Analytic structure of the ac quasienergy in the complex field plane

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Motivated by the possibility of developing new computational techniques for studying multiphoton ionization of atoms by monochromatic radiation, we have analyzed, in a nonrigorous fashion, the behavior of the ac quasienergy as a function of the complex field strength F. We elaborate upon earlier work of Manakov and Fainshtein [Theor. Math. Phys. **48**, 815 (1981)] and conjecture that the ac quasienergy is a multivalued analytic function $E_{ac}(F)$ whose branches originate from different unperturbed (real or "shadow") atomic levels for real values of F. We further conjecture that the branch points of $E_{ac}(F)$ are of the square-root type. These branch points occur at complex values of F where two branches coalesce, and, as F sweeps along the real axis, the passage past a branch point coincides with the passage past either an intermediate multiphoton resonance (R) or a multiphoton ionization threshold (T). Branch points of type-R group into quadruplets, while branch points of type-T group into pairs. The two branches that intersect at a type-R branch point are both physically accessible, and originate from real levels, while only one of the two branches that intersect at a type-T branch point is physically accessible—the unphysical type-T branch is a "shadow" eigenvalue, which corresponds to a state with unphysical boundary conditions. We discuss the probability for the atom to make a transition from one branch to another when F is a slowly varying function of time. Normally, a type-R branch point enhances the ionization signal, while a type-T branch point diminishes the signal. In partial support of some of our conjectures, we present results of an accurate numerical study of the quasienergy (and its perturbation expansion) for the ground state of both the hydrogen atom and a model atom (an electron bound to a zero-range potential).

I. INTRODUCTION

Several powerful techniques are now available for calculating the dc shift $\Delta_{dc}(F)$ and the dc width $\Gamma_{dc}(F)$ of the energy level of an atom in a dc field of arbitrary strength F. The development of these techniques (see, for example, Refs. 1–5) has rested in part on an understanding of the analytic structure, in the complex F plane, of the dc quasienergy $E_{dc}(F) \equiv E_0 + \Delta_{dc}(F) - i\Gamma_{dc}(F)/2$, where E_0 is the unperturbed energy of the level of interest. In particular, a dispersion relation[6] connecting the real and imaginary parts of $E_{dc}(F)$ has been constructed, demonstrating that $\Gamma_{dc}(F)$ can be calculated from a knowledge of $\Delta_{dc}(F)$.

In this paper, we study the analytic structure of the ac quasienergy $E_{ac}(F)$ in the complex F plane, where now $F/\sqrt{2}$ is the root-mean-square strength of a monochromatic radiation field. ($F = \sqrt{8\pi I/c}$, where I is the intensity of the light.) We write $E_{ac}(F) \equiv E_0 + \Delta_{ac}(F) - i\Gamma_{ac}(F)/2$, where $\Delta_{ac}(F)$ and $\Gamma_{ac}(F)$ are the ac shift and ac width. Our discussion is heuristic and certainly not comprehensive; we give no rigorous proofs, and we concentrate on those features of $E_{ac}(F)$ that are of physical relevance. In part, our motivation for this study is that it may open the possibility for developing techniques for calculating $E_{ac}(F)$ that are similar to, and as powerful as, those developed for calculating $E_{dc}(F)$. It is natural to expect that $E_{ac}(F)$ has physically significant singularities near real values of F at which the ionization signal exhibits structure. Two different observable phenomena come to mind. The first is the substructure in the above-threshold peaks of the photoelectron energy spectrum, seen initially by Freeman *et al.*[7] by ionizing Xe with short-pulsed light. This substructure is due to Rydberg levels of Xe shifting in and out of resonance with the light as F varies. Near a resonance an *enhancement* of the ionization signal is observed. The second phenomenon is the disappearance of an above-threshold peak in the photoelectron energy spectrum, seen initially by Kruit *et al.*[8] The disappearance of a peak is due to a channel closing as F sweeps past a multiphoton ionization threshold; the minimum number of photons that the atom must absorb to ionize increases by unity. Near a threshold a *reduction* in the ionization signal would be seen were it not for Rydberg levels accumulating at each threshold. In the case of multiphoton detachment of the negative hydrogen ion a reduction in the calculated[9] yield for neutral atoms is found as F sweeps by a detachment threshold.[10]

The analytic structure of $E_{ac}(F)$, and its relevance to the convergence of the perturbation series for $E_{ac}(F)$, was discussed earlier by Manakov and Fainshtein;[11] we amplify some of their statements. We conjecture that, in general, $E_{ac}(F)$ is nonsingular everywhere in the finite

complex F plane except for branch-point singularities of the square-root type, where two branches join. As F sweeps along the real axis, the passage past a branch point coincides with the passage past either an intermediate multiphoton resonance (R) or a multiphoton ionization threshold (T). The two branches that meet at a branch point of type R do so in the E plane at a complex value of E; both branches are physically accessible and normally originate from a real bound or quasibound level at $F = 0$. However, the two branches that meet at a branch point of type T do so in the E plane at a real value of E, coincident with a threshold; but only one of these branches is physically accessible—the unphysical branch corresponds to a "shadow" state,[12] in which the emergent photoelectron, at asymptotically large distances from the residual atomic core, is represented (in at least one channel) by an *in*going wave rather than an *out*going wave. The probability for depopulating the physical branch, when a type-T branch point is passed, is related to the probability for the atom to ionize as this branch point is passed, and this turns out to be very small if the passage is rapid. As we will see, due to symmetry with respect to a shift in the origin of time by one half-cycle, for any branch point at $F_{\rm br}$ there is another one, on the same sheet of the Riemann F surface, at $-F_{\rm br}$. Type-R branch points group into quadruplets; if the two atomic levels participating in the resonance are only weakly coupled to the continuum, the branch points occur in conjugate pairs, but in general the members of a quadruplet are not necessarily distributed over all four quadrants of the F plane. Type-T branch points group into pairs, one at $F_{\rm br}$, the other at $-F_{\rm br}$. Only type-T branch points were discussed by Manakov and Fainshtein, and their relevance to shadow states was not explored. We present a unified treatment of both types of branch points. For simplicity, in our discussion of $E_{\rm ac}(F)$ we regard the atomic potential as a *screened* Coulomb potential, which supports only a *finite* number of bound states. The situation is somewhat different, and more complicated, for an unscreened Coulomb potential since the latter potential supports an infinite number of bound states which accumulate at each threshold, and so the type-T branch points may, in fact, be essential singularities when the atomic potential is pure Coulomb. However, in practice all atomic potentials are screened.

We do not fully understand the limit in which the frequency ω of the light vanishes; but it is clear that in this limit the thresholds disappear. For any finite value of ω there are infinitely many type-T branch points (since there are infinitely many thresholds), and those branch points that are at a finite distance from the origin of the F plane, when ω is finite, accumulate at the origin as ω vanishes (see Sec. IV C). The real axis is a natural branch cut for $E_{\rm dc}(F)$, with the origin an essential singularity of $E_{\rm dc}(F)$. However, we do not know what happens to those branch points that are infinitely far from the origin of the F plane, when ω is finite; presumably they simply fade away. (Similar considerations apply to type-R branch points.) That $E_{\rm dc}(F)$ is singular at $F = 0$ is already implied by the fact that the perturbation expansion of $E_{\rm dc}(F)$, in powers of F^2, has no imaginary part, and therefore yields (erroneously) a vanishing dc width. On the other hand, the perturbation expansion of $E_{\rm ac}(F)$ does yield a nonvanishing ac width, if terms in powers of F^2 at least as large as N_0 are retained, where N_0 is the minimum number of photons that the atom must absorb to ionize in a very weak field. Thus we do not normally expect $E_{\rm ac}(F)$ to be singular at $F = 0$.[13] The radius of convergence of the perturbation expansion of a *particular* branch of $E_{\rm ac}(F)$ is the distance of the closest branch point from the point $F = 0$ *on that sheet of the Riemann F surface* on which the branch is defined; there may be branch points on other sheets that are closer to the origin, but they do not determine the radius of convergence of the perturbation expansion of the branch of interest. That $E_{\rm ac}(F)$ has a nonzero radius of convergence (at most frequencies) has been proved rigorously.[14]

In Sec. II we introduce the Floquet ansatz, which leads to the eigenvalue problem for $E_{\rm ac}(F)$, and we discuss qualitatively the nature of $E_{\rm ac}(F)$. In Sec. III we introduce the scattering operator and examine its analytic structure in the energy plane. In Sec. IV we analyse in more detail the behavior of $E_{\rm ac}(F)$ near its branch points in the F plane. In Sec. V we discuss the probability for the atom to make a transition from one branch to the other. In Sec. VI we present some results of a numerical study of the ground-state branches of the quasienergy for both the hydrogen atom in a linearly polarized field and a model atom (an electron bound to a zero-range potential) in a circularly polarized field. We numerically illustrate some of the features that we discuss in the preceding sections; in particular, we show that the branch points are of the square-root type, and that their positions determine the radii of convergence of the perturbation series.

II. THE EIGENVALUE PROBLEM

A. Quasienergy

Let H_a denote the Hamiltonian of the bare atom, with $H(t) \equiv H_a + V(t)$ the Hamiltonian of the dressed atom. We regard the atom as having only one electron, and we describe the radiation as a classical monochromatic field. We work in the velocity gauge, in which the atom-field interaction $V(t)$ is, in the dipole approximation,

$$V(t) = (-e/\mu c)\mathbf{A}(t)\cdot\mathbf{p} \tag{1a}$$

$$\equiv V_+ e^{-i\omega t} + V_- e^{i\omega t} , \tag{1b}$$

where $\mathbf{A}(t)$ is the vector potential of the field and where e, μ, and $\mathbf{p}$ are, respectively, the charge, reduced mass, and (center-of-mass) canonical momentum of the electron. We have omitted the $\mathbf{A}(t)^2$ term from $V(t)$ (it can always be removed by a simple gauge transformation). If we write the oscillating electric field as $F(\mathrm{Re}\hat{\epsilon}e^{-i\omega t})$, with ω the frequency, $\hat{\epsilon}$ the unit polarization vector ($\hat{\epsilon}^*\cdot\hat{\epsilon} = 1$),

and t the time, we have $\mathbf{A}(t) = (c/\omega)F(\mathrm{Im}\hat{\epsilon}e^{-i\omega t})$. Therefore we have $V_+ = i(e/2\mu\omega)F\hat{\epsilon}\cdot\mathbf{p}$ and $V_- = -i(e/2\mu\omega)F\hat{\epsilon}^*\cdot\mathbf{p}$. The intensity of the field is $I = cF^2/(8\pi)$.

Provided that I varies sufficiently slowly with time—that is, provided that I is almost constant over one cycle (so that the frequency bandwidth is small compared to ω), and provided that the state of the atom does not change appreciably over a cycle as I varies—we can treat F as an adiabatic parameter, and regard $V(t)$ as periodic in t with period $2\pi/\omega$. Under these conditions we can make the Floquet ansatz (see, for example, Refs. 15 and 16): Thus we replace the exact state vector of the atom by $e^{-iE_{\rm ac}(F)t/\hbar}|\mathcal{F}(t)\rangle$, where $|\mathcal{F}(t)\rangle$ is periodic in t with period $2\pi/\omega$. The ac quasienergy emerges naturally as an eigenvalue. Substituting the Floquet ansatz into the time-dependent Schrödinger equation $i\hbar(d/dt)|\Psi(t)\rangle = H(t)|\Psi(t)\rangle$, making the harmonic expansion $|\mathcal{F}(t)\rangle = \sum_n e^{-in\omega t}|\mathcal{F}_n\rangle$, and using Eq. (1b), yields the following set of coupled equations for the time-independent harmonic components $|\mathcal{F}_n\rangle$:

$$(E_{\rm ac} + n\hbar\omega - H_a)|\mathcal{F}_n\rangle = V_+|\mathcal{F}_{n-1}\rangle + V_-|\mathcal{F}_{n+1}\rangle\ , \qquad (2)$$

where, for brevity, we have dropped the argument F of $E_{\rm ac}(F)$, as we do on other occasions. Since this set of equations for the harmonic components is homogeneous, it forms, together with appropriate boundary conditions (see below), a standard eigenvalue problem. The eigenvalue $E_{\rm ac}(F)$ is determined in principle by a single characteristic equation that has no singularities in the finite F plane. [An explicit example of this characteristic equation is given by Eq. (24) below.] Presumably, therefore, $E_{\rm ac}(F)$ is a single (multivalued) analytic function of F with (infinitely) many branches, each branch being a different eigenvalue solution of the characteristic equation. The fact that $E_{\rm ac}(F)$ is a single analytic function implies that if one branch of $E_{\rm ac}(F)$ is known along a segment of the real F axis, all other branches can in principle be calculated by analytic continuation around the branch points in the complex F plane. Note that Eq. (2) is linear not only in $E_{\rm ac}$ but also in F—recall that the operators $V_\pm$ are proportional to F. Rather than regard $E_{\rm ac}$ as the eigenvalue, we could fix its value and regard F as the undetermined eigenvalue. In fact, in calculations carried out so far, we have always found the eigenvalue problem for F more easy to solve than the eigenvalue problem for $E_{\rm ac}$. Equation (2) is also linear in ω, and we can, of course, also fix E and F and regard ω as the eigenvalue; we plan to report on a study of the quasienergy in the complex ω plane elsewhere.

For any branch of the quasienergy there is another branch that differs by exactly $m\hbar\omega$, where m is any integer, since $H(t) \equiv H_a + V(t)$ is periodic in time with frequency ω.[15,16] However, branches of $E_{\rm ac}(F)$ that differ by an integral multiple of $\hbar\omega$ are not physically distinguishable (if they are of the same symmetry). In the limit of vanishing F, each branch of $E_{\rm ac}(F)$ approaches a discrete eigenvalue of H_a. A discrete eigenvalue of H_a may be real and negative, in which case it corresponds to a bound-state level of the unperturbed atom, or it may be complex, in which case it corresponds to a quasi-bound-state level of the unperturbed atom. In this paper, we are primarily interested in, and in fact we restrict our discussion to, bound-state branches—those that originate from a bound state. On a bound-state branch, $\Gamma_{\rm ac}(F)$ vanishes as F vanishes, but $\Delta_{\rm ac}(F)$ vanishes only on the branch that approaches E_0; for other bound-state branches, $E_0+\Delta_{\rm ac}(0)$ is the energy of an unperturbed bound-state level different from the one of interest, unless $\Delta_{\rm ac}(0)$ is an integral multiple of $\hbar\omega$. When we expand $E_{\rm ac}(F)$ as a Taylor series about E_0, in powers of F, we are expanding a particular branch, and so we are performing the expansion on a particular sheet of the multisheeted Riemann F surface. Therefore the radius of convergence of the perturbation expansion of the shift and width of a particular level is determined by the location of the branch point closest to the origin on the appropriate sheet of the Riemann surface.

Note that, with the $\mathbf{A}(t)^2$ term removed from $V(t)$, the threshold of the continuous spectrum of the atom, which we can choose to be at zero energy, is unshifted by the field. This can be understood as follows: If the cycle-averaged energy of the atom in the field is at or above the continuum threshold, the electron can escape to infinity, and in a stationary state it spends most of the time as a free particle, far from the residual atomic core. If the cycle-averaged energy of the atom is exactly at the continuum threshold, the drift momentum of the electron, when it is far from the atomic core, is zero. Now, in the velocity gauge the canonical momentum of a free electron *is* the drift momentum. Hence the $\mathbf{A}(t)\cdot\mathbf{p}$ term cannot shift the continuum threshold; this is true for *all* values of F in the finite complex F plane. It follows that the continuum threshold energy remains zero for all F.[17]

B. Confluence of branches

To illustrate how the branch points arise, we consider the multiphoton ionization of an atom that is initially in the bound state a. As the intensity of the light is varied, the energy level a and other levels shift. Let us suppose that at a certain intensity there is an intermediate N-photon resonance between levels a and b, that is, we suppose that N photons can almost resonantly excite the level b from a. At this (real) intensity the difference of the shifted binding energies of states a and b is close to, and may even equal, $N\hbar\omega$; but the difference of the a and b branches of the quasienergy $E_{\rm ac}(F)$ do not differ by exactly $N\hbar\omega$ (recall the "no crossing rule" of Wigner and von Neumann) since bound states decay so that the bound-state branches of the quasienergy are complex [$\Gamma_{\rm ac}(F) \neq 0$] for nonzero *real* F. However, if we allow F to become complex we introduce another degree of freedom, and in general there is a nearby *complex* value of F, which we denote by $F_{\rm res}$, for which the difference of the a and b branches of the $E_{\rm ac}(F)$ is *exactly* $N\hbar\omega$.

If we now recall that for any branch of the quasienergy there is another branch that differs by exactly an integral multiple of $\hbar\omega$, it follows that at $F_{\rm res}$ the a branch and the branch displaced by $-N\hbar\omega$ from the b branch coincide exactly. In other words, $F_{\rm res}$ is a branch point of $E_{\rm ac}(F)$, of order two (since *two* branches meet there). Therefore $F_{\rm res}$ is a square-root branch point, and if it is to have a significant physical effect, $F_{\rm res}$ must not lie far from the real axis. Note, incidentally, that there may be other physically significant $(N+2m)$-photon resonances between levels a and b, where m is an integer.[16]

There are, as discussed in the Introduction, also branch points associated with the confluence of branches at channel thresholds. Let us denote by $N_{\rm min}$ the minimum number of photons that the atom must absorb in order to ionize from the state a at a given intensity. Thus $N_{\rm min}$ is the smallest integer N for which $E_0+\Delta_{\rm ac}(F)+N\hbar\omega$ is positive. As F increases, energy level a shifts. Suppose that this level shifts downward relative to the (unshifted) continuum threshold of the atomic spectrum. Then an intensity may be reached at which the *real* part of the a branch of the quasienergy is exactly $-N_{\rm min}\hbar\omega$; at slightly higher values of the intensity, that is, at slightly larger values of real F, the atom must absorb an additional photon to ionize from state a. In general, there is a nearby complex value of F, which we denote as $F_{\rm thr}$, at which the a branch of the quasienergy, not just the real part, is exactly $-N_{\rm min}\hbar\omega$. At $F_{\rm thr}$ the branch of the quasienergy that is displaced from the a branch by $N_{\rm min}\hbar\omega$ coincides exactly with the continuum threshold. Now, at this threshold, outgoing waves in the $N_{\rm min}$th channel are indistinguishable from ingoing waves, and therefore $F_{\rm thr}$ is a square-root branch point of $E_{\rm ac}(F)$, at which a branch with outgoing-wave character meets a branch with ingoing-wave character.

C. Boundary conditions

We have already remarked that branches of $E_{\rm ac}(F)$ of the same symmetry which differ by an integral multiple of $\hbar\omega$ are not physically distinguishable. Other branches are distinguishable by the asymptotic boundary conditions on the harmonic components. The atom is initially bound, and when the field induces it to ionize, the photoelectron moves outwards, away from the residual atomic core. Thus the physically relevant boundary conditions, which pertain to F real, must describe a photoelectron that, asymptotically, moves outwards, not inwards. The position space representation $\langle\mathbf{x}|\mathcal{F}_n\rangle$ of a harmonic component must be regular at $r=0$ (where $r\equiv|\mathbf{x}|$) and, in the velocity gauge, must be a superposition of outgoing (or exponentially decaying) waves[17] at $r\sim\infty$:

$$\langle\mathbf{x}|\mathcal{F}_n\rangle\sim\sum_{m=-\infty}^{\infty}f_{mn}(\hat{\mathbf{x}},E_{\rm ac})e^{ik_m(E_{\rm ac})r}/r\ , \tag{3}$$

where $\hat{\mathbf{x}}$ is the unit vector $\mathbf{x}/r$, and where $k_m(E)$ (which we often abbreviate as k_m) is the wave number for channel m:

$$k_m(E)=(2\mu/\hbar^2)^{1/2}(E+m\hbar\omega)^{1/2}\ . \tag{4}$$

If the outgoing electron sees a charged atomic core we must include the Coulombic logarithmic distortion in the exponent on the right-hand side of Eq. (3). Each harmonic component contains open and closed channels; channel m is open or closed depending if the real part of $(E_{\rm ac}+m\hbar\omega)$ is positive or negative, respectively.[18] If, for real F, channel m is physically accessible, ${\rm Im}k_m(E_{\rm ac})$ must be positive if the channel is closed, while ${\rm Re}k_m(E_{\rm ac})$ must be positive if the channel is open; this follows because the exponential e^{ik_mr} must decay if the channel is closed (the atom has absorbed an insufficient number of photons to ionize) while e^{ik_mr} must behave as an outgoing wave if the channel is open (ingoing waves cannot be present if the atom is initially bound). The probability for finding the electron in some finite volume $\mathcal{V}$ centered at the atomic core is

$$\int_{\mathcal{V}}|e^{-iE_{\rm ac}t/\hbar}\langle\mathbf{x}|\mathcal{F}(t)\rangle|^2\ , \tag{5}$$

and this probability must decay in time, so that if F is real, $\Gamma_{\rm ac}(F)$ must be positive on physical branches of $E_{\rm ac}(F)$. Hence, if the mth channel is open, and F is real, ${\rm Im}k_m<0$; therefore e^{ik_mr} explodes as r increases. Although the divergence of e^{ik_mr} may seem unphysical at first sight, it is reasonable provided that $\Gamma_{\rm ac}(F)$ is small compared to the energy $\mu v_m^2/2$ of the outgoing photoelectron in the mth (open) channel. To see this, note first that

$$\mu v_m^2/2\equiv{\rm Re}(E_{\rm ac}+m\hbar\omega)\ , \tag{6}$$

and, with $\Gamma_{\rm ac}$ small, $\hbar k_m\approx\mu v_m-i(\Gamma_{\rm ac}/2v_m)$. Therefore the outgoing (spherical) electron wave in channel m contains the exponential $e^{ik_mr}e^{-iE_{\rm ac}t/\hbar}\approx e^{ik_m(r-v_mt)}$. The magnitude of this exponential is smaller than or equal to unity, since in a quantum-mechanical wave-packet description, the photoelectrons leaving the atom in channel m correspond to a component of the wave packet that is *bounded by the wave front* propagating according to $r\approx v_mt$. [Were $\Gamma_{\rm ac}(F)$ to be negative, for real F, on physical branches, e^{ik_mr} would decay as r increases in all channels m, and the eigenvector would be normalizable. When acting on a normalizable state, the Hamiltonian is Hermitian, and it follows that $E_{\rm ac}(F)$ should be real and hence that $\Gamma_{\rm ac}(F)$ should be zero, in contradiction with our assumption.]

Two branches that differ by $l\hbar\omega$, with l an integer, cannot be distinguished by the asymptotic boundary conditions since k_m for one branch is equal to k_{m+l} for the other, and so the two branches have the same sequence, $\{k_m\}$, of channel wave numbers. Whenever two physically distinguishable branches intersect, at a branch point of $E_{\rm ac}(F)$, the asymptotic boundary conditions for the two branches become identical, and therefore states corresponding to two different branches are *physically indistinguishable* at their common branch point. Type-R and type-T branch points have different characteristics. Two branches a and b that intersect at a type-R (resonance)

branch point F_{res} have, at F_{res}, a common value E_{res} that is complex; E_{res} lies *off* the real axis of the energy plane. Except in the neighborhood of F_{res}, the two branches are distinguishable by the atomic characters of the states a and b, as well as by the photoelectron energy spectrum. Both branches are physically accessible (for F real). On the other hand, two branches that intersect at a type-T (threshold) branch point F_{thr} have a common value E_{thr} that is real; $E_{thr} = -m\hbar\omega$, where m is the channel number of the threshold at which the two branches intersect in the energy plane. At the mth threshold k_m vanishes, and in the neighborhood of F_{thr}, the two branches have values of k_m that are almost equal but opposite. One of the branches is physically accessible, the other is not since k_m has the wrong sign; for the unphysical branch, $\mathrm{Im}k_m$ is negative if channel m is closed (so $e^{ik_m r}$ explodes), and/or $\mathrm{Re}k_m$ is negative if channel m is open (so $e^{ik_m r}$ behaves as an ingoing wave). Thus, except in the neighborhood of F_{thr}, the two branches are distinguishable by the arrow of time.

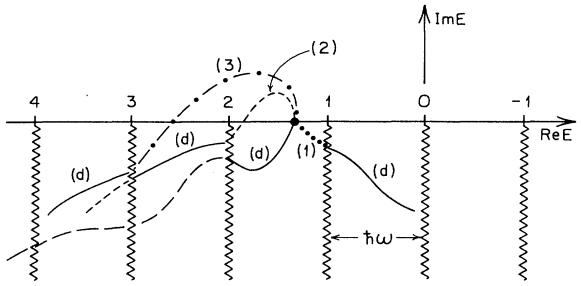

FIG. 1. A schematic diagram, in the E plane, of the trajectories followed by some different branches of the quasienergy as the field strength F is varied along the positive real axis. All of these branches originate from the same bound state at $F = 0$; we have chosen this bound state so that the atom must absorb a minimum of two photons to ionize in a weak field. A zigzag line is a branch cut emanating from the threshold whose channel number is shown at the threshold. The dominant branch (d) is shown as a solid line, and each shadow branch is specified by the label (n), where n is the channel number of the threshold at which the shadow pole changes places with the dominant pole. Note that as F increases the shadow branch (1) crosses the cut emanating from threshold 1, whence it becomes a dominant branch, so that there are simultaneously two dominant branches.

III. THE SCATTERING OPERATOR; ANALYTIC STRUCTURE IN THE E PLANE

It is useful to introduce the operator $S(E,F)$, whose matrix elements form the scattering matrix for an electron to scatter with an incident drift energy E from the atomic core in the presence of a monochromatic radiation field of strength F. In a physical scattering process, which pertains to real values of E and F, the ingoing electron is represented by a superposition of ingoing spherical waves, and $S(E,F)$ contains the information about the amplitudes of the outgoing spherical waves that represent the scattered electron. [If $S(E,F) = 1$ there is no scattering.] We can analytically continue $S(E,F)$, from the real energy axis, into the complex E plane. However, $S(E,F)$ has infinitely many branches, and must be defined on a Riemann energy surface with infinitely many sheets. The different branches arise from the ambiguity in the sign of $k_m(E)$ that appears in taking the square root of $(E + m\hbar\omega)$. Thus $k_m(E)$ is a two-valued function of E, with a branch point at $E + m\hbar\omega = 0$. Hence $S(E,F)$ has branch points in the E plane at each of the channel thresholds, that is, at the points $m\hbar\omega$ with m any integer. We draw cuts in the lower half of the E plane, along the lines $\mathrm{Re}(E + m\hbar\omega) = 0$, as indicated in Fig. 1. We define the "physical" sheet of the Riemann energy surface as the one on which $3\pi/4 > \arg(k_m) > -\pi/4$ for all m. Other ("unphysical") sheets are reached by crossing one or more of the cuts. On an unphysical sheet, $3\pi/4 < \arg(k_m) < 7\pi/4$ for at least one value of m.

In the zero-field limit, the scattering operator has simple poles[19] at the discrete eigenvalues of the atomic Hamiltonian H_a. As F is changed continuously from zero, these poles move around in the complex E plane, following the trajectories of the discrete branches of $E_{ac}(F)$. Thus $S(E,F)$ has simple poles at (generally complex) values E that coincide with the discrete branches of $E_{ac}(F)$. At these poles the analytically continued wave function satisfies asymptotic boundary conditions similar to Eq. (3), for which there are either no ingoing or no outgoing waves. (Both ingoing and outgoing waves can only be present in channels that are radiatively decoupled from all other channels.) In the zero-field limit all channels are uncoupled, and therefore $S(E,0)$ has only a single branch point, at $E = 0$. Suppose that E_0 is a bound-state eigenvalue of H_a, and let N_0 denote the value of N_{min} in the weak-field limit (so the atom must absorb at least N_0 photons in order to ionize from this bound state in a weak field). Thus $S(E,0)$ has a pole at E_0 on the negative real axis of the *physical* sheet, between the thresholds at $-N_0\hbar\omega$ and $-(N_0-1)\hbar\omega$. Suppose that we analytically continue $S(E,0)$ along a path which starts at E_0 on the physical sheet and terminates at E_0 on another sheet. Provided that this path does not encircle the branch point at $E = 0$, $S(E,0)$ is single valued, and hence $S(E,0)$ has a pole at $E = E_0$ on this other sheet. In other words, if $S(E,0)$ has a bound-state pole on the physical sheet, it also has poles at the shadows cast by the first pole on all sheets that can be reached from the physical sheet by crossing cuts emanating from any threshold, but avoiding the cut emanating from threshold with channel number 0. If, now, F is increased from zero, the bound-state pole on the physical sheet moves downward into the lower half of the energy plane, and its associated poles also move. However, in general each member of this family of poles moves in a

different way.[20,21] We refer to the pole which is on the physical sheet, at any given value of F, as the dominant pole, and, following Eden and Taylor,[21] we refer to the associated poles, on unphysical sheets, as shadow poles. If F is sufficiently small, the dominant pole is the one that was originally at E_0 on the physical sheet, but if F is increased further this pole may cross a cut, and thereby move onto another (unphysical) sheet. (Henceforth, we continue to use the term "crossing" a cut to mean that the cut is traversed so as to pass from one sheet of the Riemann surface onto another.) When this happens, a shadow pole normally crosses the cut also, at about the same value F, provided the cut emanates from the negative-energy axis; see Fig. 1. This shadow pole is the one that was originally on the sheet reached from the physical sheet by following a path that crosses the cut, and therefore when this pole crosses the same cut it moves onto the physical sheet. Hence the pole that we called dominant just before it crossed the cut becomes a shadow pole, and one of the poles that we called shadow becomes dominant. In general, each time the dominant pole is about to cross a cut emanating from a threshold on the negative-energy axis, a shadow pole moves onto the physical sheet to take its place. In Fig. 2 we show typical mth channel wave numbers k_m for the dominant and shadow poles.

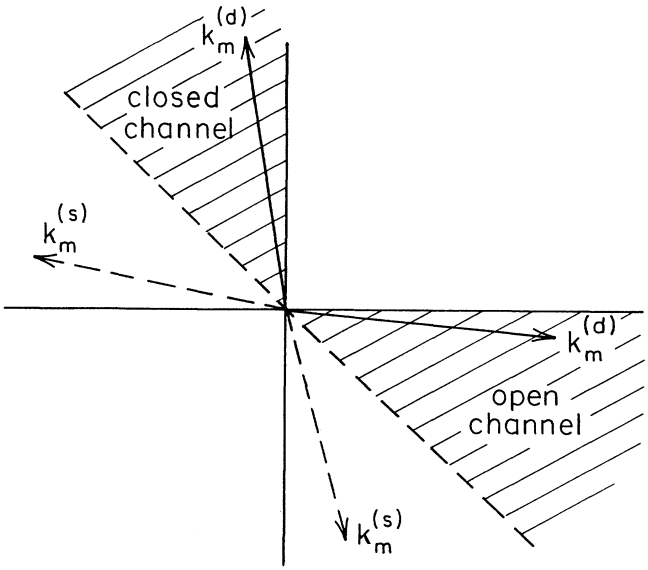

FIG. 2. A schematic diagram of the mth channel wave numbers for the dominant (d) and shadow (s) branches. The wave number $k_m^{(d)}$ for the dominant branch lies in the hatched region, in the lower right quadrant when the mth channel is open, and in the upper left quadrant when the mth channel is closed. The wave number $k_m^{(s)}$ for the shadow branch lies in the open region, and in general is not exactly equal and opposite to $k_m^{(d)}$.

Sometimes a shadow pole may stray far from the path followed by the dominant pole as F increases along the positive real axis, and eventually it may also become a dominant pole. For example, it is often observed that if $N_0 \geq 2$ a shadow pole moves in the direction of the positive energy axis as F increases along the positive real axis, and it crosses all the appropriate cuts so as to move onto the physical sheet.[12(b),22] In this circumstance, there are simultaneously two dominant poles at large values of F (stemming from the same bound state—see Fig. 1) and each of these dominant poles may come into resonance with another dominant pole, or even with each other. We conclude this section by noting that for any dominant or shadow pole of $S(E,F)$ there is a sequence of (dominant or shadow) poles, each pole separated from the next by $\hbar\omega$, since to any branch of $E_{\rm ac}(F)$ are associated physically indistinguishable branches that differ by an integral multiple of $\hbar\omega$. We call these additional poles "spontaneous" poles, since they appear only when the field is turned on; in other words, the residues of the spontaneous poles vanish as F does. When the field is turned on, channel 0 becomes radiatively coupled to channels $n \neq 0$, and a bound- (or quasi-bound-) state pole in channel 0 appears in all other channels. [However, the scattering operator is *not* periodic in the energy variable: $S(E+n\hbar\omega,F) \neq S(E,F)$.]

IV. THE QUASIENERGY; ANALYTIC STRUCTURE IN THE F PLANE

A. Symmetry

A shift in the origin of time by one half-cycle π/ω is equivalent to a change in the sign of F. However, the quasienergy is a cycle-average energy, and is therefore independent of the origin of time. It follows that for any given branch we have $E_{\rm ac}(F) = E_{\rm ac}(-F)$, and therefore the perturbation expansion of $E_{\rm ac}(F)$ is a series in *even* powers of F. Furthermore, if there is a branch point at $F_{\rm br}$, there is another one at $-F_{\rm br}$ on the same sheet of the Riemann F surface.

The Hamiltonian of the full system consisting of the atom plus the radiation field is time-reversal invariant. However, $H(t) \equiv H_a + V(t)$ includes only the *coupling* of the atom to the field; we treat the radiation field as external. Under the operation of time reversal the rotational sense of the polarization and the direction of propagation of the field are reversed. Therefore, in order to consider the effect of time reversal we must introduce the coupling of the atom to the time-reversed field; this interaction is $V_+^{\mathcal{T}} e^{-i\omega t} + V_-^{\mathcal{T}} e^{i\omega t}$, where $V_+^{\mathcal{T}} = i(e/2\mu\omega)F\hat{\epsilon}^* \cdot \mathbf{p}$ and $V_-^{\mathcal{T}} = -i(e/2\mu\omega)F\hat{\epsilon} \cdot \mathbf{p}$. Let $\mathcal{T}$ be the operator that effects time reversal on only the space of atomic coordinates. Since $\mathcal{T}$ complex conjugates c numbers and anticommutes with $\mathbf{p}$, it follows that if the harmonic components $|\mathcal{F}_n\rangle$ form a solution of Eq. (2), with eigenvalue

$E_{ac}(F)$ at field strength F, the harmonic components $\mathcal{T}|\mathcal{F}_n\rangle$ form a solution of an equation that differs from Eq. (2) through the replacement of $V_\pm$ by $V_\pm^{\mathcal{T}}$, and the eigenvalue is $E_{ac}^{\mathcal{T}}(F) = E_{ac}^*(F^*)$ (where the asterisk $*$ denotes "complex conjugate"). Consequently, if any two branches of $E_{ac}(F)$ have the same value at $F = F_{br}$, the corresponding two (time-reversed) branches of $E_{ac}^{\mathcal{T}}(F)$ must have the same value at $F = F_{br}^*$. Thus, if $E_{ac}(F)$ has a branch point at $F = F_{br}$, the quasienergy $E_{ac}^{\mathcal{T}}(F)$ has a branch point at $F = F_{br}^*$. If the polarization of the light is real, we have $\hat{\epsilon} = \hat{\epsilon}^*$, and therefore $V_\pm^{\mathcal{T}} = V_\pm$, so that the functions $E_{ac}(F)$ and $E_{ac}^{\mathcal{T}}(F)$ are one and the same. If the polarization is elliptic, $E_{ac}(F)$ and $E_{ac}^{\mathcal{T}}(F)$ are still one and the same since the Hamiltonian H_a of the atom is rotationally invariant and reflection invariant. (Reflections reverse the rotational sense of the polarization.) However, no matter what the polarization is, the time-reversed branch is physically distinguishable from the original branch since $\mathcal{T}$ changes the asymptotic boundary conditions—it changes outgoing waves into ingoing waves, and vice versa. Therefore, although $E_{ac}(F)$ has branch points at both $F = F_{br}$ and $F = F_{br}^*$, these two branch points are on different sheets of the Riemann F surface.

A conjugate pair of branches cannot be joined at any point in the F plane (except possibly at $F \sim \infty$). For a fixed value of F, we can join a conjugate pair of branches by following a path in the E plane, which encircles all open-channel thresholds. To see this, let us for simplicity take F to be real. We start our path in the E plane at $E_{ac}(F)$, we move parallel to the real negative-energy axis, crossing this axis at $-\infty$, and we move back along the other side of the real axis, terminating at the point $E_{ac}^*(F)$, opposite to $E_{ac}(F)$. This path crosses the cuts emanating from the open channels, but not the cuts emanating from the closed channels, and the wave number k_m of each (closed or open) channel changes to $-k_m^*$; hence the state vector transforms into the conjugate state vector. Now, since we have fixed F to be real, $E_{ac}^*(F)$ is an eigenvalue, corresponding to the conjugate eigenvector. Therefore, when we analytically continue $S(E,F)$ along our path in the E plane, we find that it has poles at both $E_{ac}(F)$ and $E_{ac}^*(F)$; but the pole at $E_{ac}^*(F)$ lies on a sheet different from the one on which the pole at $E_{ac}(F)$ lies, since our path crosses (infinitely) many cuts. Therefore, in order to transform a wave function into its conjugate, we have to follow a contour in the complex F plane which wraps around the appropriate type-T branch points corresponding to open channels.

We can analytically continue the shift and width from the real F axis into the the complex F plane, thereby defining analytic (multivalued) functions $\Delta_{ac}(F)$ and $\Gamma_{ac}(F)$. For any F we have $E_{ac}(F) = E_0 + \Delta_{ac}(F) + i\Gamma_{ac}(F)/2$, but in general $\Delta_{ac}(F)$ and $\Gamma_{ac}(F)$ are not real if F is not real (an exception occurs at a type-T branch point). It follows from inversion symmetry that any given branches of $\Delta(F)$ and $\Gamma(F)$ are even in F. Since the functions $\Delta_{ac}^*(F^*)$ and $\Gamma_{ac}^*(F^*)$ are also analytic on a domain including the real F axis, and since $\Delta_{ac}(F)$ and $\Gamma_{ac}(F)$ are real if F is real, we have (the Schwarz reflection principle)

$$\Delta_{ac}(F) = \Delta_{ac}^*(F^*) , \tag{7a}$$

$$\Gamma_{ac}(F) = \Gamma_{ac}^*(F^*) . \tag{7b}$$

If $E_{ac}(F)$ has a branch point at $F = F_{br}$, *both* $\Delta_{ac}(F)$ and $\Gamma_{ac}(F)$ must necessarily have branch points at F_{br}. [It follows that the perturbation expansions of $\Delta_{ac}(F)$ and $\Gamma_{ac}(F)$ have the same radii of convergence.] From Eqs. (7), it follows that $\Delta_{ac}(F)$ and $\Gamma_{ac}(F)$ have branch points at $F = F_{br}^*$, on the same sheet as that on which the branch point at F_{br} lies, but this is merely a consequence of the mathematical construction of the shift and width, and there is no reason to suppose that E_{ac} has a branch point at F_{br}^* on the same sheet as F_{br}.[23] Nevertheless, as we see in a moment, if F_{br} is a type-R branch point there is another branch point, on the same sheet, which is sometimes very close to F_{br}^*.

B. Type-R (resonance) branch points

At a type-R branch point in the F plane, two branches of the quasienergy that stem from different bound-state levels of the unperturbed atom intersect; they intersect at $F = F_{res}$, where they have a common value E_{res}, which is generally complex. For $F \approx F_{res}$, the scattering operator $S(E,F)$ has two simple poles on the physical sheet of the E plane in the neighborhood of E_{res}; these two poles correspond to the two different atomic levels, and when $F = F_{res}$ the poles coalesce at E_{res}. The singularity of $S(E,F_{res})$ at $E = E_{res}$ is also a *simple* pole, since there is only a single state when $F = F_{res}$; recall that at a branch point of $E_{ac}(F)$ the asymptotic boundary conditions for the two branches are identical.

Intermediate resonances sometimes occur between levels that have widths that are very small compared to the shifts—this may be the case, for example, when the levels a and b are connected by a one-photon transition while ionization from these levels occurs only through the absorption of many photons. In such a case, it is reasonable to ignore the widths of the a and b branches, and approximate these branches of $E_{ac}(F)$ by $E_0 + \Delta_{ac}(F)$; but the branch points of $\Delta_{ac}(F)$ occur in conjugate pairs, and therefore, within this approximation, if the function $E_{ac}(F)$ has a branch point at F_{res} it has another one on the same sheet at F_{res}^*. We can make the argument more quantitative as follows: Suppose that, at a given intensity, the atom must absorb at least $N_{min}^{(a)}$ or $N_{min}^{(b)}$ photons to ionize from the levels a or b, respectively. We can determine $E_{ac}(F)$ approximately by solving the truncated set of equations that results from excluding, in Eq. (2),

harmonic components $|\mathcal{F}_n\rangle$ with photon index n greater than or equal to the smaller of $N_{\min}^{(a)}$ and $N_{\min}^{(b)}$. Within this approximation, the widths of the a and b branches are identically zero, since we have neglected the coupling to the continuum, and therefore if these branches meet at $F_{\rm res}$ they also meet at $F'_{\rm res} = F^*_{\rm res}$. We now gradually increase the coupling to the continuum by including more and more harmonic components $|\mathcal{F}_n\rangle$. Once the photon index $n \geq N_{\min}^{(a)}, N_{\min}^{(b)}$, we still expect to find two branch points, at $F_{\rm res}$ and $F'_{\rm res}$, but there is no longer any reason for $F'_{\rm res}$ to lie exactly at $F^*_{\rm res}$. This suggests that, in general, for any type-R branch point at $F_{\rm res}$ there is a nearby "conjugate" one on the same sheet at a point $F'_{\rm res}$, where $F'_{\rm res}$ is close to $F^*_{\rm res}$ when the widths of the two branches are small compared to the shifts. Actually, we can change this latter condition to the requirement that the difference of the widths of the two branches be small compared to the difference of the shifts, since if this new condition is fulfilled the branch points occur where the shifts of the two branches are (very nearly) equal, and those points occur in conjugate pairs. Furthermore, due to inversion symmetry, there must be branch points at $-F_{\rm res}$ and $-F'_{\rm res}$, on the same sheet, so that type-R branch points group into quadruplets.

In the vicinity of the branch points at $F_{\rm res}$ and $F'_{\rm res}$ the quasienergy $E_{\rm ac}(F)$ behaves like a product, rather than a sum, of $\sqrt{F - F_{\rm res}}$ and $\sqrt{F - F'_{\rm res}}$; if $E_{\rm ac}(F)$ were to behave like a sum it would have four, rather than two, branches associated with a single resonance. Therefore we must have

$$E_{\rm ac}(F) \approx E_{\rm res}(F) \pm C_{\rm res}(F)\sqrt{(F - F_{\rm res})(F - F'_{\rm res})}\ , \tag{8}$$

where $E_{\rm res}(F)$ and $C_{\rm res}(F)$ are smoothly varying, nonsingular, functions of F in the neighborhood of $F_{\rm res}$ and $F'_{\rm res}$, and where the choice of sign preceding $C_{\rm res}(F)$ fixes the branch. If we follow a closed circuit in the F plane around just one of the branch points, $F'_{\rm res}$ say, the corresponding trajectories of each of the two branches of $E_{\rm ac}(F)$ wrap around $E'_{\rm res} \equiv E_{\rm res}(F'_{\rm res})$, but not completely[24]—the trajectories in the E plane are not closed, but rather one pole trajectory terminates where the other pole trajectory started, and vice versa. However, if we follow a closed circuit in the F plane around *both* branch points, both branches of $E_{\rm ac}(F)$ form closed circuits around $E_{\rm res}$ and $E'_{\rm res}$ in the E plane. It is reasonable that there are closed circuits in the F plane that yield closed circuits for $E_{\rm ac}$ around $E_{\rm res}$ and $E'_{\rm res}$ since $S(E,F)$ is single valued along a closed circuit[25] in the E plane about $E_{\rm res}$ or/and $E'_{\rm res}$. In fact, if F is in the neighborhood of $F_{\rm res}$ and $F'_{\rm res}$, and if E is in the neighborhood of $E_{\rm res}$ and $E'_{\rm res}$, we may approximate $S(E,F)$ by

$$\begin{aligned} 1/S(E,F) \\ \approx S_{\rm res}[\ & E - E_{\rm res}(F) \\ & + C_{\rm res}(F)\sqrt{(F - F_{\rm res})(F - F'_{\rm res})}]\ . \end{aligned} \tag{9}$$

We see that when F is different from $F_{\rm res}$ and $F'_{\rm res}$, there are, from the two-valued square root in Eq. (9), two isolated poles, but when F is equal to either $F_{\rm res}$ or $F'_{\rm res}$, Eq. (9) yields only a single simple pole.

C. Type-T (threshold) branch points

At a type-T branch point, at $F_{\rm thr}$ in the F plane, two branches of $E_{\rm ac}(F)$ intersect at a threshold in the E plane, where they have a common *real* value $E_{\rm thr} = -m\hbar\omega$, with m the channel number of the threshold. We can label these branch points by the channel number. As we explain shortly, physically significant type-T branch points occur only with positive channel number. In the neighborhood of a type-T branch point, the two branches have values of k_m that are almost equal but opposite; at $F = F_{\rm thr}$ the two branches coalesce and k_m vanishes. The physically accessible branch, which we call the dominant branch, corresponds to a dominant pole of $S(E,F)$ in the E plane. The physically inaccessible branch, which we call the shadow branch, corresponds to a shadow pole of $S(E,F)$. The dominant and shadow poles are simple, and when $F = F_{\rm thr}$ these poles presumably coalesce into one simple pole at $E = E_{\rm thr}$. [We note that $S(E,F)$ also has a branch point at $E = E_{\rm thr}$.] Two branches which intersect at a type-R branch point are both dominant, since a type-R branch point arises from the confluence of two poles that *both* lie on the physical sheet.

We can, of course, solve Eq. (2) to find a value of F for which $k_m = 0$ for *any* integer m; but it does not follow that $E_{\rm ac}(F)$ has a branch point associated with *every* threshold. We saw, in Sec. III, that if, when F is infinitesimally small, the scattering operator $S(E,F)$ has a bound-state pole, at some point on the physical sheet of the Riemann E surface, there are also shadow poles, but they lie only on those sheets that can be reached from the physical sheet without crossing the cut emanating from the threshold at channel number 0. Therefore there is no shadow branch associated with the zeroth threshold. Hence the point $F_{\rm thr,0}$ in the F plane at which $k_0 = 0$ is not a branch point. It must, in fact, be a saddle point. This follows because if $F_{\rm thr,0}$ is not a branch point, a single circuit in the F plane around $F_{\rm thr,0}$ must produce a double circuit in the E plane around $E = 0$ since $S(E,F)$ has a square-root branch point at the threshold at $E = 0$ (and we must return to the same point on the same sheet of the E plane when we follow a complete circuit in the F plane around a nonsingular point); hence, for $F \approx F_{\rm thr,0}$ we have $E_{\rm ac}(F) \approx C_{\rm thr,0}(F - F_{\rm thr,0})^2$, where $C_{\rm thr,0}$ is a constant. There might be branch points at those values of F for which $k_m = 0$ when $m < 0$, but such values of F must lie far from the real F axis (otherwise there would be real values of F for which an atom could ionize by absorbing a negative number of photons); hence physically significant type-T branch points occur only with positive

channel number.

We can easily show that $E_{ac}(F)$ cannot have type-T branch points at both F_{thr} and F^*_{thr} on the same sheet of the F plane: If this were the case, $E_{ac}(F)$ would have the same value E_{thr} at F_{thr} and F^*_{thr} since E_{thr} is real. Furthermore, we know that if $E_{ac}(F)$ has a branch point at F_{thr}, the conjugate branch has a branch point at F^*_{thr}, where its value is $E^*_{thr} = E_{thr}$. Therefore the conjugate and original branches would join at F_{thr}, in conflict with the discussion of Sec. IV A above. We can make this argument stronger, and extend it to show that not only is there no second branch point at F^*_{thr}, there is no second one very close to F^*_{thr} (or F_{thr}): For if there were two nearby branch points, at F_{thr} and F'_{thr} say, $E_{ac}(F)$ would behave as the product of $\sqrt{F - F_{thr}}$ and $\sqrt{F - F'_{thr}}$ near these branch points. [$E_{ac}(F)$ would not behave like the sum of the square roots since that would give four branches.] Therefore a closed circuit in the F plane around both F_{thr} and F'_{thr} would give closed trajectories in the E plane around $E_{thr} \equiv E_{ac}(F_{thr}) = E_{ac}(F'_{thr})$; but this is in conflict with the fact that $S(E, F)$ is not single valued along a closed path in the E plane around E_{thr}. Hence we must have

$$E_{ac}(F) \approx E_{thr}(F) \pm C_{thr}(F)\sqrt{F - F_{thr}} , \tag{10}$$

where $E_{thr}(F)$ and $C_{thr}(F)$ are smoothly varying, nonsingular, functions of F in the neighborhood of F_{thr}; we denote $E_{thr}(F_{thr})$ by simply E_{thr}, where $E_{thr} = -m\hbar\omega$, with m the channel number of the relevant threshold. Due to inversion symmetry, there is also a type-T branch point at $-F_{thr}$. Note, however, that in the F^2 plane, rather than the F plane, any branch of E_{ac} that has a type-T branch point also has a branch point at infinity; in contrast, a type-R branch point in the F^2 plane does not have an associated branch point at infinity.

If, to simplify matters, we assume that the square-root term on the right-hand side of Eq. (10) dominates when F is very close to $\mathrm{Re}F_{thr}$, we obtain

$$\Gamma_{ac}(F = \mathrm{Re}F_{thr}) \approx \mp 2(-\mathrm{Im}F_{thr})^{1/2}(\mathrm{Re}e^{-i\pi/4}C_{thr}) , \tag{11a}$$

$$\frac{d\Delta_{ac}}{dF}(F = \mathrm{Re}F_{thr}) \approx \pm\tfrac{1}{2}(-\mathrm{Im}F_{thr})^{-1/2}(\mathrm{Re}e^{-i\pi/4}C_{thr}) . \tag{11b}$$

It follows that

$$\Gamma_{ac}(F = \mathrm{Re}F_{thr}) \approx \beta(\mathrm{Im}F_{thr})[d\Delta_{ac}(F = \mathrm{Re}F_{thr})/dF] , \tag{12}$$

where $\beta = 4$. However, although the square-root term dominates when F is sufficiently close to F_{thr}, it is certainly not always true that this term dominates when F is very close to $\mathrm{Re}F_{thr}$. (At $F = \mathrm{Re}F_{thr}$ the shadow pole and the dominant pole are about to interchange their roles. In order that this interchange evolves smoothly, their imaginary parts should not be too different. However, if the square-root behavior were to dominate, their imaginary parts would be the same, but of opposite sign.) In fact, the linear behavior of $E_{thr}(F)$—viz., $E_{thr}(F) \approx E_{thr} + [dE_{thr}(F = F_{thr})/dF](F - F_{thr})$—may be more significant when F is very close to $\mathrm{Re}F_{thr}$. We expect the linear behavior to dominate when, for example, the light is circularly polarized, for then the threshold (if its channel number is larger than 1) has only a barely noticeable effect on the ionization rate, as we illustrate in Sec. VI B. Even if the linear behavior does dominate, Eq. (12) remains valid in form, provided that F_{thr} is not far from the real axis; the only difference is that β has the value 2 rather than 4. [However, if both the linear and square-root terms are important, Eq. (12) does not hold.] Now, since $\Gamma_{ac}(\mathrm{Re}F_{thr})$ is positive on a dominant (bound-state) branch, so is the right-hand side of Eq. (12). Therefore, if Eq. (12) is valid, and if, on a dominant branch, the shift decreases (increases) as F increases along the positive real axis towards $\mathrm{Re}F_{thr}$, the branch point at F_{thr} lies in the lower (upper) right quadrant of the complex F plane, provided that F_{thr} is not far from the real axis.

We can prove this last point, without resorting to the approximation of Eq. (12), by appealing to the Cauchy-Riemann relations:

$$\frac{\partial \mathrm{Im}E_{ac}(F)}{\partial \mathrm{Im}F} = \frac{\partial \mathrm{Re}E_{ac}(F)}{\partial \mathrm{Re}F}, \tag{13a}$$

$$\frac{\partial \mathrm{Im}E_{ac}(F)}{\partial \mathrm{Re}F} = -\frac{\partial \mathrm{Re}E_{ac}(F)}{\partial \mathrm{Im}F}. \tag{13b}$$

However, before doing this we briefly discuss the behavior of the lines along which $\mathrm{Re}E_{ac}(F)$ and $\mathrm{Im}E_{ac}(F)$ are constant, for a *dominant* bound-state branch of $E_{ac}(F)$. We show a fairly typical map of these lines Fig. 3. We have assumed that N_0, the minimum number of photons the atom must absorb to ionize in a weak field, is large compared to unity. We may also asssume that for F real the shift is negative, and monotonically decreases as F increases; therefore, as F increases along the positive real axis, $E_{ac}(F)$ moves past the threshold of the $(N_0 + 1)$th channel, and the minimum number of photons the atom must absorb to ionize increases to $N_0 + 1, \ldots$ and so on. Except at the origin ($F = 0$) the lines $\mathrm{Re}E_{ac}(F) = \text{const}$ are perpendicular to the lines $\mathrm{Im}E_{ac}(F) = \text{const}$ at their points of intersection. In the vicinity of the origin we have, according to perturbation theory, $E_{ac}(F) \approx E_0 + E_1F^2$ (unless a branch point sits at the origin) and therefore the origin is a saddle point of $E_{ac}(F)$, through which the lines $|E_0| + \mathrm{Re}E_{ac}(F) = 0$ and $\mathrm{Im}E_{ac}(F) = 0$ pass. Since $N_0 > 1$, E_1 is real (and negative). Therefore, at $F = 0$, the lines $\mathrm{Im}E_{ac}(F) = 0$

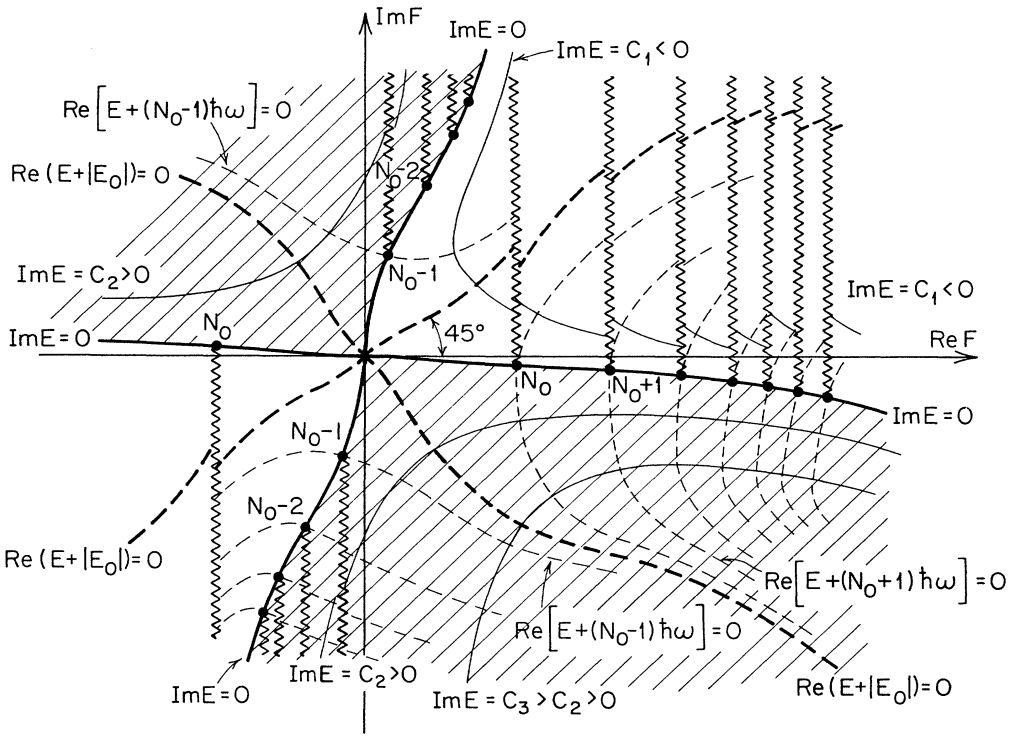

FIG. 3. A schematic diagram showing the distribution, in the F plane, of T-type branch points (the solid circles) for the dominant branch. The lines ReE=const and ImE=const are the lines along which the real and imaginary parts of the dominant branch of $E_{ac}(F)$ are constant. The dominant branch originates from a bound state in which the atom must absorb a minimum of N_0 photons to ionize in a weak field. We have assumed that $N_0 \gg 1$, and that as F increases along the positive real axis the shift decreases monotonically. The unhatched regions are those in which Im$E < 0$.

are tangential to the real and imaginary F axes, and the lines $|E_0| + \mathrm{Re}E_{ac}(F) = 0$ make angles of 45° with these axes. Along the real F axis, $\mathrm{Im}E_{ac}(F)$ is equal to $-\Gamma_{ac}(F)/2$, and this is negative if $F \neq 0$. Hence the lines $\mathrm{Im}E_{ac}(F) = 0$ cannot intersect the real F axis, except at $F = 0$. Now, the T-type branch points lie along the lines $\mathrm{Im}E_{ac}(F) = 0$, at the intersections with the lines $n\hbar\omega + \mathrm{Re}E_{ac}(F) = 0$, with n an integer. The physically significant T-type branch points lie along the line $\mathrm{Im}E_{ac}(F) = 0$ that is closest to the real F axis (the line that is tangential to the real axis at $F = 0$). In the right half of the F plane, this line is in the lower (upper) right quadrant if $\partial\mathrm{Im}E_{ac}(F)/\partial\mathrm{Im}F$ is positive (negative) when evaluated on the positive real F axis, since $\mathrm{Im}E_{ac}(F)$ becomes negative as the real F axis is approached. From Eq. (13a) we see that $\partial\mathrm{Im}E_{ac}(F)/\partial\mathrm{Im}F$ is equal to $d\Delta_{ac}(F)/dF$ on the real F axis, and therefore the physically significant type-T branch points lie in the lower (upper) right quadrant if $\Delta_{ac}(F)$ decreases (increases) as F increases along the positive real F axis. If the Nth type-T branch point (associated with the Nth threshold) lies close to the origin, we may estimate its position $F_{\mathrm{thr,N}}$ by putting $E_{ac}(F) = -N\hbar\omega \approx E_0 + E_1F^2$, to give $F_{\mathrm{thr},N} \approx \sqrt{-(E_0 + N\hbar\omega)/E_1}$, so (with E_1 negative) this branch point lies (almost) on the real axis if $N \geq N_0$ or (almost) on the imaginary axis if $N < N_0$. Since the N_0th and $(N_0 - 1)$th branch points lie closest to the origin, it is always one of these two branch points which determines the radius of convergence of the perturbation expansion of $E_{ac}(F)$, provided there is no type-R branch point which lies still closer to the origin.

We draw the type-T branch cuts in the F plane as vertical lines. As we let F vary along these branch cuts we do not reproduce the vertical branch cuts in the E plane shown in Fig. 1. Rather, as we let F vary along the cut starting from the Nth branch point, we produce a *curved* line in the lower half of the E plane starting from $N\hbar\omega$. Actually, this curved line in the E plane is not unique since the dominant and shadow branches follow different trajectories in the E plane as F varies.

Let us now turn our attention to the movement of the type-T branch points when varying the frequency. If we vary ω, the branch points and the lines $\mathrm{Re}E_{ac}(F) =$ const and $\mathrm{Im}E_{ac}(F) =$ const move. If ω is very small

(that is, $N_0 \gg 1$) we have $\Delta_{ac}(F) \approx -P$, where $P \equiv (e^2F^2/4\mu\omega^2)$ is the ponderomotive shift. In this case, we have $E_1 \approx -(e^2/4\mu\omega^2)$, and hence $F_{thr,N} \approx (2\omega/|e|)\sqrt{\mu(E_0 + N\hbar\omega)}$, so that as ω vanishes infinitely many type-T branch points accumulate at the origin, as pointed out earlier by Manakov and Fainshtein.[11] Suppose that we increase ω; those branch points that lie on the section of the line $\mathrm{Im}E_{ac}(F) = 0$ that is almost parallel to the imaginary F axis move towards the origin, and those that lie on the section of the line that is almost parallel to the real F axis move away from the origin. If we choose ω such that $N\hbar\omega + E_0 = 0$, the Nth branch point sits at the origin $F = 0$; if we increase ω slightly, the Nth branch point moves from the origin along the positive F axis, and accordingly N_0 decreases to N (fewer photons are required to ionize the atom). As we continue to increase ω, the Nth branch point cannot move back across the origin since the equation $N\hbar\omega + E_0 = 0$ has only one solution for fixed N. Furthermore, this equation cannot be satisfied for $N \leq 0$, so that branch points associated with thresholds that have channel numbers $N \leq 0$ cannot pass through the origin. This, of course, is consistent with the fact that an atom cannot ionize by absorbing a negative number of photons. Moreover, there is no singularity associated with the zeroth threshold—there is a saddle point rather than a branch point—and if this saddle point were to pass through the origin, one of the lines $\mathrm{Im}E_{ac}(F) = 0$ emanating from it would cross the positive real F axis, contradicting the fact that $\mathrm{Im}E_{ac}(F) < 0$ on the real F axis. The $N = 1$ branch point does, of course, reach the origin, and presumably passes through it as ω increases beyond $|E_0|/\hbar$. Restricting the remainder of this paragraph to $\omega > |E_0|/\hbar$, we have, at least in the weak-field limit, $N_0 = 1$, and the coefficient E_1, in the expansion $E_{ac}(F) \approx E_0 + E_1F^2$, becomes complex. Therefore the lines $\mathrm{Im}E_{ac}(F) = 0$ are no longer tangential to the real and imaginary F axes at $F = 0$, and the lines $|E_0| + \mathrm{Re}E_{ac}(F) = 0$ no longer make angles of 45° with these axes. When $\omega \gg |E_0|/\hbar$, the shift is normally positive (if F is real) and it normally (the model atom discussed in Sec. VI B is abnormal) decreases monotonically as F increases. [Thus the line $|E_0| + \mathrm{Re}E_{ac}(F) = 0$, which lies in the upper right quadrant, crosses the real axis and turns into the lower half plane as ω increases beyond $|E_0|/\hbar$.] Assuming this is the case, the minimum number of photons that the atom must absorb to ionize cannot increase as F increases along the positive real axis. Hence the lines $\mathrm{Im}E_{ac}(F) = 0$ along which the branch points specified by positive channel numbers lie, must sharply diverge from the real F axis as they move away from the origin [for otherwise the (N_0+1)th branch point would be physically significant, and the minimum number of photons required for ionization would increase to $N_0 + 1$ or more as F increases]. Presumably the lines $\mathrm{Im}E_{ac}(F) = 0$, along which the branch points specified by negative channel numbers lie, must eventually converge towards the real F axis as they move away from the origin since the ac width vanishes[26] as F increases along the positive real axis.

V. ADIABATIC VERSUS DIABATIC TRANSITIONS

A. Resonances

At any point in the neighborhood of two type-R branch points, at F_{res} and F'_{res}, the two branches are separated by the distance $|\delta E_{ac}(F)|$, where

$$\delta E_{ac}(F) = 2C_{res}\sqrt{(F - F_{res})(F - F'_{res})} \tag{14a}$$

$$= 2C_{res}\{[F - \tfrac{1}{2}(F_{res} + F'_{res})]^2 - \tfrac{1}{4}(F_{res} - F'_{res})^2\}^{1/2} , \tag{14b}$$

where here C_{res} is the (approximately) constant value of $C_{res}(F)$ in the vicinity of the branch points. There are two extreme cases of interest, which we label as (a) and (b), and which we now discuss.

In case (a) we have $F'_{res} \approx F^*_{res}$, which applies when the difference of the widths of the two branches is small compared to the difference of the shifts. In this case we have

$$\delta E_{ac}(F) \approx 2C_{res}\sqrt{(F - \mathrm{Re}F_{res})^2 + (\mathrm{Im}F_{res})^2} . \tag{15}$$

If F is confined to the real axis, the separation of the real parts of the two branches, and simultaneously the separation of the imaginary parts (the widths), are at a minimum at $F = \mathrm{Re}F_{res}$. Thus the real parts undergo an *avoided* crossing on the real axis at $\mathrm{Re}F_{res}$, as do the imaginary parts. The gap between the real parts at the avoided crossing is $2|(\mathrm{Re}C_{res})(\mathrm{Im}F_{res})|$, and this is the Rabi coupling energy $\hbar\Omega_0$ between the levels a and b (where Ω_0 is the Rabi flopping frequency).[24] The gap between the imaginary parts is $2|(\mathrm{Im}C_{res})(\mathrm{Im}F_{res})|$. Actually, C_{res} is approximately real since, in the present case, $\delta E_{ac}(F)$ is approximately the difference of the shifts, and they are real if F is real. Therefore the imaginary parts of the two branches very nearly exhibit a true, rather than an avoided, crossing at $F = \mathrm{Re}F_{res}$.[27]

In case (b) we have $F'_{res} \approx F_{res}$. Numerical studies indicate that this case may apply when the difference of the widths of the two branches is large compared to the difference of the shifts. In case (b) we have

$$\delta E_{ac}(F) \approx 2C_{res}(F - F_{res}) . \tag{16}$$

Provided that C_{res} is still approximately real, we see that now the real parts of the two branches exhibit very nearly a true crossing at F_{res}, while the imaginary parts exhibit an avoided crossing.[27]

We can join the two branch points at F_{res} and F'_{res} by a branch cut. If we follow a closed path in the F plane that encircles both branch points, without crossing the cut, any given branch returns to its original value, at the point where we started. However, suppose that we follow a continuous path along the real F axis. In case (a), where F_{res} and F'_{res} lie on opposite sides of the real axis, we must cross the cut, and this results in a switchover between branches. (Note that the discontinuity in the shift of a given branch, on each side of the cut

at $\mathrm{Re}F_{\mathrm{res}}$, is, up to a sign, $\hbar\Omega_0$.) On the other hand, in case (b), where F_{res} and F'_{res} lie on the same side of the real axis, we avoid the cut, and there is no switchover between branches. Thus, if the atom starts out in atomic level a, and if F is varied slowly along the real F axis, past a pair of type-R branch points, the atom evolves *adiabatically* into atomic level b in case (a), but remains in atomic level a in case (b). Of course, the atom can adjust to a change of character in its state, in case (a), only if the cut is crossed sufficiently slowly. If the cut is crossed rapidly, the atom evolves *diabatically*, and the character of its state is preserved. In a diabatic passage through the cut, the atom follows the *same* branch of the ac quasienergy. Mathematically, this amounts to distorting the path along the real F axis into the contour C that passes around one of the branch points, thereby avoiding the branch cut; see Fig. 4(a). Physically, this amounts to the atom undergoing a transition across the energy gap of the avoided crossing of the real parts of the two branches of the quasienergy, at $F_{\mathrm{cr}} \equiv \mathrm{Re}F_{\mathrm{res}}$, and the probability for this to occur is significant only if the avoided crossing is passed in a time δt such that $\hbar/\delta t$ is comparable to or larger than the energy gap.

In the absence of a resonance, the probability for the atom to remain in a particular state that develops *adiabatically* (continuously) as the (real) field strength varies with time is

$$P_a = |\exp\left(-\frac{i}{\hbar}\int_{-\infty}^{t} dt'\, E_{\mathrm{ac}}(F)\right)|^2 = \exp\left(\frac{2}{\hbar}\mathrm{Im}\int_{-\infty}^{t} dt'\, E_{\mathrm{ac}}(F)\right) ,$$

where $E_{\mathrm{ac}}(F)$ is the appropriate branch of the quasienergy, and where F is a function $F(t)$ of time t,

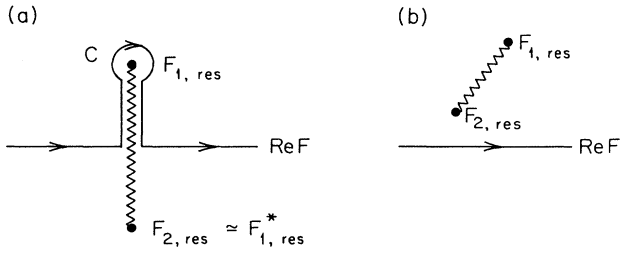

FIG. 4. A schematic diagram showing the locations, in the F plane, of two type-R branch points associated with a single resonance. In (a) the difference of the widths of the two resonant branches is negligible compared to the difference of the shifts, and the two branch points are conjugates of each other. The contour C wraps around the section of the cut in either the upper half (as we have shown) or the lower half (see text) of the F plane. In (b) the difference of the widths is large compared to the difference of the shifts, and the two branch points lie on the same side of the real F axis.

with $F(t)$ real when t is real. [We assume that $F(t)$ varies slowly on the time scale of a cycle, so that both the Hamiltonian and the state vector depend parametrically on F.] This probability decays exponentially as t increases [the imaginary part of $E_{\mathrm{ac}}(F)$ is negative if F is real] since the population in the adiabatic state is depleted by ionization according to the well-known decay law $P_a = \exp\{-(1/\hbar)\int_{-\infty}^{t} dt'\, \Gamma_{\mathrm{ac}}[F(t')]\}$. Suppose that at some value of F there is a resonance, described by case (a), with the two type-R branch points on opposite sides and equidistant from the real F axis. Let us calculate the probability for a diabatic transition at this resonance. As $F(t)$ sweeps along the real axis, the cut joining the two branch points is crossed, and the atomic character of the adiabatic state changes, from a to b say. There is another adiabatic state whose character changes from b to a as the cut is crossed. The probability for the atom to undergo a diabatic transition to this other adiabatic state, so that the atom remains on the same a branch of the quasienergy, is[28]

$$P_a = \exp\left(\frac{2}{\hbar}\mathrm{Im}\int_{C'} dt\, E_{\mathrm{ac}}(F)\right) , \tag{17}$$

where C' is the contour in the t plane that corresponds to the contour C in the F plane, which circumvents the branch point F_{res}, as shown in Fig. 4(a). We are assuming that the value of P_a obtained by integrating along C, around F_{res}, is smaller than unity (for otherwise we cannot interpret P_a as a probability); if this value is not less than unity, we must integrate around F^*_{res}, along the contour that is the image of C—this simply amounts to reversing the sign of $\mathrm{Im}F_{\mathrm{res}}$ in Eq. (19a) below, and does not alter the final result of Eq. (19b). In the neighborhood of the avoided crossing, at $F = F_{\mathrm{cr}} \equiv \mathrm{Re}F_{\mathrm{res}}$, we can write $F(t) \approx F_{\mathrm{cr}} + \dot{F}_{\mathrm{cr}}(t - t_{\mathrm{cr}})$, where t_{cr} is the (real) time at which the crossing occurs, so that $F(t_{\mathrm{cr}}) = F_{\mathrm{cr}}$, and where $\dot{F}_{\mathrm{cr}}$ is the (real) time derivative of $F(t)$ at t_{cr}. Therefore, provided that F_{res} is sufficiently close to the real axis, we have

$$P_a = \exp\left(-\frac{2}{\hbar\dot{F}_{\mathrm{cr}}}\mathrm{Im}\int_{F_{\mathrm{cr}}}^{F_{\mathrm{res}}} dF\, \delta E_{\mathrm{ac}}(F)\right) , \tag{18}$$

where $\delta E_{\mathrm{ac}}(F)$ was defined by Eq. (15); we have chosen a sign convention so that if F is on the cut, the discontinuity in the a branch of $E_{\mathrm{ac}}(F)$ as the cut is crossed, in the direction that $F(t)$ changes as t increases from just below to just above t_{cr}, is $-\delta E_{\mathrm{ac}}(F)$. Setting $F = F_{\mathrm{cr}} + ix$, we obtain

$$P_a = \exp\left(-\frac{4}{\hbar\dot{F}_{\mathrm{cr}}}\mathrm{Re}\int_0^{\mathrm{Im}F_{\mathrm{res}}} dx\, C_{\mathrm{res}}\sqrt{(\mathrm{Im}F_{\mathrm{res}})^2 - x^2}\right) \tag{19a}$$

$$= e^{-\Omega_0\delta t} , \tag{19b}$$

where $\hbar\Omega_0 = 2|(\mathrm{Re}C_{\mathrm{res}})(\mathrm{Im}F_{\mathrm{res}})|$ and where $\delta t \equiv (\pi/2)|\mathrm{Im}F_{\mathrm{res}}/\dot{F}_{\mathrm{cr}}|$. Therefore the probability P_a for the atom to make a diabatic transition, in case (a), is less

than unity, as it should be, but it is close to unity if the field varies sufficiently rapidly that the avoided crossing is passed in a time δt such that $\hbar/\delta t$ is large compared to the energy gap $\hbar\Omega_0$. (In other words, $P_a \approx 1$ if the resonance is passed so rapidly that there is no time for the atom to undergo Rabi flopping.) In the adiabatic limit, where $\dot{F}_{cr} \to 0$, the time δt becomes infinite, P_a vanishes, and the atom switches branches as F passes along the real axis past the branch points.

Note that Eq. (19b) is analogous to the usual Landau-Zener formula for the probability of a diabatic transition in an atomic collision. In that context, F is replaced by the internuclear separation R of the two colliding atoms. A study of the quasienergy for one atom perturbed by another, in the complex R plane, has been carried out by Solov'ev,[29] and of course there are many analogies with the present study. There are also important differences: First, F, unlike R, is under the control of the experimentalist. Second, the energy transfer that results from absorbing photons is quantized. (The energy transfer that results from varying F, like that which results from varying R, is not quantized; it arises from stimulated photon scattering between modes within the finite frequency bandwidth of the light.) Third, this quantization leads to a series of resonance branch points associated with the same two atomic levels. Fourth, this quantization also implies the existence of channel thresholds, separated in the energy plane by units of $\hbar\omega$, which gives rise to a series of shadow branch points associated with the same single atomic level. It is towards the shadow branches that we now turn our attention.

B. Thresholds

At any point in the neighborhood of a type-T branch point, at F_{thr}, the two branches are separated by the distance $|\delta E_{ac}(F)|$, where now

$$\delta E_{ac}(F) = 2C_{thr}\sqrt{F - F_{thr}}, \tag{20}$$

with C_{thr} the (approximately) constant value of $C_{thr}(F)$ in the vicinity of the branch point.[30] We draw the branch cut from a type-T branch point so that it vertically crosses the real F axis and extends to infinity, as in Figs. 2 and 5. As $F(t)$ sweeps along the real axis, the real and imaginary parts of $E_{ac}(F)$ each exhibit avoided crossings at $F_{cr} \equiv F(t_{cr})$, where now $F_{cr} \equiv \mathrm{Re}F_{thr}$.

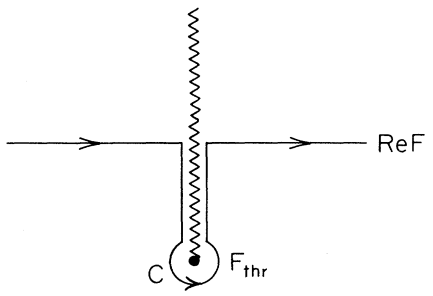

FIG. 5. A schematic diagram showing the location, in the F plane, of a T-type branch point. The contour C wraps around the branch cut in the lower half of the F plane.

The atom starts out on a dominant branch. However, if, as $F(t)$ increases, a threshold is passed, the cut emanating from a type-T branch point is crossed, and the character of the adiabatic state changes (continuously) to that of a shadow branch. The probability for making a diabatic transition so as to remain on the dominant branch (i.e., the survival probability) is

$$P_d = \exp\left(-\frac{2}{\hbar\dot{F}_{cr}}\mathrm{Im}\int_{F_{cr}}^{F_{thr}} dF\ \delta E_{ac}(F)\right), \tag{21}$$

where $\delta E_{ac}(F)$ is defined by Eq. (20), with the sign convention chosen so that $-\delta E_{ac}(F)$ is the discontinuity in the *dominant* branch of $E_{ac}(F)$ as the cut is crossed in the direction that $F(t)$ changes as t increases from just below to just above t_{cr}, and where again $\dot{F}_{cr}$ is the time derivative of $F(t)$ evaluated at $t = t_{cr}$. Setting $F = F_{cr} + ix$, we obtain

$$P_d = \exp\left(-\frac{4}{\hbar\dot{F}_{cr}}\mathrm{Re}\int_0^{\mathrm{Im}F_{thr}} dx\ C_{thr}e^{i\pi/4}\sqrt{x - (\mathrm{Im}F_{thr})}\right) \tag{22a}$$

$$= \exp\left(\frac{8}{3\hbar\dot{F}_{cr}}(-\mathrm{Im}F_{thr})^{3/2}\mathrm{Re}(C_{thr}e^{i\pi/4})\right) \tag{22b}$$

$$= \exp\left(-\frac{4\mathrm{Im}F_{thr}}{3\hbar\dot{F}_{cr}}\delta\Delta_{ac}(F_{cr})\right), \tag{22c}$$

where $-\delta\Delta_{ac}(F_{cr}) \equiv -2C_{thr}\sqrt{-i\mathrm{Im}F_{thr}}$ is the discontinuity in the *shift* of the *dominant* branch as the cut is crossed at $F_{cr} = \mathrm{Re}F_{thr}$ in the direction that $F(t)$ changes as t increases from just below to just above t_{cr}. If we define δF so that, with the branch the dominant one, $-\delta\Delta_{ac}(F_{cr}) = [d\Delta_{ac}(F)/dF]\delta F$, where $d\Delta_{ac}(F)/dF$ is evaluated just before the threshold is passed, we can rewrite Eq. (22c), assuming the validity of Eq. (12), as

$$P_d = e^{-\Gamma_{ac}\delta t/\hbar}, \tag{23}$$

where now $\delta t = 4\delta F/(3\beta\dot{F}_{cr})$. The quantity δt is the characteristic time associated with the passage past the threshold, although this cannot make sense if δt is negative. Note, however, that if δt were negative, the shadow pole would lag behind the dominant pole as (real) t increases towards t_{cr}—the shadow pole would lag behind as the poles move, in the E plane, towards the threshold—and therefore the shadow pole would not be "ready" to take over its role as a dominant pole. Thus we assume that δt is positive. If the threshold is passed very slowly, so that $\dot{F}_{cr}$ is small, and δt is long compared to $\hbar/\Gamma_{ac}$, we have $P_d \approx 0$—the atom completely ionizes. On the other hand, if the threshold is passed in a time δt that is short compared to $\hbar/\Gamma_{ac}$, we have $P_d \approx 1$—the atom remains on the dominant branch, as normally happens in practice.

VI. NUMERICAL STUDY

In this section we present results of a numerical study of the ground-state branch of the quasienergy. We first report results for the hydrogen atom, irradiated by linearly polarized light, to illustrate some features of type-R branch points. The calculations for hydrogen were carried out by Potvliege by solving the system of equations (2) using a complex Sturmian basis[12(b),16] with a single wave number[12(b),31] for the basis functions. In order to detect shadow branches, a basis containing more than one wave number must be used,[12(b)] and since the computer code has not yet been adapted to utilize a basis containing more than one wave number, type-T branch points have not been studied for hydrogen. However, we have studied type-T branch points for a model atom, irradiated by circularly polarized light. This model, originally studied in depth by Berson[32] and by Manakov and Rapoport,[33,34] consists of an electron initially bound to the zero-range potential

$$\frac{4\pi\hbar^2}{2\mu\kappa}\delta(\mathbf{x})\frac{\partial}{\partial r}r \ .$$

This potential supports a bound state with energy $E^{(0)} = \hbar^2\kappa^2/(2\mu)$, but no other discrete (bound or quasibound) levels. Hence, within this model, the quasienergy has only type-T branch points, and no type-R branch points. (See, however, the end of Sec. III.)

A. Hydrogen atom

When the hydrogen atom is irradiated by 204-nm linearly polarized light there is an intermediate two-photon resonance, at an intensity close to 1×10^{14} W/cm^2, between the 1s state and a superposition of $3s$ and $3d$ states. In the zero-field limit there are two different degenerate superpositions of $3s$ and $3d$ states; both are of even parity, but one superposition, which we label as $3s$, is predominantly $3s$, while the other, which we label as $3d$, is predominantly $3d$. As the intensity is varied past 1×10^{14} W/cm^2, the real parts of the $1s$ branch and the $3d$ branch undergo an avoided crossing. The branch points that correspond to this $1s$-$3d$ resonance occur in the complex F plane at $F_{1s\text{-}3d} \approx 0.0175 + i0.0040$ and at $F'_{1s\text{-}3d} \approx F^*_{1s\text{-}3d}$, where we use atomic units (a.u.). The $3s$ and $3d$ branches meet at $F = 0$, but this is not a branch point since the two branches are uncoupled there. However, these two branches meet again at $F_{3s\text{-}3d} \approx 0.0165 + i0.0043$, in a.u. (The uncertainty in the numerical estimates of the positions of all the branch points is 0.0005 a.u. The branch points were located by allowing F to vary around circles of small radii, and different centers, and by inspecting, each time, whether the basis-set composition of a branch has changed after a complete circuit in the F plane—if it did, the new branch was identified by determining the bound state from which it originates in the limit of vanishing F.) All three branch points are shown in Fig. 6. Note that while $F_{3s\text{-}3d}$ is numerically very close to $F_{1s\text{-}3d}$,

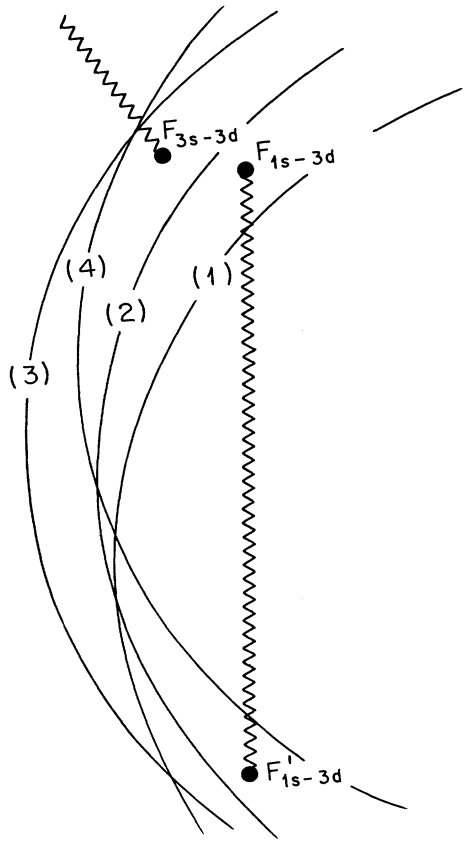

FIG. 6. The branch points associated with the two-photon resonance between the $1s$ and a superposition of $3s$ and $3d$ states of the hydrogen atom irradiated by 204-nm linearly polarized light. The curves labeled (1)–(4) are sections of circles of the same radius 0.0065 and centers at (1) $0.0220 - i0015$, (2) $0.021\,90 - i0.000\,25$, (3) $0.021\,00 + i0.000\,25$, (4) $0.021\,70 + i0.001\,25$. Units are a.u.

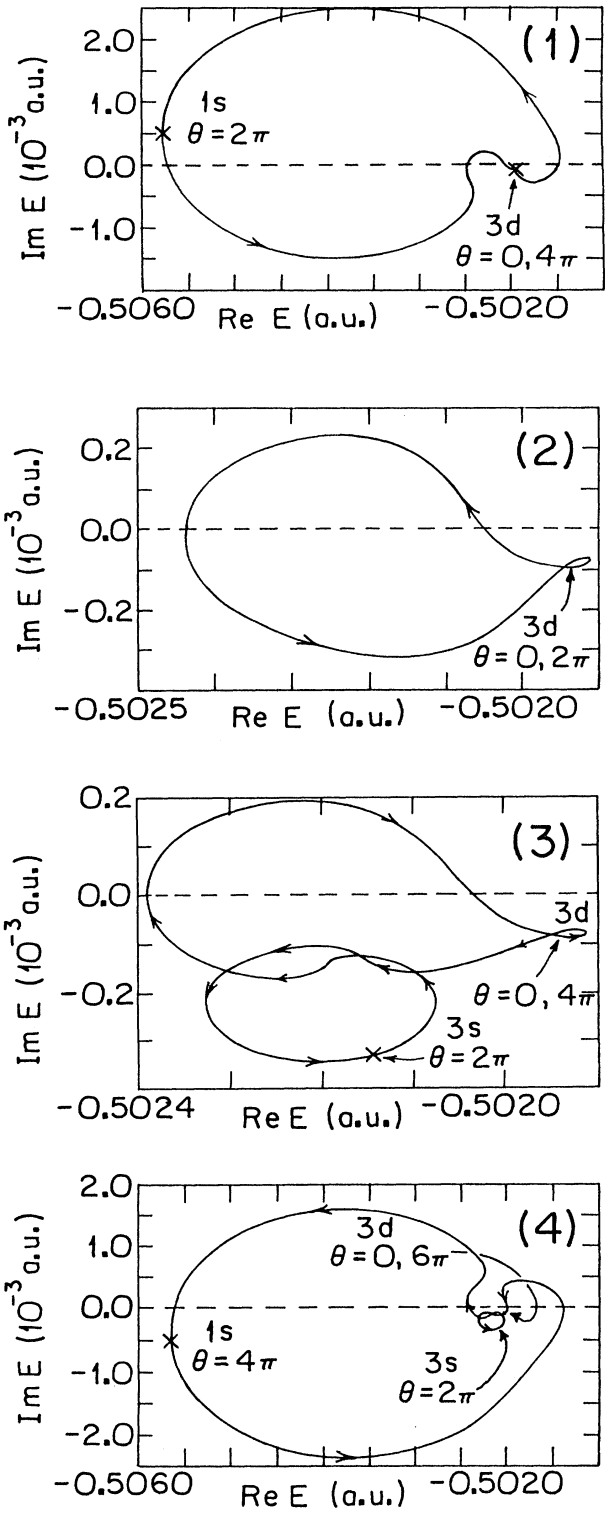

FIG. 7. Trajectories in the E plane, starting on the $3d$ branch and following one or more loops of a circle in the F plane, for atomic hydrogen irradiated by 204-nm linearly polarized light. Each trajectory in the E plane is numbered according to the number labeling the circle in the F plane, as given by the caption to Fig. 6.

the branch point at $F_{3s\text{-}3d}$ lies on a different sheet from the one at $F_{1s\text{-}3d}$ since the latter branch point is at the intersection of the $1s$ and $3d$ branches, while the former is at the intersection of the $3s$ and $3d$ branches. Thus, although $F_{3s\text{-}3d}$ is (slightly) closer to the origin than $F_{1s\text{-}3d}$, it does not influence the radius of convergence of the perturbation series for the $1s$ branch.

In Fig. 7 we show the trajectories in the E plane, starting on a particular dominant branch of $E_{\rm ac}$, for various closed trajectories in the F plane. The trajectories in the F plane are circles, all of the same small radius 0.0065 a.u., but with different centers. Thus we write $F = F_c + 0.0065e^{i\theta}$, where θ varies from 0 to $2m\pi$, with m a positive integer equal to the number loops in the F plane. We start, at $\theta = 0$, where $F = F_c + 0.0065$, on the (dominant) $3d$ branch. (This branch can be reached by starting on the $1s$ branch at $F = 0$ and following a trajectory that crosses the cut joining $F_{1s\text{-}3d}$ and $F'_{1s\text{-}3d}$, or it may be reached by starting on the $3d$ branch at $F = 0$ and following a trajectory that avoids the cut.) Circle (1) in the F plane encloses the branch point at $F_{1s\text{-}3d}$, but no other branch point. After one loop around this circle, the $3d$ branch becomes the $1s$ branch. After a second loop around this circle, the $3d$ branch is recovered, indicating that the branch point is indeed of the square-root type. Circle (2) encloses the branch points at both $F_{1s\text{-}3d}$ and $F'_{1s\text{-}3d}$, but not the branch point at $F_{3s\text{-}3d}$. The branch does not change after one loop around this circle (it remains $3d$). Circle (3) encloses all three branch points. After one loop around circle (3) the $3d$ branch becomes the $3s$ branch, and after a second loop the $3d$ branch is recovered. Finally, circle (4) encloses the branch points at $F_{1s\text{-}3d}$ and $F_{3s\text{-}3d}$, but not the branch point at $F'_{1s\text{-}3d}$. After one loop around circle (4) the $3d$ branch becomes the $3s$ branch; this change occurs before the cut joining $F_{1s\text{-}3d}$ and $F'_{1s\text{-}3d}$ is crossed, and so this cut has no effect. After a second loop around circle (4) the $3s$ branch becomes the $1s$ branch; the $3s$ branch first switches to the $3d$ branch, and from there to the $1s$ branch. After a third loop around circle (4), the $3d$ branch is recovered.

If we express the Rayleigh-Schrödinger perturbation series for the $1s$ branch as $E_{\rm ac}^{(1s)}(F) = \sum_{n=0}^{\infty} E_{2n}^{(1s)} F^{2n}$, the series converges for $|F| < R$, where, by the ratio test, the radius of convergence R is the limit, as n tends to infinity, of $|E_{2n}^{(1s)}/E_{2n+2}^{(1s)}|^{1/2}$. In Table I we show the ratios $|E_{2n}^{(1s)}/E_{2n+2}^{(1s)}|^{1/2}$ versus n for various wavelengths. Let us focus on the wavelength 204 nm. (For 204 nm, we calculated the perturbation series through 34th order in F. The coefficients $E_{2n}^{(1s)}$ are complex for $n \geq 3$ since the atom ionizes if it absorbs three or more photons. We note that the perturbation series for the $1s$ branch of the hydrogen atom has been studied by Pan, Taylor, and Clark[35] over various wavelengths, but only up to the term in F^{2N_0}, where N_0 is the minimum number of photons the atom must absorb to ionize). At the wavelength 204 nm the ratios $|E_{2n}^{(1s)}/E_{2n+2}^{(1s)}|^{1/2}$ behave erratically with n, and have not settled down by $n = 17$; nor does Padé extrapolation yield a converged result. However, the results

TABLE I. The square root of the absolute ratio of successive coefficients $|E_{2n}^{(1s)}/E_{2n+2}^{(1s)}|^{1/2}$ of the power series expansion of the ground-state branch of the quasienergy for the hydrogen atom irradiated by linearly polarized light of wavelengths 204, 355, and 140 nm. The fifth column is a Padé extrapolation (Ref. 36). Units are a.u.

n	204 nm	355 nm	140 nm	140 nm (Padé)
1	0.0388	0.165 656	0.078 622	
2	0.0188	0.030 241	0.068 485	
3	0.0193	0.018 730	0.061 081	
4	0.0198	0.017 886	0.057 633	
5	0.0207	0.017 542	0.055 653	0.048 382
6	0.0221	0.017 590	0.054 361	0.048 249
7	0.0251	0.017 585	0.053 452	0.048 180
8	0.0395	0.017 586	0.052 776	0.048 165
9	0.0115	0.017 587	0.052 253	0.048 169
10	0.0133	0.017 588	0.051 838	0.048 168
11	0.0164	0.017 589	0.051 499	0.048 168
12	0.0182	0.017 589		
13	0.0200	0.017 590		
14	0.0225			
15	0.0292			
16	0.0174			
17	0.0122			

are not inconsistent with a radius of convergence equal to $|F_{1s-3d}| \approx 0.017$. We note that at 204 nm three photons only barely ionize the hydrogen atom from the $1s$ state, and so there is a type-T branch point not far from the origin of the F plane; we do not know exactly where it is, but perhaps its presence is responsible for the erratic behavior of the ratios of the coefficients. It is not so difficult at every wavelength to estimate the radius of convergence of the perturbation series from the coefficients of the series. For example, in Table I we see that for 355 nm the ratios $|E_{2n}^{(1s)}/E_{2n+2}^{(1s)}|^{1/2}$ converge steadily, and yield a radius of convergence of 0.0176 a.u. For light of wavelength 355 nm there is a three-photon $1s$-$2p$ resonance at an intensity of about 1×10^{13} W/cm^2. This resonance gives rise to a *true* crossing of the real parts of the $1s$ and $2p$ branches, and the two branch points *both* lie in the upper right quadrant of the F plane, and therefore on the same side of the real F axis. These branch points are at about $0.0172 + i0.0030$ and $0.0175 + i0.0027$ a.u., in good agreement with the radius of convergence of 0.0176 a.u. At the wavelength of 140 nm the ratios $|E_{2n}^{(1s)}/E_{2n+2}^{(1s)}|^{1/2}$ decrease monotonically, and although they converge rather slowly as n increases, the convergence can be readily accelerated by Padé extrapolation,[36] to yield a radius of convergence of 0.048 168 4. In fact, 140-nm light is only slightly detuned from an intermediate one-photon resonance at weak intensities. The relative shift of the $1s$ and $2p$ levels is in the wrong direction for the resonance to become more sharply tuned as the intensity increases. One of the branch points corresponding to this resonance lies almost on the imaginary F axis, at $F_{1s-2p} \approx i0.048$ a.u., in accord with the previous estimate of the radius of convergence. (Another branch point lies, of course, at $-F_{1s-2p}$, which is almost at F_{1s-2p}^{*}, but we do not know where the "conjugate" branch point F'_{1s-2p} lies.) Further details will be presented elsewhere.[37]

B. Model atom

We now consider the model atom, whose potential was defined above. Following Berson[32] we introduce the scaled quantities $\epsilon = E_{\rm ac}(F)/|E_0|$, $v = -e\hbar F/(\mu^{3/2}|E_0|^{3/2})$, $w = \hbar\omega/|E_0|$, and $\rho = \sqrt{2}v/w^2$. The characteristic equation from which the bound-state branch of the quasienergy $E_{\rm ac}(F)$ may be determined is[32]

$$1 + i\sum_{s=0}^{\infty}\sum_{n=-s}^{s}\frac{(-1)^{n+s}(\epsilon+nw)^{s+1/2}\rho^{2s}}{(s-n)!(s+n)!(2s+1)} = 0 \ . \tag{24}$$

This equation gives not only the dominant branch but also the shadow branches, as can be seen by writing $(\epsilon+nw)^{s+1/2} = (2k_n)^{2s+1}$ and choosing the sign of k_n as discussed above. To find the branch point associated with the confluence of a shadow branch and a dominant branch at the mth channel threshold we numerically solve Eq. (24) for the value of F for which $\epsilon = -mw$. We specify energies in units in which $E_0 = -1$ (so that $E_{\rm ac} = \epsilon$) and we specify field strengths in units in which $e\hbar/(\mu^{3/2}|E_0|^{3/2}) = 1$ (so that $v = -F$). Throughout our calculations for the model atom, we set the ratio of the frequency to the binding energy to be $w = 0.525$, so that (at weak fields) the atom must absorb a minimum of two photons to ionize.

For this value of w the quasienergy has a type-T branch point very close to the real axis in the F plane, at

$$F_{\rm thr,2} = 0.164\,420\,289\,825\,400\,2 - i0.000\,026\,465\,838\,787\,61 \ ; \tag{25}$$

at this branch point the dominant branch coincides with a shadow branch at the channel 2 threshold in the E plane, at $E_{\mathrm{thr},2} = -1.05$. In Fig. 8 we demonstrate that this type-T branch point is of the square-root type. We find that $C_{\mathrm{thr}}(F_{\mathrm{thr},2})$ is extremely small; in fact, the numerical evidence is that $C_{\mathrm{thr}}(F)$ vanishes as $(F - F_{\mathrm{thr},2})^2$ when F approaches $F_{\mathrm{thr},2}$ so that $E_{\mathrm{ac}}(F) - E_{\mathrm{thr},2}$ behaves approximately as $(F - F_{\mathrm{thr},2})^{5/2}$, as indicated in Fig. 8(a). This result is not altogether surprising, and is related to the Wigner threshold law, as we now explain. When $E_{\mathrm{ac}}(F)$ approaches the threshold with channel number N, in a direction such that the Nth channel changes from being open to closed, the partial rate for ionization by N photons vanishes as $(E_{\mathrm{ac}} - E_{\mathrm{thr},N})^{(2l+1)/2}$, where $E_{\mathrm{thr},N} = -N\hbar\omega$, and where l is the lowest allowed value of the orbital angular momentum quantum number; for circularly polarized light $l = N$. Now, provided that $F_{\mathrm{thr},N}$, the position of the branch point at which $E_{\mathrm{ac}}(F) = E_{\mathrm{thr},N}$, is not far from the real F axis, and provided $(E_{\mathrm{ac}} - E_{\mathrm{thr},N})$ is approximately proportional to $(F - F_{\mathrm{thr},N})$ as F varies along the real axis, we expect that the partial rate for N-photon ionization behaves as $(F - F_{\mathrm{thr},N})^{(2l+1)/2}$ near the threshold; this suggests that $C_{\mathrm{thr}}(F)$ behaves as $(F - F_{\mathrm{thr},N})^l$. However, we cannot draw this conclusion for all of the branch points; we have verified it holds also for $N = 1$, but it may not hold for $N > 2$ since $F_{\mathrm{thr},N}$ may be insufficiently close to the real axis. Note that the background term $E_{\mathrm{thr}}(F)$—see Eq. (10)—*is* linear in the neighborhood of $F_{\mathrm{thr},2}$, as is apparent from Fig. 8(b), and that this term swamps the square-root behavior.

As noted in Sec. IV, the point $F_{\mathrm{thr},0}$ in the F plane at which $k_0 = 0$ is not a branch point but a saddle point. We demonstrate this numerically in Fig. 9. We denote the second branch, which meets the dominant branch $E^{(d)}$ when $F = F_{\mathrm{thr},0}$, by $E^{(s)}$, but we are not sure where this second branch originates from. Note that the difference of $E^{(d)}$ and $E^{(s)}$ behaves as $(F - F_{\mathrm{thr},0})^4$.

It is instructive to look at the behavior of the quasienergy in the E plane as F moves by the branch point at $F_{\mathrm{thr},2}$. In the upper box of Fig. 10 we show four different trajectories, in the E plane. These trajectories start from either a point d on the dominant branch or a point s on the shadow branch, and they terminate at either a point d' on the dominant branch or a point s' on the shadow branch. The trajectories in

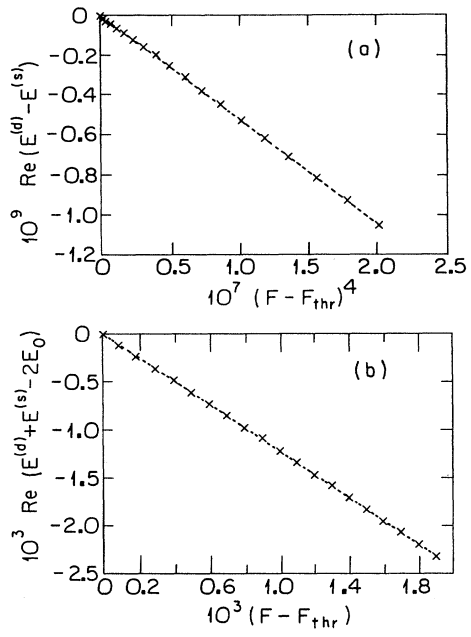

FIG. 8. Behavior in the F plane of the dominant ($E^{(d)}$) and shadow ($E^{(s)}$) branches of the bound-state quasienergy near a type-T branch point for the zero-range model atom, for the case where w, the ratio of the frequency to the unperturbed binding energy, is 0.525. The shadow branch is the one associated with the threshold of channel 2, and the branch point is at $F_{\mathrm{thr}} \equiv F_{\mathrm{thr},2}$, given by Eq. (25) of the text. Note that E_0, the unperturbed binding energy, is -1. The units are defined in the text.

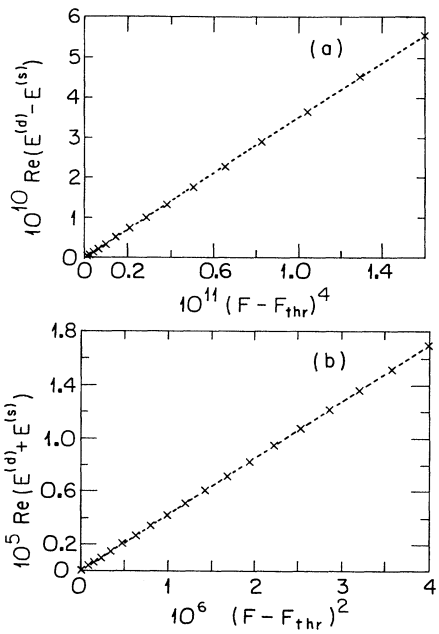

FIG. 9. Behavior in the F plane of the dominant ($E^{(d)}$) and shadow ($E^{(s)}$) branches of the bound-state quasienergy near a type-T saddle point for the zero-range model atom (with $w = 0.525$). The shadow branch is the one associated with the threshold of channel 0, and the saddle point is at $F_{\mathrm{thr}} \equiv F_{\mathrm{thr},0} = 0.125\,234 + i0.774\,159$. The units are defined in the text.

the E plane correspond to the trajectories in the F plane shown in the lower box of Fig. 9. If a trajectory in the F plane crosses a branch cut, the corresponding trajectory in the E plane also crosses a branch cut, and there is a switchover of branches. This occurs for the trajectories marked by open circles and open squares. In order to remain on the same branch, a trajectory in the F plane must be chosen so as to avoid the branch cut; this occurs for the trajectories marked by crosses and asterisks. Let us illustrate this for a path in the F plane along which the branch remains dominant. It is helpful to turn to Fig. 3, taking $N_0 = 2$ to be specific. In order to arrive on the dominant branch at a point F on the real axis on the right of $\mathrm{Re}F_{\mathrm{thr}}$, starting at a point on the left of $\mathrm{Re}F_{\mathrm{thr}}$, we see from Fig. 3 that $\mathrm{Im}E$ must first increase through zero and become positive, while $\mathrm{Re}E$ decreases; then $\mathrm{Im}E$ must become negative again as the real F axis is approached. This is indeed the behavior that we find in the upper half of Fig. 10.

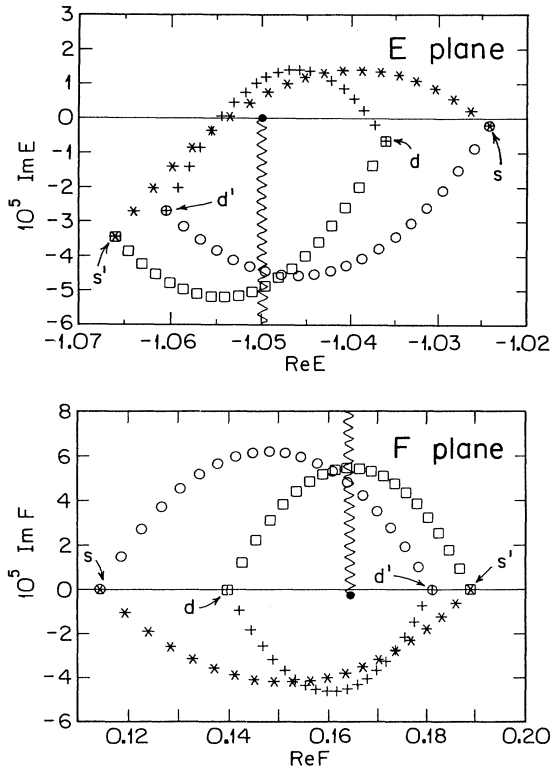

FIG. 10. In the upper box we show four different trajectories, in the E plane, of the bound-state quasienergy for the zero-range model atom (with $w = 0.525$). These trajectories correspond to the trajectories in the F plane shown in the lower box, and they start from either a point d on the dominant branch or a point s on the shadow branch, terminating at either a point d' on the dominant branch or a point s' on the shadow branch. Each trajectory in the E plane corresponds to the trajectory in the F plane with similar markings. The units are defined in the text; the vertical scale is marked in these units reduced by 10^{-5}.

We can also generate, from Eq. (24), the Rayleigh-Schrödinger perturbation series for the ground-state branch of $E_{\mathrm{ac}}(F)$; this can be done most easily by iteration, to successively higher powers of v, using symbolic manipulation software. Expressing the perturbation series as $E_{\mathrm{ac}}(F) = \sum_n E_{2n}F^{2n}$, we show, in Table II, the ratios $|E_{2n}/E_{2n+2}|$. The ratios $|E_{2n}/E_{2n+2}|$ converge only very slowly as n increases, but, as before, we accelerated the convergence by Padé extrapolation.[36] The radius of convergence of the power series is $\lim_{n\to\infty}|E_{2n}/E_{2n+2}|^{1/2} = \sqrt{0.027\,034\,03} = 0.164\,420\,3$. We found that the branch point of $E_{\mathrm{ac}}(F)$ closest to the origin is the one associated with the confluence of the dominant and shadow eigenvalues at the two-photon threshold (which is not surprising since E_0 is not far from this threshold), and the distance of this branch point from the origin—see Eq. (25)—is in excellent agreement with the radius of convergence of the power series. Fur-

TABLE II. Absolute ratio of successive coefficients of the power series expansion of the bound-state quasienergy for the zero-range model atom, for the case where w, the ratio of the frequency to the unperturbed binding energy, is 0.525. The third column is a Padé extrapolation (Ref. 36) of these ratios. For units see text.

n	$\lvert E_{2n}/E_{2n+2}\rvert$	Padé
1	23.519 71	
2	0.052 61	
3	0.106 44	
4	0.174 59	
5	0.168 97	0.194 744 255 9
6	0.091 36	−0.034 205 319 4
7	0.062 76	0.044 576 020 9
8	0.054 36	0.055 478 134 3
9	0.048 56	0.025 595 805 8
10	0.044 42	0.020 255 726 4
11	0.041 67	0.014 965 386 3
12	0.039 70	0.027 377 022 0
13	0.038 19	0.026 577 118 9
14	0.037 01	0.027 125 721 3
15	0.036 05	0.027 382 299 9
16	0.035 27	0.027 011 480 3
17	0.034 61	0.027 034 170 1
18	0.034 04	0.027 034 414 8
19	0.033 56	0.027 033 895 8
20	0.033 14	0.027 034 093 8
21	0.032 77	0.027 034 040 9
22	0.032 44	0.027 034 012 5
23	0.032 15	0.027 034 005 8
24	0.031 88	0.027 034 006 9
25	0.031 65	0.027 034 010 0
26	0.031 43	0.027 034 013 6
27	0.031 24	0.027 034 016 8
28	0.031 06	0.027 034 019 6
29	0.030 89	0.027 034 021 8

thermore, we studied the perturbation series for the shift and width functions, $\Delta_{ac}(F)$ and $\Gamma_{ac}(F)$, and we confirmed that these series have exactly the same radii of convergence, equal to that of the perturbation series of $E_{ac}(F)$. Calculations of the radius of convergence of the Rayleigh-Schrödinger perturbation series for this model atom were recently carried out over a wide range of frequencies by Alvarez and Sundaram.[38] As noted by them, as ω varies, the radius of convergence rises and falls, dropping to zero each time the condition $N\hbar\omega + E_0 = 0$ is satisfied, where N is a (positive) integer.

It is not difficult to find the positions of the type-T branch points in the F plane since we know their location in the E plane, and, as noted in Sec. II, the eigenvalue problem for F is relatively easy to solve. Once we know where these branch points lie in the F plane, we can (if there are no type-R branch points to worry about—and there are not for the zero-range potential) *analytically continue the original perturbation series* for $E_{ac}(F)$ beyond its radius of convergence. There are many ways to do this, and we give just one example. We expand $E_{ac}(F)$ about $E_{ac}(F_{thr})$ in powers of $(F^2 - F_{thr}^2)$, where $\pm F_{thr}$ are the branch points nearest to the origin. Thus we write

$$E_{ac}(F) = \sum_{n=0}^{n_0} A_n (F^2 - F_{thr}^2)^n + \sqrt{F^2 - F_{thr}^2} \sum_{n=0}^{n_0} B_n (F^2 - F_{thr}^2)^n \ . \quad (26)$$

If we expand the right-hand side of Eq. (26) about the origin of the F plane, we can determine the $2n_0 + 2$ coefficients A_n and B_n by matching this series to the original Rayleigh-Schrödinger series $\sum_{n=0}^{2n_0+2} E_{2n} F^{2n}$ truncated after the term in F^{2n_0+2}. Of course, we should find that $A_0 = E_{thr,2}$, where, in the present case, $E_{thr,2} = -1.05$; we should also find, in the present case, that the coefficients B_0 and B_1 are (close to) zero. In fact, choosing $n_0 = 29$ we find that $A_0 = -1.050\,000\,000\,000\,041 + i2.3 \times 10^{-15}$, whereas $A_1 = -1.850\,37 - i0.001\,64$, and that $B_0 = -4.5 \times 10^{-13} - i6.7 \times 10^{-12}$ and $B_1 = 2.4 \times 10^{-9} + i2.7 \times 10^{-8}$, whereas $B_2 = -0.039\,233 - i0.000\,086$. The series of Eq. (26) has a rather large radius of convergence since there are no other type-T branch points close to F_{thr}. Furthermore, due to the presence of the (two-valued) square root in Eq. (26), this series yields both the dominant and shadow branches. In Table III we show estimates of the dominant and shadow branches, based on the series (26) with $n_0 = 29$, for various values of F along the positive real axis, and we compare these estimates with the exact values to illustrate the high accuracy of the estimates. As F increases along the positive real axis, from below $\mathrm{Re}F_{thr,2}$ to above, the branch that was a dominant one becomes a shadow one, and vice versa. The point at which this switchover takes place depends on where we draw the branch cut of the square root $\sqrt{F^2 - F_{thr,2}^2}$. We continue to draw the cut along a vertical line that starts from $F_{thr,2}$ and crosses the real F axis. Thus the switchover takes place at $\mathrm{Re}F_{thr,2} = 0.164$, and

TABLE III. The dominant (d) and shadow (s) branches of the bound-state quasienergy for the zero-range model atom (with $w = 0.525$) vs F^2. The shadow branch is the one associated with the threshold of channel 2. The third column contains the results based on the series expansion given by Eq. (26) of the text, while the fourth column contains the exact results. (Units are given in text.)

F^2	Type	$E_{ac}(F)$ (series)	$E_{ac}(F)$ (exact)
0.00	d	$-1.000\,000\,000\,000\,0 - i0.000\,000\,000\,000\,0$	$-1.000\,000\,000\,000\,0 - i0.000\,000\,000\,000\,0$
	s	$-1.000\,005\,393\,441\,4 + i0.000\,001\,602\,662\,2$	$-1.000\,000\,000\,000\,0 - i0.000\,000\,000\,000\,0$
0.01	d	$-1.018\,488\,354\,266\,1 - i0.000\,001\,155\,921\,9$	$-1.018\,488\,354\,266\,1 - i0.000\,001\,155\,921\,9$
	s	$-1.018\,488\,649\,362\,2 - i0.000\,000\,676\,480\,7$	$-1.018\,488\,354\,233\,2 - i0.000\,000\,749\,501\,4$
0.02	d	$-1.036\,985\,641\,032\,8 - i0.000\,007\,043\,833\,0$	$-1.036\,985\,641\,032\,8 - i0.000\,007\,043\,833\,0$
	s	$-1.036\,985\,643\,682\,6 - i0.000\,006\,865\,219\,2$	$-1.036\,985\,640\,787\,8 - i0.000\,006\,865\,748\,9$
0.03	d	$-1.055\,488\,318\,824\,0 - i0.000\,021\,416\,699\,9$	$-1.055\,488\,318\,824\,1 - i0.000\,021\,416\,700\,0$
	s	$-1.055\,488\,365\,134\,1 - i0.000\,021\,417\,146\,3$	$-1.055\,488\,365\,134\,3 - i0.000\,021\,417\,146\,1$
0.04	d	$-1.073\,995\,360\,701 - i0.000\,046\,318\,566$	$-1.073\,995\,360\,707 - i0.000\,046\,318\,560$
	s	$-1.073\,998\,659\,588 - i0.000\,046\,333\,594$	$-1.073\,998\,659\,597 - i0.000\,046\,333\,601$
0.05	d	$-1.092\,506\,165\,47 - i0.000\,082\,547\,10$	$-1.092\,506\,165\,44 - i0.000\,082\,547\,57$
	s	$-1.092\,527\,791\,01 - i0.000\,082\,642\,20$	$-1.092\,527\,790\,93 - i0.000\,082\,642\,26$
0.06	d	$-1.111\,020\,505\,3 - i0.000\,130\,182\,3i$	$-1.111\,020\,506\,9 - i0.000\,130\,181\,0i$
	s	$-1.111\,097\,932\,1 - i0.000\,130\,533\,9i$	$-1.111\,097\,935\,0 - i0.000\,130\,535\,6i$
0.07	d	$-1.129\,538\,264 - i0.000\,188\,733$	$-1.129\,538\,251 - i0.000\,188\,722$
	s	$-1.129\,744\,711 - i0.000\,189\,698$	$-1.129\,744\,692 - i0.000\,189\,721$
0.08	d	$-1.148\,059\,29 - i0.000\,257\,23$	$-1.148\,059\,20 - i0.000\,257\,29$
	s	$-1.148\,520\,48 - i0.000\,259\,71$	$-1.148\,520\,31 - i0.000\,259\,63$
0.09	d	$-1.166\,583\,0 - i0.000\,334\,3$	$-1.166\,583\,0 - i0.000\,334\,8$
	s	$-1.167\,498\,3 - i0.000\,340\,6$	$-1.167\,498\,2 - i0.000\,339\,6$
0.10	d	$-1.185\,107 - i0.000\,419$	$-1.185\,109 - i0.000\,420$
	s	$-1.186\,777 - i0.000\,432$	$-1.186\,780 - i0.000\,429$

if we were to show, in Table III, values of the branches on the left and right edges of the cut we would see a very small discontinuity, determined by the second term on the right-hand side of Eq. (26).

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[20] Shadow poles that, at $F = 0$, lie on the negative-energy axis of a sheet reached by encircling the threshold at channel N_0 initially move into the upper half of the E plane as F increases from zero. This follows because if F is small, k_{N_0}, for the dominant pole, is approximately *real* and *positive*; therefore changing the sign of k_{N_0} in the asymptotic boundary condition amounts to complex conjugating $\exp(ik_{N_0}r)$—hence the shadow poles that have a negative sign for k_{N_0} initially behave like the complex conjugate of the dominant pole.
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[23] Any function $f(z)$ that is nonsingular throughout a domain $\mathcal{D}$ that includes part of the real z axis can be expressed as $f_1(z)+if_2(z)$, where $f_1(z)$ and $f_2(z)$ are real on the section of the real axis inside $\mathcal{D}$. If $f(z)$ has a branch point at z_{br}, both $f_1(z)$ and $f_2(z)$ necessarily have branch points at z_{br} and z_{br}^* [unless $f(z)$ has also a branch point at z_{br}^*].
[24] If $E_{ac}(F)$ were nonsingular at $F = F_{res}$ it would have a Taylor series expansion about $E_{ac}(F_{res})$ in powers of $F - F_{res}$, and therefore would trace out one or more complete circuits about E_{res} as F traces out a complete circuit around F_{res}.
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[30] Actually, $C_{thr}(F)$ may be extremely small, or may even vanish at $F = F_{thr}$; see Sec. VI of the text. If $C_{thr}(F)$ vanishes as $(E-E_{thr})^l$, we simply replace C_{thr} by $C_{thr}(E-E_{thr})^l$ on the right-hand side of Eqs. (10) and (20).
[31] The radial Sturmian functions behave, for asymptotically large values of the radial coordinate r, as $r^n e^{i\kappa r}$, where, in the calculations reported on here, the wave number κ was

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$$f_k(j) \equiv \frac{c_{1,k} + c_{2,k}(j+k) + c_{3,k}(j+k)^2}{1 + c_{4,k}(j+k) + c_{5,k}(j+k)^2} ,$$

where j and k are integers with j ranging from 1 to 5 and with $k = 0, 1, 2, \ldots$. For a fixed value of k, choose the coefficients $c_{i,k}, i = 1, \ldots, 5$ so that $f_k(j) = b_{k+j}, j = 1, \ldots, 5$. We want the limit, as k approaches infinity, of $f_k(j)$, but this is the limit of $c_{3,k}/c_{5,k}$; in the third column of Table I we show $c_{3,k}/c_{5,k}$, with the integer n in the first column related to k by $n = k + 5$.

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