Classical dynamics for a class of SU(1,1) Hamiltonians

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(Received 5 September 1990; revised manuscript received 30 October 1990)

Following a recent revival of interest in the properties of SU(1,1) Hamiltonians, we discuss the properties of classical-limit dynamics for a class of such models. Using a coherent-states technique, we analyze the classical energy function determining the motion and discuss the existence of stable and unstable stationary solutions. Appropriate phase diagrams are also presented. Comparison with previous works on similar models is made.

I. INTRODUCTION

The identification of the dynamical group for a given physical system often leads to deeper understanding of that system's quantum and classical properties. The analysis of the spin-system spectra is greatly facilitated by the fact that the appropriate group for these systems is the compact SU(2) group. The fact that the noncompact SU(1,1) group might also be physically relevant was noticed already in the late 1950s.[1] The first application, we believe, of the SU(1,1) group to the analysis of a many-body system is due to Solomon.[2] Right after that it was noticed that the same group plays a role in a construction of a simple model for the Josephson effect.[3] It was shown in Refs. 2 and 3 that the SU(1,1) Hamiltonian emerges in a natural way in the Bogoliubov-like analysis of a Bose-Einstein condensate system.[4,5] The toy model Hamiltonian for such a system, the Foldy-like Hamiltonian,[3,4] can indeed be written down entirely in terms of the SU(1,1) group algebra generators. SU(1,1) models attract considerable attention in connection with the theory of squeezed states in quantum optics.[6] Recently, Gerry and Kiefer[7] noticed that the SU(1,1) model analogous to the SU(2) Lipkin-Meshkov-Glick model,[8] exhibits interesting ground-state properties, sometimes referred to as the ground-state phase transformations.

The Lipkin-Meshkov-Glick model borrowed from the nuclear physics is SU(2) invariant. We shall see that the nonlinear SU(1,1) model discussed in Ref. 7 is closely related to the nonlinear generalization of the Foldy-like model from Ref. 3. Indeed, the leading terms in the interaction of *bogolons* can be imitated by adding terms quadratic in the SU(1,1) generators to the Foldy Hamiltonian. The important difference between the model discussed in Ref. 7 and those in Refs. 2 and 3 is that in the latter the SU(1,1) generators were *constructed* from bosonic creation and annihilation operators representing original degrees of freedom. From the mathematical point of view this procedure is similar to using the Schwinger boson representation for SU(2) spins.[9,10] The latter representation was used in the analysis of bound magnons in the Heisenberg ferromagnet[11] and was shown to be particularly useful in the analysis of nonlinear excitations in one-dimensional magnetic models.[12,13]

In this paper we would like to discuss a bosonic SU(1,1) model which generalizes those from Refs. 3 and 7. The plan of the paper is as follows. In Sec. II we shall recall the model from Ref. 3 and perform its classical-limit analysis using the coherent-states representation. In Sec. III we will show how the bosonic representation permits us to discuss the classical limit of the nonlinear model from Ref. 7. In Sec. IV we shall combine both models and discuss the resulting properties. Section V is devoted to final comments and conclusions.

II. THE FOLDY-LIKE MODEL (REF. 3)

In the conventional Bogoliubov analysis of the Bose-Einstein condensed system one assumes that the condensate occupation number is large. Thus, it is permissible to replace the creation and annihilation operators corresponding to the condensate state (assumed to be of zero momentum) by a c number proportional to $\sqrt{n_0}$. Following standard procedure,[4,5] we write

$$\hat{H}_B=\sum_{\mathbf{q}\neq 0}[\varepsilon_{\mathbf{q}}+(V_0+V_{\mathbf{q}})\hat{a}_0^{\dagger}\hat{a}_0]\hat{a}_{\mathbf{q}}^{\dagger}\hat{a}_{\mathbf{q}} +\tfrac{1}{2}\sum_{\mathbf{q}\neq 0}V_{\mathbf{q}}(\hat{a}_{\mathbf{q}}^{\dagger}\hat{a}_{-\mathbf{q}}^{\dagger}\hat{a}_0^{2}+\text{H.c.})+\tfrac{1}{2}V_0\hat{a}_0^{\dagger 2}\hat{a}_0^{2}+\Gamma\ , \tag{2.1}$$

where, as usual, $\varepsilon_{\mathbf{q}}$ is the free boson kinetic energy, $V_{\mathbf{q}}$ is the matrix element of the interbosonic interaction, and $\mathbf{q}$ is the momentum vector. The Γ term contains all the parts of the Hamiltonian usually neglected in the Bogoliubov analysis. Replacing zero-momentum condensate creation and annihilation operators by $\sqrt{n_0}$ and restricting the set of momentum vectors to just two values, labeled + and −, and neglecting all higher-order and constant terms, we obtain from Eq. (2.1) the Foldy-like Hamiltonian

$$\hat{H}=\omega(\hat{a}^{\dagger}_{+}\hat{a}_{+}+\hat{a}^{\dagger}_{-}\hat{a}_{-}+1)+\frac{\gamma}{2}(\hat{a}^{\dagger}_{+}\hat{a}^{\dagger}_{-}+\text{H.c.}) . \quad (2.2)$$

The physical meaning of all terms in the Hamiltonian (2.2) is transparent. The first term describes the kinetic energy and the second one represents simultaneous creation and/or annihilation of (zero total momentum) pairs of excitations. The Hamiltonian (2.2) was used in Ref. 3. A similar model was discussed in Ref. 6 to describe *two mode up-convertors.*

Following the original observation of Solomon[2] we define the SU(1,1) generators in terms of the boson operators $\hat{a}_{\pm}$:

$$\hat{J}_1=-\tfrac{1}{2}(\hat{a}^{\dagger}_{+}\hat{a}^{\dagger}_{-}+\text{H.c.}) ,$$
$$\hat{J}_2=\frac{i}{2}(\hat{a}^{\dagger}_{+}\hat{a}^{\dagger}_{-}-\text{H.c.}) , \quad (2.3)$$
$$\hat{J}_3=\tfrac{1}{2}(\hat{a}^{\dagger}_{+}\hat{a}_{+}+\hat{a}^{\dagger}_{-}\hat{a}_{-}+1) .$$

One can easily check that $\hat{J}_i$ satisfies the SU(1,1) commutation relations,

$$[\hat{J}_1,\hat{J}_2]=-i\hat{J}_3,\quad [\hat{J}_2,\hat{J}_3]=i\hat{J}_1,\quad [\hat{J}_3,\hat{J}_1]=i\hat{J}_2 . \quad (2.4)$$

The Foldy Hamiltonian, Eq. (2.2), can now be written as

$$\hat{H}=2\omega\hat{J}_3-\gamma\hat{J}_1 . \quad (2.5)$$

Note that Eq. (2.3) is noting else but the Schwinger boson representation for SU(1,1) "spins"[9] and that the Hamiltonian (2.5) is just a Zeeman term in SU(1,1) geometry with the "magnetic field" not parallel to the quantization axis. The standard Bogoliubov transformation diagonalizing the Hamiltonian (2.2) is then just a rotation aligning the magnetic field with the quantization axis. In our further analysis we shall assume that the coupling constant γ is positive corresponding to the repulsive character of interbosonic interactions.

It follows from the above that the SU(1,1) degrees of freedom are constructed here from the original bosonic degrees of freedom, in contrast to the procedure used in Refs. 9, 12, and 13, where the original SU(2) spin degrees of freedom were mapped on the bosonic ones via the Schwinger boson representation. It seems, therefore, appropriate that the classical-limit dynamics be analyzed using the bosonic coherent states and not the generalized coherent states for the SU(1,1) group as done in Ref. 7. Indeed, the relation between the generalized coherent states for SU(1,1) and various bosonic representations for SU(1,1) generators was thoroughly studied.[14] It was shown in Ref. 14 that the generalized coherent states of SU(1,1) corresponding to the discrete representations of this group are the eigenstates of the Foldy Hamiltonian (cf. also Ref. 3). This property precludes their use for the analysis of the classical limit for those systems for which the bosonic degrees of freedom are the building blocks of the model.

Having formulated our model we can now analyze its classical limit following conventional procedure[15] and derive equations of motion for complex amplitudes $\alpha_{\pm}$ defined as eigenvalues of the annihilation operators $\hat{a}_{\pm}$,

$$\hat{a}_{\pm}|\alpha_{\pm}\rangle=\alpha_{\pm}|\alpha_{\pm}\rangle , \quad (2.6)$$

where $|\alpha_{\pm}\rangle$ are usual coherent states for the ± oscillators. We also denote $|\tilde{\alpha}\rangle=|\alpha_{+}\rangle\otimes|\alpha_{-}\rangle$. The classical equations of motion for this model are obtained from the "classical" Hamiltonian

$$\mathcal{H}(\alpha_{+},\alpha_{-})=\langle\tilde{\alpha}|\hat{H}|\tilde{\alpha}\rangle \quad (2.7)$$

using the Poisson brackets between α variables

$$\{\alpha_{\sigma},\alpha^{*}_{\sigma'}\}=i\delta_{\sigma\sigma'} , \quad (2.8)$$

where $\sigma=+$ or $-$. We have then $\dot{\alpha}_{\pm}=\{\alpha_{\pm},\mathcal{H}(\tilde{\alpha})\}$ and the classical Hamiltonian corresponding to the Foldy model, Eq. (2.2), reads

$$\mathcal{H}(\tilde{\alpha})=\omega(|\alpha_{+}|^2+|\alpha_{-}|^2+1)+\frac{\gamma}{2}(\alpha^{*}_{+}\alpha^{*}_{-}+\alpha_{+}\alpha_{-}) . \quad (2.9)$$

It is convenient to write the resulting equations of motion for α's using polar decomposition $\alpha_{\pm}=\sqrt{2\rho_{\pm}}\exp(i\phi_{\pm})$. We obtain the set of four equations for real variables ρ and $\phi_{\pm}$,

$$\dot{\rho}_{\pm}=\gamma\sqrt{\rho_{+}\rho_{-}}\sin(\phi_{+}+\phi_{-}) ,$$
$$\dot{\phi}_{\pm}=\omega+\frac{\gamma}{2}\left[\frac{\rho_{\mp}}{\rho_{\pm}}\right]^{1/2}\cos(\phi_{+}+\phi_{-}) . \quad (2.10)$$

It follows from Eqs. (2.10) that there exists an additional constant of motion responsible for maintaining the constant *difference* in the ± oscillator occupation numbers (densities),

$$\frac{d}{dt}(\rho_{+}-\rho_{-})=0 . \quad (2.11)$$

Denoting $\rho_0=\rho_{+}-\rho_{-}\geq 0$ we obtain from Eq. (2.10) two equations of motion for the density $\rho\equiv\rho_{-}$ and phase $\psi=\phi_{+}+\phi_{-}$,

$$\dot{\rho}=\gamma\sqrt{\rho(\rho+\rho_0)}\sin\psi ,$$
$$\dot{\psi}=2\omega+\frac{\gamma}{2}\frac{2\rho+\rho_0}{\sqrt{\rho(\rho+\rho_0)}}\cos\psi . \quad (2.12)$$

Equations (2.12) are the canonical equations for the Hamiltonian $\mathcal{F}_F(\rho,\psi)$,

$$\mathcal{F}_F=2\omega\rho+\gamma\sqrt{\rho(\rho+\rho_0)}\cos\psi , \quad (2.13)$$

and the Poisson bracket $\{\rho,\psi\}=-1$.

The stationary points $(\bar{\rho},\bar{\psi})$ of Eqs. (2.12) are important for the analysis of the ground-state properties of the system.[6] We obtain

$$[\bar{\rho}(\bar{\rho}+\bar{\rho}_0)]^{1/2}\sin\bar{\psi}=0 , \quad (2.14a)$$

$$2\omega+\cos\bar{\psi}\frac{d}{d\bar{\rho}}[\bar{\rho}(\bar{\rho}+\bar{\rho}_0)]^{1/2}=0 . \quad (2.14b)$$

Equation (2.14a) gives $\bar{\psi}=n\pi$, and from (2.14b) it follows that only odd values of n are relevant. We obtain then

$$\bar{\rho}=\rho_0\left\{\left[1-\left[\frac{\gamma}{\gamma_c}\right]^2\right]^{-1/2}-1\right\}, \quad (2.15)$$

where $\gamma_c=2\omega$. Note that the only physically relevant solutions are these with $\rho\neq 0$. Equation (2.15) implies that the coupling constant γ has to be smaller than γ_c. The critical value of γ has a clear quantum-mechanical meaning:[3] for $\gamma<\gamma_c$ the quantum Foldy-like Hamiltonian possesses a discrete spectrum and for $\gamma>\gamma_c$ there is a continuous spectrum only. Similar behavior is observed in the present classical case: standard analysis reveals that the stationary points of the Hamiltonian $\mathcal{F}_F$ at $\rho=\bar{\rho}$ and $\bar{\psi}=(2l+1)\pi$, present only when $\gamma<\gamma_c$, correspond to the *minima* of the total energy, so these stationary points are *stable*. For larger values of the coupling constant γ there are no stationary solutions of Eqs. (2.12). In Fig. 1 we present the behavior of $\mathcal{F}_F(\rho,\bar{\psi}=(2l+1)\pi)$, $l=0,1\ldots$ for values of γ at, above, and below the critical value. For γ less than the critical value the potential develops a minimum. For $\gamma>\gamma_c$ there are no bounded trajectories.

The physical meaning of the critical value of the coupling constant γ_c can be understood within the Bogoliubov theory of the Bose condensed many-boson system. In the grand canonical formulation of this theory[16] it is appropriate to replace the free boson kinetic energy ε_q by $\varepsilon_q-\mu$, were μ is the chemical potential. By using the form of the Hamiltonian (2.1) modified in this way, one obtains a relation between γ_c and the basic ingredients of the model: the interaction potential, condensate density, and the typical momentum of the excitation. Following the analysis of Ref. 16 we find that the condition $\gamma<\gamma_c$ is equivalent to the condition $n_0V<\hbar^2/2mr_0^2$ for the condensate density, where r_0 is the range of the interaction potential.

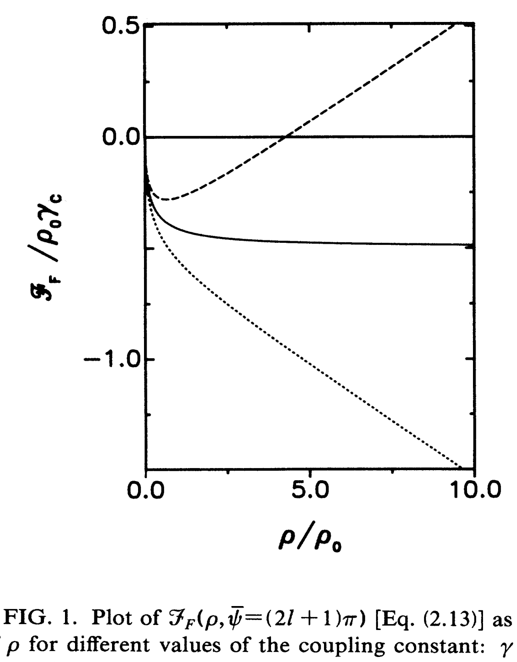

FIG. 1. Plot of $\mathcal{F}_F(\rho,\bar{\psi}=(2l+1)\pi)$ [Eq. (2.13)] as a function of ρ for different values of the coupling constant: $\gamma=\gamma_c$ (continuous), $\gamma=1.1\gamma_c$ (dotted), and $\gamma=0.9\gamma_c$ (dashed).

III. THE NONLINEAR MODEL (REF. 7)

We will now recapitulate the model of Ref. 7. The SU(1,1) invariant nonlinear Hamiltonian is

$$\hat{H}=2\omega\hat{J}_3+\lambda(\hat{J}_1^2-\hat{J}_2^2), \quad (3.1)$$

with the positive coupling constant λ. In the boson language of Sec. II it assumes the typical form of a four-wave mixing interaction,

$$\hat{H}=\omega(\hat{a}_+^\dagger\hat{a}_++\hat{a}_-^\dagger\hat{a}_-+1)+\frac{\lambda}{2}[(\hat{a}_+^\dagger\hat{a}_-^\dagger)^2+(\hat{a}_+\hat{a}_-)^2]. \quad (3.2)$$

The classical Hamiltonian for complex fields α defined as before is

$$\mathcal{H}(\alpha_+,\alpha_-)=\omega(|\alpha_+|^2+|\alpha_-|^2+1)+\frac{\lambda}{2}[(\alpha_+^*\alpha_-^*)^2+(\alpha_+\alpha_-)^2]. \quad (3.3)$$

Following precisely the same procedure as in Sec II and again using the polar decomposition of α's, we obtain the following set of equations for four real variables $\rho_\pm$ and $\phi_\pm$:

$$\dot{\rho}_\pm=4\lambda\rho_+\rho_-\sin[2(\phi_++\phi_-)],$$
$$\dot{\phi}_\pm=\omega+2\lambda\rho_\mp\cos[2(\phi_++\phi_-)]. \quad (3.4)$$

As before, the difference between densities ρ_+ and ρ_- is a constant equal to ρ_0. Recalling that the Casimir $\hat{C}$ for the SU(1,1) group is $\hat{C}=\hat{J}_3^2-(\hat{J}_1^2+\hat{J}_2^2)$, we observe that $\rho_0=\frac{1}{2}\sqrt{1+4\langle\hat{C}\rangle}$, where $\langle\hat{C}\rangle=\langle\bar{\alpha}|\hat{C}|\bar{\alpha}\rangle$. Equations of motion analogous to Eqs. (2.12) are now

$$\dot{\rho}=4\lambda[\rho(\rho+\rho_0)]\sin(2\psi),$$
$$\dot{\psi}=2\omega+2\lambda(2\rho+\rho_0)\cos(2\psi). \quad (3.5)$$

These equations are again Hamiltonian, the Poisson bracket is as before,and the corresponding Hamiltonian $\mathcal{F}_{NL}(\rho,\psi)$ is now

$$\mathcal{F}_{NL}(\rho,\psi)=2\omega\rho+2\lambda\rho(\rho+\rho_0)\cos(2\psi). \quad (3.6)$$

Stationary points of Eqs. (3.5) are now different. We have $\bar{\psi}=n\pi/2$, and only odd values of n are relevant. We obtain from $\partial\mathcal{F}_{NL}/\partial\bar{\rho}=0$ that $\bar{\rho}=(\omega-\lambda\rho_0)/2\lambda$ and thus the coupling constant λ has to be smaller than $\lambda_c\equiv\omega/\rho_0$. In Fig. 2 we have shown the behavior of the $\mathcal{F}_{NL}(\rho,\bar{\psi}=(2l+1)\pi/2)$ for values of λ below, at , and above the critical value λ_c. For $\lambda<\lambda_c$ these stationary solutions are unstable because $\mathcal{F}_{NL}(\rho,\psi)$ has *saddle points* at $\rho=\bar{\rho}$, $\psi=\bar{\psi}$. This is in contrast to the case discussed in Sec. II, where the stationary solutions were stable. For λ larger than the critical value those saddle points disappear. The critical value λ_c depends on the value of the constant of motion ρ_0. For large ρ_0 the value of λ_c de-

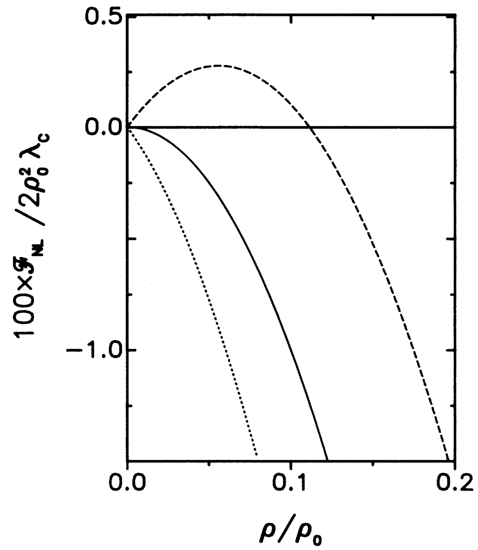

FIG. 2. Plot of $\mathcal{F}_{\rm NL}(\rho,\bar{\psi}=(2l+1)\pi/2)$ [Eq. (3.6)] as a function of ρ for different values of the coupling constant $\lambda=\lambda_c$ (continuous), $\gamma=1.1\lambda_c$ (dotted), and $\gamma=0.9\lambda_c$ (dashed).

creases to zero. In the following section we shall analyze properties of the model which results from supplementing the Foldy Hamiltonian from Sec. II with nonlinear terms discussed above.

IV. THE NONLINEAR FOLDY MODEL

The Hamiltonian for this particular generalization of the Foldy model reads

$$\hat{H}=2\omega\hat{J}_3-\gamma\hat{J}_1+\lambda(\hat{J}_1^2-\hat{J}_2^2)\ . \tag{4.1}$$

Following the same procedure as in two preceding sections we first express operators $\hat{J}_i$ in terms of the bosonic operators $\hat{a}$ and then derive equations of motion for $\rho_\pm$ and $\phi_\pm$. As before the difference $\rho_0=\rho_+-\rho_-$ is a constant of motion. The equations for ρ and $\psi=\phi_++\phi_-$ are

$$\begin{aligned}\dot{\rho}&=4\lambda\rho(\rho+\rho_0)\sin(2\psi)+\gamma\sqrt{\rho(\rho+\rho_0)}\sin\psi\ ,\\ \dot{\psi}&=2\omega+2\lambda(2\rho+\rho_0)\cos(2\psi)+\frac{\gamma}{2}\frac{2\rho+\rho_0}{\sqrt{\rho(\rho+\rho_0)}}\cos\psi\ ,\end{aligned} \tag{4.2}$$

being canonical equations of motion for the Hamiltonian

$$\mathcal{F}(\rho,\psi)=2\omega\rho+2\lambda\rho(\rho+\rho_0)\cos(2\psi)+\gamma\sqrt{\rho(\rho+\rho_0)}\cos\psi\ . \tag{4.3}$$

The stationary point conditions are now more complicated than before. From $\dot{\rho}=0$ we obtain

$$\sin\bar{\psi}[8\lambda\bar{\rho}(\bar{\rho}+\bar{\rho}_0)\cos\psi+\gamma\sqrt{\bar{\rho}(\bar{\rho}+\bar{\rho}_0)}]=0\ . \tag{4.4}$$

We have now two types of stationary points. The first, generically related to the minima in the Foldy model (Sec. II), and the other corresponding to the saddle points in the Gerry and Kiefer model (Sec. III). They will be referred to as F and GK stationary points, respectively. We begin our discussion with the GK points.

The GK points correspond to vanishing of the expression inside the square brackets in Eq. (4.4), resulting in

$$\begin{aligned}\cos(\bar{\psi}_{\rm GK})&=-\frac{\gamma}{2\gamma_c}\left[1-\left[\frac{\lambda}{\lambda_c}\right]^2\right]^{-1/2},\\ \bar{\rho}_{\rm GK}&=\frac{\rho_0}{2}\left[\frac{\lambda_c}{\lambda}-1\right]\ .\end{aligned} \tag{4.5}$$

provided that $\lambda<\lambda_c$ and

$$\frac{\gamma}{\lambda}<2\frac{\gamma_c}{\lambda_c}\left[\left[\frac{\lambda_c}{\lambda}\right]^2-1\right]^{1/2}\ . \tag{4.6}$$

Standard analysis shows that $\mathcal{F}_{\rm NL}$ has saddle points at the GK points which for $\gamma\to0$ go over to the saddle points discussed in Sec. III. In Fig. 3 we display the phase diagram for the GK stationary points in the $(\gamma/\lambda,\lambda)$ plane. There are two regions in this diagram. The "forbidden" one, for $\lambda>\lambda_c$ for which no GK stationary points exists, and the "allowed" one, in which the GK saddle points do occur. In the limit of $\gamma\to0$ we recover the results of Sec. III in which case the allowed region collapses to the [0,1] interval of the λ/λ_c axis. We emphasize that there is no region in the parameter space in which the GK stationary solution becomes stable (minimum).

The F stationary solutions are more interesting. They correspond to the vanishing of the sine factor in Eq. (4.4) and lead to the nonlinear equation for $\bar{\rho}$ which has a real root if $\bar{\psi}=(2l+1)\pi$, as in Sec. II. This particular family of stationary points is generically related to the energy minima in the model of Sec. II. To see whether the present solutions also correspond to the $\mathcal{F}$-energy minima, one looks at the determinant of the second derivatives of $\mathcal{F}$. Across the line in the parameter space

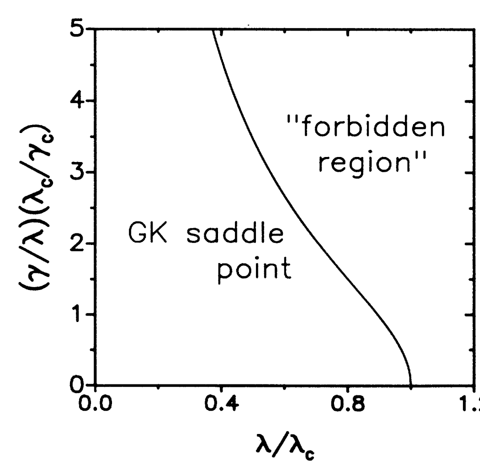

FIG. 3. Phase diagram for the GK stationary points in the nonlinear Foldy-like model.

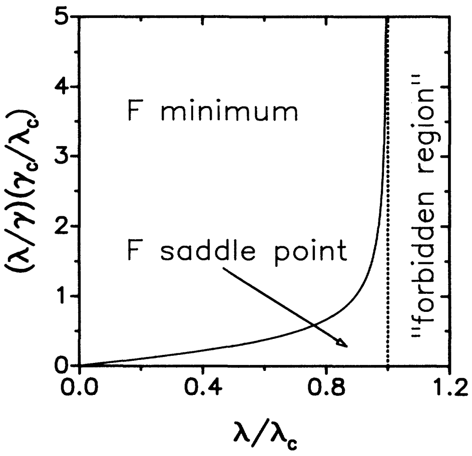

FIG. 4. Phase diagram for the F stationary points in the nonlinear Foldy-like model.

$$\frac{\lambda}{\gamma}=\frac{1}{2}\frac{\lambda_c}{\gamma_c}\left\{\left[\left[\frac{\lambda_c}{\lambda}\right]^2-1\right]^{1/2}\right\}^{-1}, \tag{4.7}$$

this determinant changes sign from positive (F minimum) above that line to negative (F saddle point) below it. Thus this line represents the genuine ground-state phase-transition line in our model across which the stable stationary solution becomes unstable. In Fig. 4 we have plotted the corresponding phase diagram exhibiting three regions: of stable F points (minima), unstable (saddle points), and the forbidden region in which no stationary solutions do exist. When the nonlinear coupling constant λ vanishes, the phase-diagram line collapses to the point at the origin of the graph at which only the stable stationary point is present. For this case we regain the results of Sec. II and, therefore, the condition $\gamma<\gamma_c$ is necessary for the the existence of this stationary point. Note that the phase boundary line in Fig. 4 [Eq. (4.7)] is formally given by the same equation as the line separating regions of existence and nonexistence of the GK saddle points in Fig. 3 [Eq. (4.6)]. We conclude that the full nonlinear Foldy-like model behaves differently then any of its constituent submodels.

V. COMMENTS AND CONCLUSIONS

In the previous sections we have discussed the ground-state properties of a class of SU(1,1) models using the coherent-states representation. We have shown that the nonlinear Foldy-like model exhibits ground-state phase transformation quite differently than these in the SU(1,1) generalized Lipkin-Meshkov-Glick model discussed by Gerry and Kiefer.[7] Some of its properties, namely the existence of stable minima of the F type, are reminiscent of those in the Foldy model. Discussion of the physical consequence of the instability of the F points for the condensed many-boson system, which was the starting point for construction of our toy Hamiltonian in Sec. II, is quite complex. For example, the analysis in Sec. II shows that the interpretation of the critical value of the coupling constant γ_c depends on the choice of the momentum vector used in reduction of the Hamiltonian (2.1) to its toylike form Eq. (2.2). If we choose that momentum from the range which corresponds for ^{4}He to the region between the phonon maximum and the roton minimum then we can relate the classical instability to the decay of the excitations from that range. The nonlinear interactions modify this instability in a way discussed in Sec. IV. In a recent work Jezek and Hernandez[17] analyze several nonlinear SU(1,1) models, which are referred to as superfluid models with quasiparticle interactions. Their model differs from ours in several respects (sign of coefficients in the linear model, symmetry of nonlinear terms, etc.) but the main difference is that they use the generalized SU(1,1) coherent states instead of the boson coherent states. We emphasize again that for the many-boson system the latter approach is the proper one. One can see the conceptual difference very clearly by analyzing the SU(2) symmetric model—the one-dimensional Heisenberg ferromagnetic chain—using either the SU(2) coherent states[18] or analysis based on Schwinger-boson mapping.[12,13]

The analysis of the ground-state phase transition given in this work is analogous to the mean-field treatment in the usual phase-transformation language. Using the ordinary coherent-states approach to the analysis of the classical limit of the quantum system dynamics we have tacitly assumed that the number of excitations (or quanta) involved is large and that quantum fluctuations can be neglected. The analysis of the quantum fluctuations in our model requires a different approach than presented in this work, for example one should write down a proper Fokker-Planck equation and follow the appraoch widely used in quantum optics.[19] For truly nonlinear systems, such as discussed in this work, that kind of discussion is not free from its own problems and clearly requires additional studies.

The model studied in this paper remains also interesting in its own right. From the point is view of possible field-theoretical generalization, this model exhibits several tempting features. Indeed, it is relatively easy to propose a whole new class of nonlinear Schrödinger-like equations by considering equations for complex amplitudes α,

$$\dot{\alpha}_{\pm}=i\omega\alpha_{\pm}+i\lambda\alpha_{\pm}^{*}(\alpha_{\mp}^{*})^2, \tag{5.1}$$

which follow from the classical Hamiltonian $\mathcal{H}(\alpha_+,\alpha_-)$ of Sec. III, or even more complicated equations following from the model of Sec. IV. Indeed, replacing $\tilde{\alpha}(t)$ by $\tilde{\alpha}(x,t)$, where x stands for spatial coordinate, and replacing ω by $\omega+\mu\partial^2/\partial x^2$, one obtains a new class of nonlinear partial-differential equations. The stationary points discussed in Secs. III and/or IV will describe now the spatially homogeneous density configuration for this new model. The generalization of the model just proposed is not the only one possible. Our suggestion is to replace Eqs. (2.10), (3.5), or (4.2) for ρ and ψ by the quantum hydrodynamic ones (analogues of the Madelung equations from the linear Schrödinger equation analysis[20]). The other possibility would be to replace variables ρ and ψ in the Hamiltonian (3.6) or (4.3) by the fields $\rho(x,t)$, $\psi(x,t)$, and then supplement that Hamiltonian by gradient terms similar to those in the Ginzburg-Landau theory.[21] This generalization will bring about three new coupling constants and therefore opens up possibilities for richer structure of spatially dependent phases and patterns.

ACKNOWLEDGMENTS

This work was supported by an operating grant from National Science and Engineering Research Council (NSERC) of Canada and Polish Grant No. CPBP 0.12. One of us (Ł.A.T.) would like to express his special appreciation to the Theoretical Physics Institute at the University of Alberta in Edmonton for hospitality extended to him in the course of preparation of this paper. His stay was supported in part by the University of Alberta Central Research Fund and NSERC International Scientific Exchange Award Grant.

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