\tau=0$, i.e., for the initial state $|\alpha_0\rangle$, the QPD function takes the form

$$Q(\alpha,\alpha^*,0)=\exp(-|\alpha-\alpha_0|^2)\ , \tag{14}$$

which is a Gaussian bell distribution centered on α_0. As the evolution of the oscillator proceeds, the states of the field can become squeezed and the Gaussian shape (14) of the Q function is deformed. Contours of the deformed Q function are conventionally considered as graphical representations of squeezing.[23]

Another graphical representation of squeezing consists of a plot of the variance of the field in a polar coordinate system. In the case of "ordinary" or squeezing that produces elliptical contours, some ellipses can be associated with the two representations: the contours of the Q functions are ellipses, and ellipses are built into the Booth elliptical lemniscate. Thus both graphical representations can, in this case, be considered as equivalent. This is not the case for anharmonic oscillator squeezing, as we shall prove in this paper.

III. FIELD VARIANCES, SQUEEZING, AND PRINCIPAL SQUEEZING

To discuss squeezing in our model, we define the Hermitian operator

$$X_\theta=ae^{-i\theta}+a^\dagger e^{i\theta} \tag{15}$$

which for $\theta=0$ corresponds to the in-phase quadrature component of the field and for $\theta=\pi/2$ to the out-of-phase component. We will also use the notation

$$X_{\theta=0}=X_1,\quad X_{\theta=\pi/2}=X_2 \tag{16}$$

to describe the two field quadrature components, the commutator of which is equal to

$$[X_1,X_2]=2i\ . \tag{17}$$

The variance of the operator (15) is given by

$$\begin{aligned}V_\theta&=\langle(\Delta X_\theta)^2\rangle\\&=\langle X_\theta^2\rangle-\langle X_\theta\rangle^2\\&=\langle(\Delta a)^2\rangle e^{-2i\theta}+\langle(\Delta a^\dagger)^2\rangle e^{2i\theta}+\langle\{\Delta a^\dagger,\Delta a\}\rangle\ ,\end{aligned} \tag{18}$$

where

$$\begin{aligned}\langle(\Delta a^\dagger)^2\rangle&=\langle a^{\dagger 2}\rangle-\langle a^\dagger\rangle^2\ ,\\ \langle\{a^\dagger,\Delta a\}\rangle&=\langle a^\dagger a+aa^\dagger\rangle-2\langle a^\dagger\rangle\langle a\rangle\ .\end{aligned} \tag{19}$$

The variance (18) is dependent on the angle θ. For the vacuum as well as for coherent states this variance is equal to unity independently of θ. The state of the field is said to be squeezed if for some θ the variance (18) becomes smaller than unity, and perfect squeezing is obtained if $V_\theta=0$. Since, for a given quantum state, the variance is still dependent on θ, the phase θ can be chosen in such a way as to maximize (or minimize) the variance. Differentiation with respect to θ leads to the angles θ_+ and θ_- for maximum and minimum variance given by the relation[32]

$$\exp(2i\theta_\pm)=\pm[\langle(\Delta a)^2\rangle/\langle(\Delta a^\dagger)^2\rangle]^{1/2}\ . \tag{20}$$

On inserting (20) into (18) one obtains the extremal variances in the form[32]

$$V_\pm=\langle(\Delta X_\pm)^2\rangle=\pm2[\langle(\Delta a)^2\rangle\langle(\Delta a^\dagger)^2\rangle]^{1/2}+\langle\{\Delta a^\dagger,\Delta a\}\rangle \tag{21}$$

where $X_\pm=X_{\theta=\theta_\pm}$. This immediately gives the condition for principal squeezing introduced by Lukš *et al.*:[30]

$$\langle\Delta a^\dagger\Delta a\rangle-|\langle(\Delta a)^2\rangle|<0\ . \tag{22}$$

It has been shown by Loudon[32] that the variance (18) can be written as

$$V_\theta=\langle(\Delta X_\theta)^2\rangle=\langle(\Delta X_-)^2\rangle\cos^2(\theta-\theta_-)+\langle(\Delta X_+)^2\rangle\sin^2(\theta-\theta_-)\ , \tag{23}$$

which is the equation for Booth's elliptical lemniscate in polar coordinates.

In homodyne detection the phase can be chosen at will by choosing the local oscillator phase. This allows for minimizing the variance V_θ, which means geometrically the choice of a coordinate system coinciding with the principal axes of the variances lemniscate (or the variances ellipse built into the lemniscate).

In the anharmonic oscillator model, assuming that the initial state of the field is a coherent state $|\alpha_0\rangle$, we have[10]

$$\begin{aligned}V_\theta(\tau)=2|\alpha_0|^2\{&\exp[|\alpha_0|^2(\cos2\tau-1)]\cos[2(\theta-\varphi_0)+\tau+|\alpha_0|^2\sin2\tau]\\&-\exp[2|\alpha_0|^2(\cos\tau-1)]\cos[2(\theta-\varphi_0)+2|\alpha_0|^2\sin\tau]+1-\exp[2|\alpha_0|^2(\cos\tau-1)]\}+1\end{aligned} \tag{24}$$

where we have used the relation

$$\alpha_0 = |\alpha_0| e^{i\varphi_0} \tag{25}$$

which takes into account the initial phase φ_0 of the field. From Eq. (24) it is clear that the in-phase and out-of-phase components are in fact defined with respect to the initial phase of the field, i.e., in phase means $\theta - \varphi_0 = 0$ and out of phase stands for $\theta - \varphi_0 = \pi/2$. From Eq. (25), we easily obtain the relation

$$\exp(2i\theta_\pm) = \pm\exp(2i\varphi_0 + i\tau + |\alpha_0|^2 \sin 2\tau) \ , \tag{26}$$

which gives

$$\theta_+ = \varphi_0 + \tfrac{1}{2}(\tau + |\alpha_0|^2 \sin 2\tau) \tag{27}$$

and

$$\theta_- = \theta_+ + \pi/2 \ . \tag{28}$$

This means that in the case of the anharmonic oscillator the phases $\theta_\pm = \theta_\pm(\tau)$ for which the variance approaches its extremal values depend on τ, i.e., they evolve in the course of the evolution of the oscillator. The intensity-dependent change in phase can easily be identified in (27). The extremal variances defined by (21) are given by the formula[36]

$$\begin{aligned} V_\pm(\tau) = \pm 2|\alpha_0|^2 \{ & \exp[2|\alpha_0|^2(\cos 2\tau - 1)] + \exp[4|\alpha_0|^2(\cos\tau - 1)] \\ & - 2\exp[|\alpha_0|^2(\cos 2\tau + 2\cos\tau - 3)]\cos[\tau + |\alpha_0|^2(\sin 2\tau - 2\sin\tau)] \}^{1/2} \\ & + 2|\alpha_0|^2\{1 - \exp[2|\alpha_0|^2(\cos\tau - 1)]\} + 1 \ . \end{aligned} \tag{29}$$

The extremal variances (29) describe the principal squeezing introduced by Lukš *et al.*[30] for the case of the anharmonic oscillator. It is obvious that principal squeezing evolves with τ showing periodic recurrences for $\tau = k \times 2\pi$ $(k = 1, 2, \ldots)$.

The time (or length) dependence of the variances (24) and (29) for a small mean number of photons $|\alpha|^2$ has already been discussed by Lukš *et al.*[36] and the destruction of quantum coherence due to attenuation and amplification has been discussed by Daniel and Milburn.[20]

Formulas (24) and (29) for squeezing and principal squeezing in the anharmonic oscillator model are exact analytical formulas. However, if the model is used to describe field propagation in a Kerr medium for which realistic values of τ are very small ($\tau \sim 10^{-6}$ is a rather optimistic estimation[9]) and the numbers of photons are very large ($|\alpha_0|^2 \gg 1$), it is possible to obtain much simpler approximate analytical formulas describing the variances. For $\tau \ll 1$ and $|\alpha_0|^2 \gg 1$, one can introduce a new variable

$$x = |\alpha_0|^2 \tau \tag{30}$$

which properly describes the scale of essential changes in the variances.[10] On expanding the variances (24) and (29) in power series in τ and retaining only leading terms of the expansions, we get the following formulas:

$$V_\theta(x) = 1 + 2x\{x[1 - \cos 2\varphi(x)] - \sin 2\varphi(x)\} \ , \tag{31}$$

where

$$\varphi(x) = \varphi_0 - \theta + x \tag{32}$$

and

$$V_\pm(x) = 1 + 2x[x \pm (1 + x^2)^{1/2}] \ . \tag{33}$$

Our expressions (31) and (33) are very simple analytical formulas approximately describing squeezing and principal squeezing in the system. The smaller τ is the better is the approximation used in deriving our formulas. The expression (33) for the extremal values of the variances can be alternatively obtained [instead of the power expansion of (29)] directly from Eq. (31) by finding its extremal values with respect to θ. The first derivative of (31) with respect to θ gives us the condition for the extrema:

$$x \sin 2\varphi(x) = \cos 2\varphi(x) \ , \tag{34}$$

and

$$\sin 2\varphi(x) = \pm(1 - x^2)^{-1/2}$$

which directly leads to Eq. (33).

As previously, $\theta - \varphi_0 = 0$ means the in-phase component of the field and $\theta - \varphi_0 = \pi/2$ the out-of-phase component in the variance (31). The only quantum term in Eq. (31) that appeared due to commutation of the field operators is the last term with $\sin 2\varphi(x)$. This term reappears in Eq. (33) as the unity under the square root. Had we omitted these terms, the variances (31) and (33) would never drop below the coherent (or vacuum) state variance. These are the terms responsible for squeezing in the anharmonic oscillator model for short time ($\tau \ll 1$) and a large number of photons ($|\alpha_0|^2 \gg 1$). From (27), for $\tau \ll 1$, we get

$$\theta_+ \approx \varphi_0 + x \ , \tag{35}$$

which can be found from Eq. (32) describing the intensity-dependent change in phase of the field. Were we to compensate dynamically this change in phase by changing the phase of the local oscillator so as to keep $\varphi(x)=0$ or $\pi/2$, we would never obtain squeezing.

Our formulas (31) and (33) are illustrated graphically in Figs. 1 and 2. In Fig. 1 the variances $V_1(x)$ and $V_-(x)$ are plotted against x. The oscillatory behavior of the variance $V_1(x)$, which is obvious from formula (31), is clearly seen and the principal squeezing variance $V_-(x)$ is the envelope that sets the lower limit for all the variances $V_\theta(x)$ [the upper limit is set by $V_+(x)$]. The envelope $V_-(x)$, as given by Eq. (33), is a monotonically decreasing function of x which asymptotically approaches zero. So, the subsequent minima of $V_1(x)$ become deeper and deeper as x increases and their values become very close to zero. This means that almost perfect squeezing can be obtained in the model. The asymptotic behavior of $V_\theta(x)$ and $V_-(x)$ for large x cannot, however, be taken too seriously because the approximations used to obtain formulas (31) and (33) break down when x becomes large. In fact, a high-precision numerical evaluation of the exact expression (29) carried out by us shows that, for given $|\alpha_0|^2(|\alpha_0|^2 \gg 1)$, $V_-(x)$ has a minimum around $x_{\min} \approx 10$. In Fig. 2 we present plots of $V_-(x)$ evaluated according to Eq. (29) for several values of $|\alpha_0|^2$ in order to show the position of this minimum. Our approximate formulas (31) and (33) work well for x less than $x_{\min}$. This is the price we have to pay for the striking simplicity of our formulas. The high degree of squeezing that can be obtained in the model has been reported by Tanaś and Kielich[9] for the two-mode version of the nonlinear propagation problem, and they used the term "self-squeezing" to denote the squeezing obtained in such a model. The meaning of self-squeezing comes from the fact that the strong optical field propagating through the nonlinear Kerr medium can squeeze its own quantum fluctuations.

In Fig. 3 we present examples of the lemniscates defined in Eq. (31) by plotting $\frac{1}{2}[V_\theta(\tau)]^{1/2}$ in polar coordinates for the values of x that correspond to the first and second minimum of $V_1(x)$. Again, the circle of unit diameter marks the vacuum fluctuations. Squeezing appears for all energies θ for which the lemniscate is contained inside the circle of vacuum fluctuations. A high degree of squeezing corresponds to a very short diameter

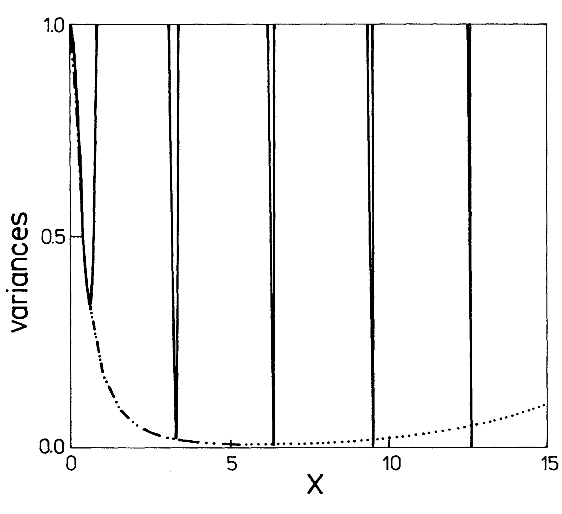

FIG. 1. Plot of the approximate variances $V_1(x)$, (solid line), $V_-(x)$ (dashed line), and the exact variance V_- (dotted line) according to (31), (33), and (29), respectively.

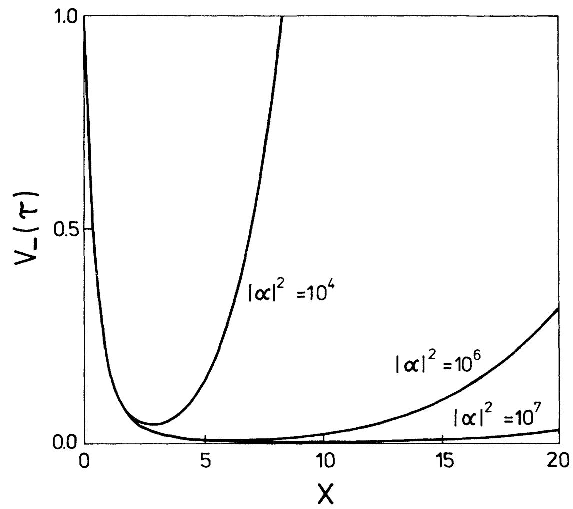

FIG. 2. Plots of the variance $V_-(\tau)$ [formula (29)] against the variable $x=|\alpha_0|^2\tau$ for given values of $|\alpha_0|^2$: (a) $|\alpha_0|^2=10^4$, (b) $|\alpha_0|^2=10^6$, (c) $|\alpha_0|^2=10^7$.

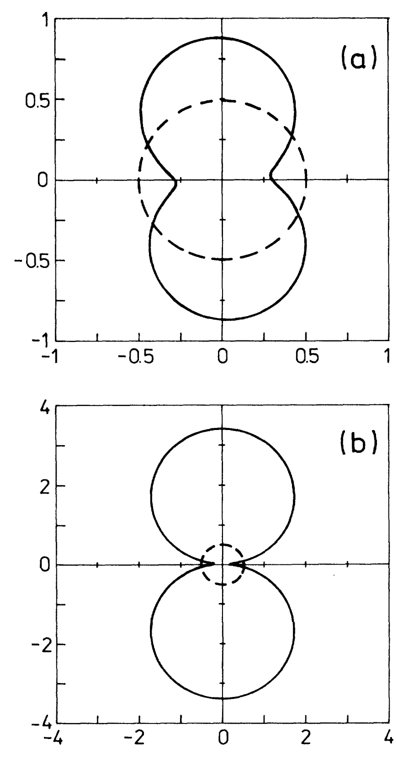

FIG. 3. Plot of $\frac{1}{2}[V_\theta(\tau)]^{1/2}$ as a function of θ [according to (31)] for (a) $x=0.59$ [first minimum of $V_1(x)$] and (b) $x=3.29$ [second minimum of $V_1(x)$].

of the lemniscate as is clearly visible in the second case.

Formula (31) retains its form even if higher-order nonlinearities are taken into account, except for the fact that x and $\varphi(x)$ have to be modified. This has been shown[37] with the use of saddle-point techniques to evaluate the sums appearing in the exact formulas for large numbers of photons.

IV. THE QUASIPROBABILITY DISTRIBUTION

The geometrical representations of variances discussed so far exhibit twofold rotational symmetry which reflects the fact that the variance is a quadratic function of the field operators. The variance forms a lemniscate in the polar coordinate system. The lemniscate is a projection of the variance ellipse which describes principal squeezing. As another convenient geometrical representation of squeezing, one often considers the quasiprobability distribution $Q(\alpha,\alpha^*,t)$ the contours of which, in the case of "ordinary" squeezing, form ellipses that can be treated as equivalent to the ellipses formed from principal squeezing. In the case of self-squeezing, i.e., squeezing obtained from the anharmonic oscillator model, the situation is not that simple, because the contours of the Q function are no longer ellipses, and an identification of such contours with the variance ellipses is not possible. To illustrate this situation more clearly, we present in Fig. 4 examples of the QPD contours obtained according to formula (13). These contours have crescent shape, first obtained by Kitagawa and Yamamoto,[13] who introduced the term "crescent" for squeezing with crescent-shaped contours of the QPD in order to contrast it with "ordinary" squeezing which produces elliptical contours. Such contours do not exhibit twofold rotational symmetry, which means that terms like the direction of squeezing, which is unambiguously related to the direction of the minor axis of the lemniscate (or ellipse), cannot be

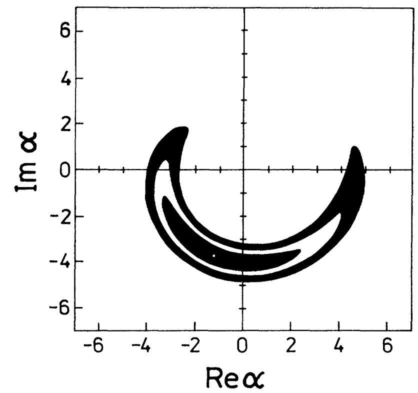

FIG. 4. Contours of the QPD in the complex α plane at $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$ of the maximum height obtained from formula (13) for $|\alpha_0|^2=16$ and $\tau=0.30$.

uniquely defined with respect to the shapes of the contours. The crescent shape of the QPD contours suggests that it is rather the amplitude of the field that is squeezed at the expense of the increased uncertainty of the phase, instead of the quadrature X_θ of the field. This idea has been exploited by Kitagawa and Yamamoto[13] as a means for reducing the photon-number uncertainty. If the number of photons in the field is not very big the quadrature squeezing obtained in the model is not very pronounced. It has been shown,[9,10] however, that for great numbers of photons a high degree of squeezing can be obtained. Unfortunately, direct numerical evaluation of the QPD according to formula (13) is impossible for $|\alpha_0|^2 \gg 1$. In this case, however, the saddle-point technique can be applied to evaluate the sums appearing in Eq. (13). We need to evaluate the sum

$$S=\sum_{n=0}^{\infty}\exp[-\tfrac{1}{2}(|\alpha|^2+|\alpha_0|^2)]\frac{(\alpha^*\alpha_0)^n}{n!}\exp[\tfrac{1}{2}i\tau n(n-1)]$$
$$=\exp[-\tfrac{1}{2}(|\alpha|-|\alpha_0|)^2]\sum_{n=0}^{\infty}\exp(-|\alpha|\,|\alpha_0|)\frac{(|\alpha|\,|\alpha_0|)^n}{n!}\exp[in(\varphi_0-\varphi)+\tfrac{1}{2}i\tau n(n-1)]\ , \tag{36}$$

where

$$\alpha_0=|\alpha_0|e^{i\varphi_0}\ ,\qquad \alpha=|\alpha|e^{i\varphi}\ . \tag{37}$$

According to (36), we have to evaluate the sum

$$S'=\sum_{n=0}^{\infty}p_n\exp[in(\varphi_0-\varphi)+\tfrac{1}{2}i\tau n(n-1)]\ , \tag{38}$$

where

$$p=e^{-N}N^n/n! \tag{39}$$

is the Poissonian weighting factor with $N=|\alpha|\,|\alpha_0|$. When $N\gg 1$, the technique applied in calculations of the collapses and revivals[38] in the Jaynes-Cummings model can be used to calculate the sum (38). Of course, our assumption $N\gg 1$, i.e., $|\alpha|\,|\alpha_0|\gg 1$, means that the result obtained in this way may be incorrect for small $|\alpha|$. However, the overall exponential factor in Eq. (36) makes the QPD essentially different from zero only if $|\alpha|$ is sufficiently close to $|\alpha_0|$, which for $|\alpha_0|\gg 1$ means also large $|\alpha|$. So we believe that our results are valid whenever the number of photons of the beam is large ($|\alpha_0|^2\gg 1$). We can write the sum (38) as the integral

$$S\approx\left[\frac{2N}{\pi}\right]^{1/2}\int_0^{\infty}\exp[Nf(y)]dy \tag{40}$$

where we have introduced the notation $y^2 = n/N$, and the function $f(y)$ is given by

$$f(y) = y^2(1 - 2\ln y) - 1 + iy^2(\varphi_0 - \varphi) + \tfrac{1}{2}i\tau y^2(y^2 N - 1) . \tag{41}$$

The saddle points of $f(y)$ are given by

$$\frac{\partial f(y)}{\partial y} = 0 . \tag{42}$$

which leads us to the following equation for finding the saddle points:

$$-4y \ln y + 2iy(\varphi_0 - \varphi) + 2i\tau N y^3 - i\tau y = 0 . \tag{43}$$

On introducing $y = \rho \exp(i\vartheta/2)$, we have

$$\begin{aligned} &\ln\rho + \tfrac{1}{2}\tau N \rho^2 \sin\vartheta = 0 , \\ &\vartheta - (\varphi_0 - \varphi) + \tfrac{1}{2}\tau - \tau N \rho^2 \cos\vartheta = 0 . \end{aligned} \tag{44}$$

From Eq. (44) one obtains the following exact relation:

$$\rho = \exp\{ -\tfrac{1}{2}[\vartheta - (\varphi_0 - \varphi) + \tfrac{1}{2}\tau]\tan\vartheta \} , \tag{45}$$

which after insertion back into one of the equations (51) gives us the following equation for ϑ:

$$\vartheta' \tan\vartheta \exp(\vartheta' \tan\vartheta) = \tau |\alpha|\,|\alpha_0| \sin\tau , \tag{46}$$

where

$$\vartheta' = \vartheta - (\varphi_0 + \varphi) + \tfrac{1}{2}\tau . \tag{47}$$

Equation (46) can be solved numerically for given values of the parameters, defining the saddle point $y = \rho \exp(i\vartheta/2)$ after inserting ϑ obtained from (46) into (45). We are interested in solutions for small τ only, and ignore all saddle points that can appear for large values of τ. If τ is very small and $|\alpha_0|$ as well as $|\alpha|$ are great, high precision is needed in solving Eq. (46). We should also emphasize that the saddle-point location depends on the actual coordinates $|\alpha|$ and φ for which we want to evaluate the QPD function.

When the saddle point has been found, we can evaluate the QPD according to the formula

$$Q(\alpha,\alpha^*,\tau) = \frac{4\exp[-(|\alpha| - |\alpha_0|)^2 + 2|\alpha|\,|\alpha_0| \mathrm{Re} f(y)]}{|f^{(2)}(y)|} \tag{48}$$

where the function $f(y)$ and its second derivative $f^{(2)}(y)$ are taken at the saddle point. This gives us a relatively simple formula for the Q function in the case of a large number of photons, which reads

$$\begin{aligned} Q(\alpha,\alpha^*,t) = D^{-1} &\exp(-|\alpha|^2 - |\alpha_0|^2) \\ &+ 2|\alpha|\,|\alpha_0| \cos\vartheta (1 + \vartheta' \tan\vartheta) \exp(-\vartheta' \tan\vartheta) \end{aligned} \tag{49}$$

with

$$D = [(1 + \vartheta' \tan\vartheta)^2 + (\vartheta')^2]^{1/2} . \tag{50}$$

If $\tau = 0$, from (46) we have $\vartheta = \varphi_0 - \varphi$, and formula (49) turns into

$$\begin{aligned} Q(\alpha,\alpha^*,t) &= \exp[-|\alpha|^2 - |\alpha_0|^2 + 2|\alpha|\,|\alpha_0| \cos(\varphi_0 - \varphi)] \\ &= \exp(-|\alpha - \alpha_0|^2) , \end{aligned} \tag{51}$$

which is the initial Gaussian distribution given by Eq. (14).

Our formula (49) allows for numerical evaluation of the QPD in the case $\tau \ll 1$ and $|\alpha_0|\,|\alpha \gg 1$ which is impossible when direction summations are to be performed according to Eq. (13). To illustrate formula (49), we show in Fig. 5 contours of the QPD obtained from formula (49), together with (46) and (47) for the parameters τ and $|\alpha_0|$ chosen in such a way as to get $x = |\alpha_0|^2 \tau$ equal to the values of the first and second minima of the variance $V_1(x)$. The corresponding contours have quite regular elliptic shapes that arise from ordinary squeezing rather than the crescent-shape contours from squeezing obtained for a small number of photons. At the first minimum 66% squeezing is obtained, and it is seen from Fig. 5(a) that this result can be slightly improved because the ellipses are still slightly inclined from the axes of the coordinate system. The second minimum corresponds to 98% squeezing, and in Fig. 5(b) the ellipses are much

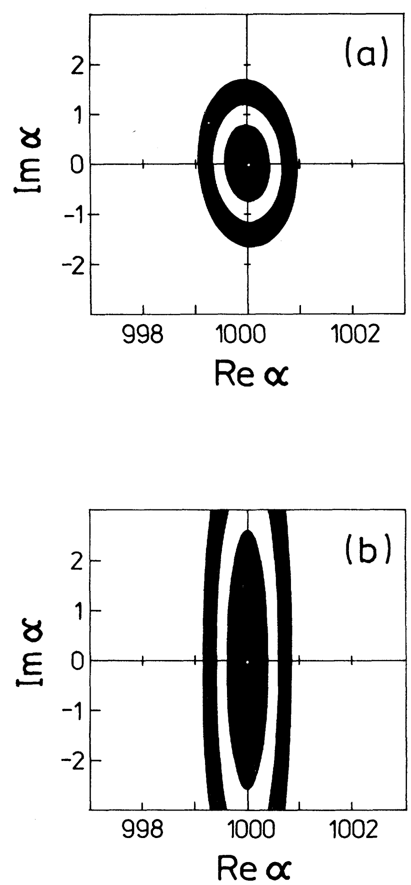

FIG. 5. Same as Fig. 4 but obtained from formula (49) for $|\alpha_0|^2 = 10^6$ and (a) $x = 0.59$ and (b) $x = 3.29$

more "squeezed" and almost perfectly fit into the coordinate system. So, in the case of large photon numbers and small anharmonicity, the QPD can again be considered as a good graphical representation of the uncertainty ellipse of the quadrature field components. This elliptic shape of the QPD contours does not mean, however, a change of the nature of the quantum state of the field when $|\alpha_0|^2$ is large; it rather reflects the fact that for $|\alpha_0| \gg 1$ the crescent shape is hardly visible, because the essentially nonzero values of the QPD are concentrated around a circle of radius $|\alpha_0|$, which means very small curvature of the crescent shape when $|\alpha_0| \gg 1$.

Generally, the QPD carries more information than the variance, which is the second statistical moment only. The shape of the QPD contours can be much more complicated than an ellipse, because it contains information about the higher statistical moments as well.

V. CONCLUSIONS

We have considered the problem of squeezing and its graphical representation for the anharmonic oscillator model. We have obtained approximate analytical formulas describing the quadrature field variances (squeezing), extremal variances (principal squeezing), and the quasiprobability distribution function $Q(\alpha,\alpha^*,\tau)$ for the most interesting case from the experimental point of view, namely, that of small nonlinearity of the medium and a large number of photons of the field. The degree of squeezing that can be obtained in this case is much higher than in the case of a small number of photons. The graphical representations of squeezing such as lemniscates (or uncertainty ellipses) representing variances of the field and the QPD contours are used to illustrate the results. It is shown that the QPD contours are not a very good graphical representation of squeezing when squeezing is defined with respect to the values of the second statistical moment, i.e., the variance. This is most strikingly visible in the case of a small number of photons in which the QPD contours have crescent shapes. As the number of photons increases the crescent shape of the contours becomes less pronounced, and they become more and more elliptic. In this case the QPD contours can again be considered as a good graphical representation of squeezing. Generally, however, the QPD contains information about the higher-order statistical moments as well, and can be roughly treated as a representation of squeezing. It is known[27–29] that for some special choices of τ the state of the field becomes a discrete superposition of coherent states, and the QPD has even a multipeak structure.

ACKNOWLEDGMENTS

This work was supported by the Polish Research Project No. CPBP 01.06.

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