Theory of fuzzy transitions between quantum and classical mechanics and proposed experimental tests

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In a generalized quantum-mechanical framework with fuzzy dynamical variables, there exists a small radical length $R < 10^{-17}$ cm, which is related to the physical impossibility of unlimited accuracy in measuring a particle's position. It sheds light on the problem of a detailed and gradual transition, as the mass increases, from the probabilistic behavior of microscopic objects ($mcR \ll \hbar$) to the deterministic behavior of macroscopic objects ($mcR \gg \hbar$), a problem unaddressable within the framework of ordinary quantum mechanics. A generalized equation of motion for a particle with an arbitrary mass m is postulated. It possesses a new local gauge symmetry and reduces to the Schrödinger and the Hamilton-Jacobi equations in the limits of small m and large m, respectively. For the double-slit experiment, the theory predicts that, as the mass increases with the momentum fixed, the interference pattern will have roughly half the usual intensity and two bright spots when $mcR/\hbar \sim 1$.

I. INTRODUCTION

It is well known that the relation between quantum and classical mechanics resembles that of wave and geometric optics. Although one can discuss the "classical limit $\hbar \to 0$" of the relation, say, $Px - xP = -i\hbar$, it is difficult to discuss and to test experimentally the gradual transition from the classical deterministic behavior to the quantum probabilistic behavior when particles change from macro-objects to micro-objects due to the decrease of their masses. (Note that fundamental equations of motion involve masses, but they do not explicitly involve the size of objects.) The difficulty stems from the lack of a basic length scale in the conventional quantum mechanics, so that one simply does not have a quantity which involves mass and can be compared with the Planck constant $\hbar$. In sharp contrast, the relation between relativistic mechanics and classical mechanics can be easily discussed because of the existence of v and c which can be directly compared. In this case, the departure of a particle, moving with a velocity v, from the classical mechanics can be discussed in terms of the parameter v/c.

In a previous paper,[1] we discussed a generalized quantum mechanics in which there exists a very small radical length R and, hence, we naturally have a dimensionless parameter $mcR/\hbar$, similar to the parameter v/c in the relativistic mechanics. The presence of such a parameter $mcR/\hbar$ enables us to discuss a generalized equation of motion which can describe both the quantum behavior of micro-object ($mcR/\hbar \ll 1$) and the classical behavior of macro-object ($mcR/\hbar \gg 1$). Furthermore, we can also explore the transition between quantum and classical mechanics when the parameter $mcR/\hbar$ is about one, for which the objects are neither microscopic nor macroscopic. The formalism of quantum mechanics based on such a generalized equation of motion is more in harmony with the fact that, in principle, classical mechanics is required for the formulation of basic concepts related to the microscopic physical world.[2] (See Sec. VI.)

We may remark that the radical length R is introduced to discuss possible modifications of physics at short distances or at extremely high energies.[1] It turns out that, in a generalized quantum-mechanical framework, the radical length R enables us to avoid the unphysical coordinate eigenstates which cannot be normalized and, therefore, do not have probabilistic meaning. As a result, the coordinate x of a quantum particle becomes a fuzzy dynamical variable because of the basic assumption

$$\Delta x_{\min} = R \ . \tag{1}$$

This implies that a quantum particle's position by itself cannot be measured with unlimited accuracy, even if we do not measure its momentum at the same time. A physical argument for (1) is as follows: The position state of a quantum particle must be determined by the Schrödinger equation with a suitable potential rather than by some other assumption. Classically, a particle can be localized at a certain position x_0 by an idealized δ-function potential $-V_0\delta(x-x_0)$. However, the corresponding position for a quantum particle with a mass m is determined by the Schrödinger equation with the same δ-function potential. One finds that its position is not exactly at x_0. Rather, it is described by the wave function

$$\varphi(x) = A \exp[-|x-x_0|/(2^{1/2}R)], \quad R = \hbar^2/2^{1/2}mV_0 \ , \tag{2}$$

which has a position uncertainty $\Delta x \approx R$. This result indicates that a quantum particle can never be forced to locate precisely at one point, although a classical particle can be forced to be at a certain point.[1] This amounts to abandoning the concept of the coordinate eigenstate or the δ function $\delta(x)$ for a particle's position. The natural mathematical framework for this inherent fuzziness is

Klauder's continuous representation[3,4] in Hilbert space, which excludes the usual eigenstates $\langle x|$ and $\langle p|$.

II. A FUZZY CRITERION FOR MICROSCOPIC AND MACROSCOPIC OBJECTS

Evidently, a ping-pong ball ($m_{\rm PPB} \sim 10^{-3}$ kg) is a "macroscopic object whose motion can be described by the Newtonian laws, while an electron ($m_e \sim 10^{-30}$ kg) is a microscopic object described by the Schrödinger equation. However, it appears that there is no clear-cut boundary between microscopic and mesoscopic objects. A particle which is neither microscopic nor macroscopic may be termed a "mesoscopic object." Since the mass of an object appears in both Newton's and Schrödinger's equations of motion, it is reasonable and convenient to use $mcR/\hbar$ as the criterion for a microscopic object. We may remark that the size of an object is not a convenient criterion because it does not appear explicitly in basic equations of motion.

With the help of the parameter $mcR/\hbar$, we can now discuss the basic commutation relation as the mass m varies from ~ 0 to $\sim \infty$. Instead of $Px - xP = -i\hbar$, let us assume

$$P_T x - x P_T = -i\hbar Q, \quad Q \equiv f(mcR/\hbar) , \tag{3}$$

where the "true momentum" P_T, which is defined for an object with an *arbitrary* mass m, depends on $mcR/\hbar$. The function $f(mcR/\hbar)$ should satisfy

$$\begin{aligned} &f(mcR/\hbar) \to 1 \quad \text{as } m \to 0 , \\ &f(mcR/\hbar) \to 0 \quad \text{as } m \to \infty . \end{aligned} \tag{4}$$

We stress that if the limit $Px - xP \to 0$ exists for macroscopic objects, then the true momentum P_T in (3) should be a mixture of the quantum momentum P (a q number) and the classical momentum P_c (a c number):

$$\begin{aligned} &\mathbf{P}_T = Q\mathbf{P} + C\mathbf{P}_c , \\ &Q = f(mcR/\hbar) \geq 0, \quad C = 1 - Q \geq 0 . \end{aligned} \tag{5}$$

We postulate to interpret the true momentum P_T in (5) as follows: For an object, the "quantum fraction" Q is the probability of finding it behaving like a quantum particle and the "classical fraction" C is that of finding it behaving like a classical particle. Such an interpretation is clearly consistent with the two limits in (4). In analogy to (5), true energy is $E_T = QE + CE_c$.

Because of the probabilistic nature of $f(mcR/\hbar)$ and the limiting properties (4), it is reasonable to assume that $f(mcR/\hbar)$ satisfies

$$df(\Lambda) = -f(\Lambda) d\Lambda, \quad \Lambda \equiv mcR/\hbar , \tag{6}$$

in analogy to the law of inherent probability in the radioactive decay. It follows from (6) that

$$f(mcR/\hbar) = \exp(-mcR/\hbar) . \tag{7}$$

For the purpose of discussions, let us assume $R \sim 10^{-20}$ cm.[5] If particles have masses $m \lesssim 10^4 m_p$, where $m_p = 1.6 \times 10^{-27}$ kg is the proton mass, we have

$$f(mcR/\hbar) \gtrsim \exp(-5 \times 10^{-3}) . \tag{8}$$

These are quantum particles for all practical purposes. On the other hand, if particles have masses $m \gtrsim 10^{-9}$ kg, we have

$$f(mcR/\hbar) \lesssim \exp(-3 \times 10^{11}) . \tag{9}$$

They are classical particles.[6]

It is interesting to see whether the radical length R in fuzzy quantum mechanics could be identified with the Planck length $\sim 10^{-33}$ cm. In this case, a particle with a mass $m \sim 10^{-9}$ kg will behave like a quantum particle because

$$mcR/\hbar \sim 3 \times 10^{-2} . \tag{10}$$

This appears to be unlikely.[7]

We remark that, theoretically, the "quantum fraction" Q may be some other functions of $mcR/\hbar$.[8] For example, if Λ in (6) is identified with $(mcR/\hbar)^{-1}$, then one gets a different function $\exp(-\hbar/mcR)$. In this case, one can make the identifications $Q = 1 - \exp(-\hbar/mcR)$ and $C = \exp(-\hbar/mcR)$. However, predictions of experimental results to be discussed later are insensitive to the specific forms of the function f.

III. GENERALIZED EQUATION OF MOTION FOR OBJECTS WITH ARBITRARY MASSES

Based on the idea of the true momentum (5), we postulate a generalized equation of motion for a physical object having an arbitrary mass $m > 0$ and moving in a potential field $A^\mu = (A^0, \mathbf{A}) = (A_0, \mathbf{A})$:

$$\left[Q \left[i\hbar \frac{\partial}{\partial t} - A_0 \right] + CE_c \right] \Phi = \frac{1}{2mQ} [Q(-i\hbar\nabla - \mathbf{A}) + C\mathbf{P}_c]^2 \Phi , \tag{11}$$

where

$$E_c = -\frac{\partial S}{\partial t} = \frac{1}{2m}(\mathbf{P}_c - \mathbf{A})^2 + A_0, \quad \mathbf{P}_c = \nabla S .$$

S is Hamilton's principal function.[9] When A^μ do not involve time explicitly, we can write the solution Φ in the form[10]

$$\Phi = \psi \exp[-iCS/Q\hbar], \quad S = -E_c t + \int_{T(\mathbf{r})}^{\mathbf{r}} \mathbf{P}_c(\mathbf{r}') \cdot d\mathbf{r}' , \tag{12}$$

where $T(\mathbf{r})$ denotes that the integration is carried out over the actual trajectory of motion of the classical object and the end point of $T(\mathbf{r})$ is $\mathbf{r}$ itself. The trajectory $T(\mathbf{r})$ is determined by the Hamilton-Jacobi equation in (11). The equation for ψ is[11]

$$\left[i\hbar \frac{\partial}{\partial t} - A_0 \right] \psi = \frac{1}{2m}(-i\hbar\nabla - \mathbf{A})^2 \psi \tag{13}$$

for any Q. According to the idea of quantum fraction in (5), we must have a new normalization condition for the wave function ψ,

$$\int |\psi(\mathbf{r})|^2 d^3r = Q \ . \tag{14}$$

Thus the probabilistic nature of a particular will gradually fade away as its mass increases.[12]

We have the basic equation (11) which describes the motion of microscopic, mesoscopic, and macroscopic objects. Evidently, (11) reduces to the Schrödinger equation in the limit $mcR/\hbar \to 0$. Nevertheless, we stress that the Schrödinger equation (13) holds for all values of the quantum fraction Q and that the basic equation (11) involves S which satisfies the Hamilton-Jacobi equation $-\partial S/\partial t = (\nabla S - \mathbf{A})^2/2m + A_0$ for all values of the classical fraction C.[10] These indicate that the two equations are, in principle, equally fundamental. In other words, the probability property $[\psi(\mathbf{r},t)]$ of micro-objects and the deterministic property $[\mathbf{r}(t)]$ of macro-objects are equally fundamental in the physical world within the present formalism. Their manifestation in different objects depends on its mass (or the quantum fraction Q), as shown in (14).

It is interesting to note that the generalized equation of motion (11) is invariant under the transformation

$$S \to S' = S - C^{-1}\hbar Q Z(\mathbf{r},t) \ ,$$

$$\Phi \to \Phi' = \Phi \exp[iZ(\mathbf{r},t)] \ ,$$

where $Z(\mathbf{r},t)$ is an arbitrary function.

IV. SIMPLE MOTIONS OF MESOSCOPIC OBJECTS IN ONE DIMENSION

Let us first consider a free mesoscopic particle with a mass m moving on the x axis. According to Eq. (11) with $A_\mu = 0$, we have

$$\left[iQ\hbar\frac{\partial}{\partial t} + CE_c \right]\Phi = \frac{1}{2mQ}(-iQ\hbar\nabla_x + CP_c)^2\Phi$$

where $E_c = P_c^2/2m$. The solution Φ can be written in the form

$$\Phi = \psi\exp(iCE_c t/Qh - iCS_0/Q\hbar), \quad S_0 = \int_{T(x)}^{x} P_c dx' \ , \tag{15}$$

where S_0 is Hamilton's characteristic function which is explicitly independent of time. The wave function ψ satisfies

$$i\hbar\frac{\partial}{\partial t}\psi = \frac{1}{2m}\left[-i\hbar\frac{\partial}{\partial x}\right]^2 \psi \ . \tag{16}$$

Its solution is given by

$$\psi = A\exp(-iEt/\hbar + iPx/\hbar), \quad E = P^2/2m \ , \tag{17}$$

where $P = P_c$ for the plane waves. In analogy to (5), the true energy E_T is given by

$$E_T = QE + CE_c = P^2/2m = E_c \ . \tag{18}$$

Next, for a simple harmonic oscillator with the potential $V(x) = Kx^2/2$, we have the generalized equation

$$\left[iQ\hbar\frac{\partial}{\partial t} + CE_c \right]\Phi = \left[\frac{1}{2m}\left[-iQ\hbar\frac{\partial}{\partial x} + CP_c \right]^2 + \tfrac{1}{2}QKx^2 \right]\Phi \ , \tag{19}$$

$$P_c = [2m(E_c - \tfrac{1}{2}Kx^2)]^{1/2} \ . \tag{20}$$

The solution for Φ is found to be

$$\Phi_n = \psi_n \exp(-iCS/Q\hbar), \quad S = -E_c t + \int_{T(x)}^{x} P_c(x')dx' \ , \tag{21}$$

$$\psi_n = \left[\frac{\alpha Q}{\sqrt{\pi}2^n n!} \right]^{1/2} H_n(\alpha x)\exp(-\alpha^2x^2/2 - iE_n t/\hbar),$$

$$\alpha^4 = mK/\hbar^2 \ , \tag{22}$$

where the energy eigenvalue E_n for ψ_n, $i\hbar Q\partial\psi_n/\partial t = QE_n\psi_n$, is

$$E_n = (n + \tfrac{1}{2})\hbar\omega, \quad \omega^2 = K/m \ . \tag{23}$$

The true energy of this mesoscopic harmonic oscillator is

$$E_{Tn} = QE_n + CE_c = Q(n + \tfrac{1}{2})\hbar\omega + CKx_{\max}^2/2 \ . \tag{24}$$

The probability density for a mesoscopic oscillator satisfies the "new normalization":

$$\int_{-\infty}^{\infty} dx\,|\Phi_n|^2 = \int_{-\infty}^{\infty} |\psi_n|^2 dx = Q \ . \tag{25}$$

In the classical limit ($m \to \infty$), the factor $\alpha\exp(-\alpha^2x^2)$ in $|\psi_n|^2$ becomes a δ function, which is not interesting physically. However, we have Hamilton's principal function

$$S = -E_c t + \int_{T(x)}^{x} P_c(x')dx', \quad E_c = H_c(P_c, x) \ , \tag{26}$$

which determines the motion of the macroscopic object. We may remark that the classical case corresponds to large phase in Φ_n.[10]

V. EXPERIMENTAL TESTS OF FUZZY TRANSITIONS

Let us consider the experimental implication of the new plane wave Φ of mesoscopic particles given by (15) and (17):

$$\Phi = A\exp\left[-i(\omega - CE_c/\hbar Q)t + i\left[kx - (CP_c/\hbar Q)\int_{T(x)}^{x} dx' \right] \right] \ . \tag{27}$$

Suppose one carries out the double-slit experimental with particles having the same momentum $P = \hbar k = \text{const}$. One can vary the masses of particles and keep their momenta fixed. In this case, the classical momentum P_c must be the same as the quantum momentum $\hbar k$, i.e., $P_c = \hbar k$. As the mass changes, the ratio $CP_c/\hbar Q = kC/Q = k[1 - \exp(-mcR/\hbar)]/\exp(-mcR/\hbar)$ also changes. On the screen at the position x, which can be reached by the particles following the classical trajectory, we have the phase

$$(k - CP_c/\hbar Q)x = xk(1 - C/Q) \equiv xk_{\text{eff}} . \tag{28}$$

As m changes, we have different quantum fraction Q for normalizing the wave function in (14) and have the effective wave number k_{eff}:

$$k_{\text{eff}} \approx k \quad \text{for } mcR/\hbar \ll 1 , \tag{29a}$$

$$k_{\text{eff}} \approx 0 \quad \text{for } mcR/\hbar \approx \ln 2 , \tag{29b}$$

$$k_{\text{eff}} \approx -Ck/Q = -V \quad \text{for } mcR/\hbar \gg 1 , \tag{29c}$$

where V represents a very large number. In (29a), one has the usual interference pattern on the screen. For (29c), the screen has only two bright spots at a and b, which correspond to the end points of classical trajectories passing through the two splits. However, in (29b) the wave function Φ leads to two dark spots at a and b due to the destructive interference there; nevertheless, at other positions it gives the usual interference pattern with half the intensity of (29a). On the other hand, half of the total particles (corresponding to the classical fraction $C \approx \frac{1}{2}$) follow the Hamilton-Jacobi equation and reach spots a and b, so that effectively there are also two bright spots at a and b with half the intensity of (29c).

This is an interesting prediction of the present theory for the double-slit experiment with mesoscopic particles with different masses and the same momenta. It is an interesting experimental test of this theory and, if confirmed, one can determine the radical length R without relying on high-energy experiments to find out the presence of a very small fundamental length.

Next, let us consider a charged particle with a mass m moving on the x-y plane with $\mathbf{v} = (v_x, v_y, 0)$ and in a constant magnetic field B in the z direction. We take the vector potentials $\mathbf{A}$ of the magnetic field as

$$\mathbf{A} = (0, Bx, 0), \quad (A_0 = 0) . \tag{30}$$

The electromagnetic gauge invariant equation of motion is [for the electromagnetic potential A^μ, we replace A^μ in (11) by eA^μ/c]

$$\left[i\hbar Q \frac{\partial}{\partial t} + CE_c \right] \Phi = \frac{1}{2mQ} \left[-i\hbar Q \nabla + C\mathbf{P}_c - \frac{e}{c} Q \mathbf{A} \right]^2 \Phi ,$$

$$E_c = \frac{1}{2m} (\mathbf{P}_c - \frac{e}{c} \mathbf{A})^2, \quad \mathbf{P}_c = m\mathbf{v} + \frac{e}{c} \mathbf{A} . \tag{31}$$

The solution Φ to (31) can be written in the form

$$\Phi = \psi \exp(-iCS/\hbar Q), \quad S = -E_c t + \int_{T(\mathbf{r})}^{\mathbf{r}} \mathbf{P}_c(\mathbf{r}') \cdot d\mathbf{r}' ,$$

where ψ satisfies

$$i\hbar Q \frac{\partial}{\partial t} \psi = QH\psi, \quad H = \frac{1}{2m} \left[\mathbf{P} - \frac{e}{c} \mathbf{A} \right]^2 . \tag{32}$$

We can write H in the form

$$H = \frac{1}{2m} (\bar{P}^2 + \bar{Q}^2) \tag{33}$$

where

$$\bar{P} = P_x, \quad \bar{Q} = (eB/c)x - P_y .$$

Because $\bar{P}\bar{Q} - \bar{Q}\bar{P} = -i\hbar eB/c$, we find that the operator QH has the following energy spectrum:

$$QE_n = (n + \tfrac{1}{2}) Q\hbar eB/mc .$$

The true energy E_T of the charged particle is given by

$$E_T = QE_n + CE_c = Q(n + \tfrac{1}{2})\hbar eB/mc + Cm\mathbf{v}^2/2 , \tag{34}$$

where $\mathbf{v}$ is its classical velocity. In the future, we hope that the new result (34) can be tested experimentally by varying the mass m of the charged particle moving in a strong magnetic field.

VI. DISCUSSIONS AND REMARKS

For a free relativistic particle, the generalized equation of motion takes the four-dimensional form[13]

$$\left[\left[i\hbar Q \frac{\partial}{\partial t} + CE_c \right]^2 - c^2(-i\hbar Q \nabla + C\mathbf{P}_c)^2 - Q^2 m^2 c^4 \right] \Phi = 0, \tag{35}$$

where $E_c^2/c^2 = p_c^2 + m^2c^2$ and c is the speed of light. Similarly, the relativistic equation for a Dirac particle can be written down formally:[13]

$$\left[i\hbar Q \frac{\partial}{\partial t} + CE_c - c\boldsymbol{\alpha} \cdot (-i\hbar Q \nabla + C\mathbf{P}_c) - Q\beta mc^2 \right] \Psi = 0 . \tag{36}$$

Equations (35) and (36) are invariant under the new transformation $S \to S - C^{-1} Q\hbar Z(\mathbf{r}, t)$, $\Phi \to \Phi \exp[iZ(\mathbf{r}, t)]$, as discussed in Sec. III. This transformation can be written in the form of a local gauge transformation:

$$\begin{aligned} &E_c \to E_c + C^{-1} Q\hbar \frac{\partial Z}{\partial t} , \\ &\mathbf{P}_c \to \mathbf{P}_c - C^{-1} Q\hbar \nabla Z , \\ &\Phi \to \Phi \exp(iZ) . \end{aligned} \tag{37}$$

The physical implications of this new gauge symmetry deserve to be further studied.

The generalized equation of motion appears to be relevant to problems of measurements and more precise formulation of quantum mechanics.[14] In ordinary quantum mechanics, there is only the wave function and no other variables $\mathbf{r}(t)$ to express macroscopic definiteness. This gives rise to problems concerning the role of "measurements" and so on:[15,14] To do experiments, the wave function must be narrow as far as macroscopic variables are concerned. But the Schrödinger equation does not preserve such narrowness, so that there must be some kind of "collapse" to enforce it. However, "what are macroscopic objects" and "how the collapse occurs" are ambiguous in principle in the conventional framework. In order to have a precise quantum mechanics, it was suggested by Bell that both ψ and $\mathbf{r}(t)$ should refer to the world as a whole.[14,16]

We note that the usual Heisenberg commutation relation is valid only from $m=0$. For the electron with the mass $m_e=0.5$ MeV/c^2, there will be a small deviation from Heisenberg's relation. Nevertheless, the effective deviation $\Delta\hbar/\hbar$ is too small to be detected because $R<10^{-17}$ cm:

$$\Delta\hbar/\hbar=1-Q\approx 2.5\times 10^{-7} \quad \text{for } R\approx 10^{-17} \text{ cm} .$$

Within the present generalized framework of quantum mechanics, there exists a very small radical length R, whose length has yet to be determined. This radical length enables us to define microscopic objects ($mcR/\hbar \ll 1$), mesoscopic objects ($mcR/\hbar \sim 1$), and macroscopic objects ($mcR/\hbar \gg 1$), and to describe their motions by the same basic equation. Furthermore, it brings about an inherent fuzziness (i.e., $\Delta x_{\min}=R$) at short distances on the microscopic level and a sharply defined position $\mathbf{r}(t)$ on the macroscopic level. As we have seen from the generalized equation of motion (11) for all objects, as the mass becomes very large, the momentum operators and the wave function become unimportant and physically uninteresting. The classical variables $\mathbf{P}_c$, $\mathbf{r}(t)$ and the Hamilton-Jacobi equation emerge to play main roles for describing the motion of macroscopic objects. Thus the theory has both the probabilistic feature for the microscopic world and the deterministic feature for the macroscopic classical world. In this way, we can avoid the puzzling idea that a macroscopic pointer in an apparatus can point simultaneously in different directions.[14] A very interesting consequence is that the theory predicts new and unexpected phenomena related to a fuzzy transition between quantum and classical mechanics. We use double-slit experiment and simple harmonic oscillators (a charged particle moving in a constant magnetic field) to illustrate such fuzzy transitions. They can be and should be tested experimentally.

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[3]J. R. Klauder, in *Path Integrals and Their Applications in Quantum, Statistical and Solid State Physics,* edited by G. J. Papadopoulos and J. T. Devreese (Plenum, New York, 1978), pp. 5–38.

[4]J. R. Klauder, J. Math. Phys. **4**, 1055 (1963); **4**, 1058 (1963); **5**, 177 (1964).

[5]We may remark that the value $R\approx 10^{-20}$ cm is not an estimate and, therefore, it should not be taken seriously. It is arbitrarily chosen for convenience and definiteness of discussions. We estimate that $R<10^{-17}$ cm, otherwise there will be a contradiction to high-energy tests of quantum electrodynamics (cf. Ref. 1). Moreover, a pollen with the size of a few micrometers (in Brownian motion) appears to be a classical particle. This leads to a lower limit $R>10^{-33}$ cm.

[6]It may appear strange that the speed of light c enters the relations ($mcR \gtrless \hbar$) that allows us to distinguish between microscopic and macroscopic objects in nonrelativistic quantum mechanics. But there is no conclusive reason for it within the usual conceptual framework. We may remark that sometimes c appears in nonrelativistic quantum mechanics, e.g., Rydberg energy $=hcR_\infty=m_e^2c^2\alpha^2/2$. One can redefine a quantity such that c disappears from a relation. For example, $mcR>\hbar$ can be written as $m/m_s>1$, where $m_s=\hbar/cR$.

[7]We are unable to make a more precise determination of the value of R other than those given in Ref. 5.

[8]A function $A/[A+(mcR/\hbar)^n]$ for any A and any positive n also fulfills the limiting properties (4). There is no really good reason that (7) should be preferred. It is essentially just a simple example for convenience of discussion.

[9]See, for example, T. Y. Wu, *Quantum Mechanics* (World Scientific, Singapore, 1986), pp. 136–140.

[10]One must be careful in taking the limit $Q\to 0$ because of the infinite phase in Φ, as shown in (12). Suppose one calculates the amplitude $K(a,b)$ of a particle to go from $(\mathbf{r}_a,t_a)$ to $(\mathbf{r}_b,t_b)$. In the limit $Q\to 0$, only when a path and a nearby path all give the same phase (i.e., S does not vary) in (12) in the first approximation, does one have a nonvanishing amplitude. This path is the one given by the Hamilton-Jacobi equation. M. C. Gutzwiller, in Ref. 3, pp. 179–181.

[11]Note that for a stationary state we have $i\hbar\partial\psi/\partial t=E\psi$, so that the time-dependent part of the "total wave function" Φ in (11) is $\exp(-iEt/\hbar+iCE_ct/Q\hbar)$ rather than $\exp(-iEt/\hbar)$.

[12]Within the present formalism, the normalization of the wave function is modified, so that the definition of the expectation value should be modified as $\langle A\rangle=Q^{-1}\int\Phi^* A\Phi d^3r$.

[13]It is understood that $E_c=-\partial S/\partial t$ and $\mathbf{P}_c=\nabla S$.

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[16]For a recent discussion of the relationship of classical and quantum mechanics in Okubofest, see E. C. G. Sudarshan, *The Quantum Envelope of a Classical System, Rochester, 1990,* From Symmetries to Strings: Forty Years of Rochester Conferences (World Scientific, Singapore, 1990).