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Comments

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Comment on "Boltzmann equation and the conservation of particle number"

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A recent paper by Banggu [Phys. Rev. A **42**, 761 (1990)] claims that there are spherical solutions of the Boltzmann equation that violate particle-number conservation. The examples presented in this paper do not violate particle-number conservation, and the modification of the Boltzmann equation proposed by Banggu is incorrect.

The Boltzmann equation can be written as

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} + \mathbf{a}\cdot\frac{\partial f}{\partial \mathbf{v}} = I_c(\mathbf{v},\mathbf{r},t), \tag{1}$$

where f is the distribution function, I_c is the collision integral (which can be ignored in the discussion that follows), and $\mathbf{a}$ is the acceleration. Banggu[1] expands the distribution function as (C_t is a constant)

$$f(\mathbf{v},\mathbf{r},t) = \sum_{i=0}^{\infty}(C_t t)^i f_i(\mathbf{v},\mathbf{r}), \tag{2}$$

and substitutes this into Eq. (1) to derive the recurrence formula

$$(i+1)C_t f_{i+1}(\mathbf{v},\mathbf{r}) + \mathbf{v}\cdot\frac{\partial f_i}{\partial \mathbf{r}} + \sum_{j=0}^{i}\mathbf{a}_j\cdot\frac{\partial}{\partial \mathbf{r}}f_{i-j} = I_{ci}. \tag{3}$$

Starting with f_0 one can use this formula to derive f_1, f_2, etc. If we define N_i to be the integral of f_i over the six-dimensional phase space, then particle-number conservation requires $N_i = 0$ for $i > 0$. Banggu presents three simple choices of f_0, describing spherical clusters of particles interacting through gravitational forces, for which he claims the calculations lead to $N_2 > 0$, violating particle-number conservation [although he admits "It is well known that the Boltzmann equation (1) can be derived from the conservation of particle number in phase space"].

The first example is $f_0(\mathbf{v},\mathbf{r}) = M^*(v)n_0(r)$, where $M^*(v)$ is a Maxwell-Boltzmann velocity distribution (with constant temperature T) and $n_0(r)$ is a constant n_0 for $r \le R$ and is zero for $r > R$. If one substitutes this f_0 into Eq. (3) and ignores the sharp boundary at $r = R$ then one indeed finds $N_2 > 0$, as claimed by Banggu. However, one must remember that the spatial derivatives of f_0 contain a delta-function singularity at $r = R$,

$$\frac{\partial f_0}{\partial \mathbf{r}} = -M^*(v)n_0\delta(r-R)\frac{\mathbf{r}}{r}. \tag{4}$$

When this is properly included in the calculation one finds $N_2 = 0$. In Banggu's second example he replaces the discontinuous density profile from the first example by the exponential $n_0(r) = \exp(-k_1 r)$. He then claims to find

$$N_2 = 4\pi n_0 k_1^{-3}(k_1/C_r)^2 + 7.5 \neq 0, \tag{5}$$

where $C_r = C_t\sqrt{m/kT}$, with k being Boltzmann's constant and m the mass of a particle. This result is incorrect. I have calculated N_2 myself, both by hand and with the help of the symbolic manipulation program MAPLE, and I find $N_2 = 0$, consistent with particle-number conservation. Note that in the limit $k_1 \to \infty$ the total number of particles must go to zero, but Banggu's N_2 goes to 7.5 in this limit. In Banggu's third example he replaces the constant temperature T of the second example by a spatially varying temperature $T(r) = T\exp(-k_2 r)$, and claims to find

$$N_2 = 4\pi n_0 k_1^2 C_r^{-2}(k_1+k_2)^{-3} + 3n_0 k_1^{-4}k_2 + 36 n_0 k_1^{-3} \neq 0. \tag{6}$$

I have not calculated N_2 for this example, but the calculation does not appear to differ in any essential way from that of the second example. Note that Banggu's result for the third example does not reduce (as it should) to his result for the second example in the limit $k_2 \to 0$.

Thus Banggu's evidence for violation of particle-number conservation results from errors in his calculations. To explain his (erroneous) results Banggu states "The Boltzmann equation itself includes a contradiction: The velocity of a particle is independent of the position of the particle, but the acceleration, namely the time derivative of the velocity, is related to the position." He then proposes a modified Boltzmann equation that contains a term $\partial \mathbf{v}/\partial \mathbf{r}$ to allow for the dependence of a particle's velocity on its position. This is completely wrong. It is true that if we follow the motion of one specific particle its velocity will depend on its position, but the $\mathbf{v}$ that appears in the Boltzmann equation does not belong to a specific particle; it is simply an Eulerian phase-space coordinate, independent of the position $\mathbf{r}$, and $\partial \mathbf{v}/\partial \mathbf{r} = 0$. Moreover, it is wrong to separate the $\mathbf{v}$ in the Boltzmann equation into mean and random components, as Banggu does. This would make sense if we were working with moments of the Boltzmann equation (e.g., the Jeans equations in stellar dynamics), but it makes no sense in the Boltzmann equation itself.

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[1] Z. Banggu, Phys. Rev. A **42**, 761 (1990).