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Statistical properties of dynamically generated anomalous diffusion

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We consider a one-dimensional Lorenz-type mapping with slowly decaying correlation function, which produces anomalous diffusion with squared displacement growing faster then linearly. The statistical properties of the process are investigated by means of diffusion-index calculation. A simple approximation for probability distribution gives a good estimate for these indices.

It is well known that the dynamical models with chaotic behavior may give rise to the processes of random-walk type,[1] obeying the diffusion law

$$\langle r^2 \rangle \sim t \ . \tag{1}$$

In this paper we investigate statistical properties of dynamically generated anomalous diffusion where the averaged square of displacement grows faster than the linear law [Eq. (1)]:

$$\langle r^2 \rangle \sim t^{1+\beta} , \quad 0<\beta<1 \ . \tag{2}$$

Dynamically generated anomalous diffusion was investigated for some statistical models.[2] The deviation from the linear law is closely connected with the correlation properties of random walks. Indeed, the averaged square of displacement may be expressed through the integral of the velocity autocorrelation function $c(t)$:

$$\langle r^2 \rangle \sim t \int_0^t c(t) dt \tag{3}$$

and if the integral in (3) diverges then the anomalous diffusion occurs. The power-law tail of correlation function is known to exist in some Hamiltonian systems[3,4] and the corresponding dynamically generated anomalous diffusion is described in Refs. 5–9.

In this paper we investigate dynamically generated anomalous diffisuion originating from the simplest dynamical system—the one-dimensional mapping. Some models of this type were previously studied in Refs. 10 and 11. Consider a dynamical system

$$x_{t+1} = f(x_t) , \tag{4}$$

$$y_{t+1} = y_t + x_t , \tag{5}$$

where $x \to f(x)$ is a symmetrical Lorenz-type mapping of the interval $[-1,1]$ [see Fig. 1(a)]. The crucial point is that in order to obtain power-law behavior of the correlations in this map, we must impose specific power-law behavior of the mapping both near the discontinuity point $f(x) = \mp 1 \pm \text{const}_1 \times |x|^{1/z}$ for $x \approx 0$ and near the ends of the interval $f(x) = \pm 1 \mp \text{const}_2 \times (\pm 1 \mp x)^z$ for $|x| \approx 1$, where $z>1$. The weakly unstable points $x = \pm 1$ are responsible for the intermittent behavior in this map. We used the mapping, which analytically may be represented in the following implicit form:

$$\begin{aligned} x &= \frac{1}{2z}[1+f(x)]^z \quad \text{for } \frac{1}{2z} > x > 0 , \\ x &= f(x) + \frac{1}{2z}[1-f(x)]^z \quad \text{for } 1 > x > \frac{1}{2z} , \\ f(-x) &= -f(x) . \end{aligned} \tag{6}$$

Here the index z is the main parameter determining the correlation properties of the mapping. It is not very difficult to check that the mapping (6) has the uniform invariant probability density [that is why we used the im-

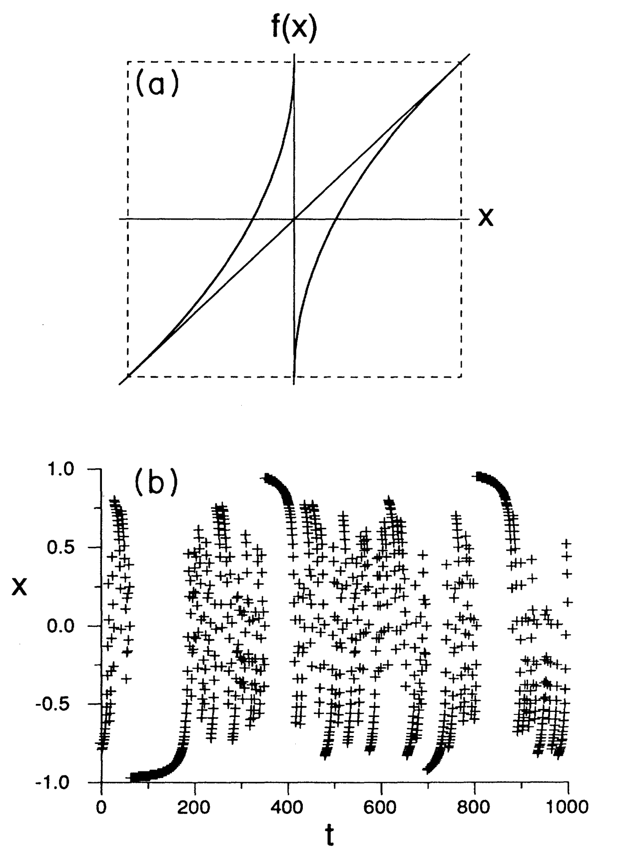

FIG. 1. (a) The Lorentz-type mapping and (b) the intermittent motion, produced by it.

plicit relations in (6)]

$$P(x)=\tfrac{1}{2}, \quad -1<x<1 \;, \tag{7}$$

while the models studied in Refs. 10 and 11, have singular invariant density. One can easily see from Fig. 1(b) that the dynamics of x is intermittent, and this causes slow decay of correlations. In Ref. 12 it was shown that for the mapping (6)

$$c(t)\sim t^{-1/(z-1)} \;.$$

For $z<2$ the integral of correlation function diverges and we obtain according to (3)

$$\langle y^2\rangle \sim t^{2-1/(z-1)} \;.$$

This formula does not allow one to reconstruct the probability distribution of the process y_t. We will use the approach of Ref. 9 and will calculate the generalized diffusion indices according to the relation

$$\langle |y|^q\rangle \sim t^{D_q} \;. \tag{8}$$

It is clear that for ordinary random walks described by the Gaussian distribution one has $D_q=q/2$. We obtained the diffusion indices for the system (6) numerically for different z (see Fig. 2). Below we would like to present a simple theory describing the obtained diffusion indices behavior.

Consider a set of initial points $\{x_0\}$, distributed according to the invariant density (7). The points lying in the center of the interval $[-1,1]$ wander in chaotic manner, while those lying close to the boundaries move "laminarly." Suppose that a point x is close to -1. Duration of the laminar phase starting at this point may be estimated from (6): $t\simeq(x+1)^{-z+1}$. Thus during the time interval t there will remain $Q\simeq t^{-1/(z-1)}$ points in the laminar state. Thus we may approximate the probability distribution $W(y,t)$ of the quantity y by the sum of

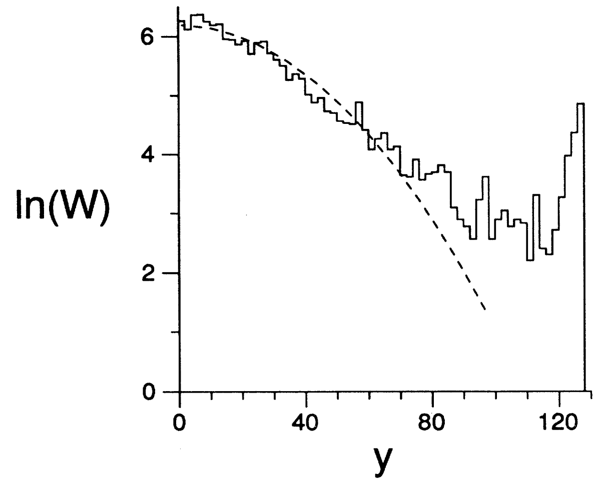

FIG. 3. The logarithm of probability distribution $W(y,t)$ for $t=128$. The dashed line approximates the main part of the distribution by the Gaussian law.

two parts. One part is produced by chaotic wanderings and is nearly Gaussian, while the second one produced by the laminar motions may be approximated by the δ functions:

$$W(y,t)=[1-Q(t)]W_G(y,t)+Q(t)[\delta(y-t)+\delta(y+t)] \;. \tag{9}$$

The numerically obtained probability distribution for y is presented in Fig. 3; one can see at least qualitative agreement with formula (9). Calculation of D_q using (8) and (9) gives

$$D_q=\begin{cases} q/2 & \text{if } q<\dfrac{2}{z-1} \\ q-\dfrac{1}{z-1} & \text{if } q\geq\dfrac{2}{z-1} \;. \end{cases} \tag{10}$$

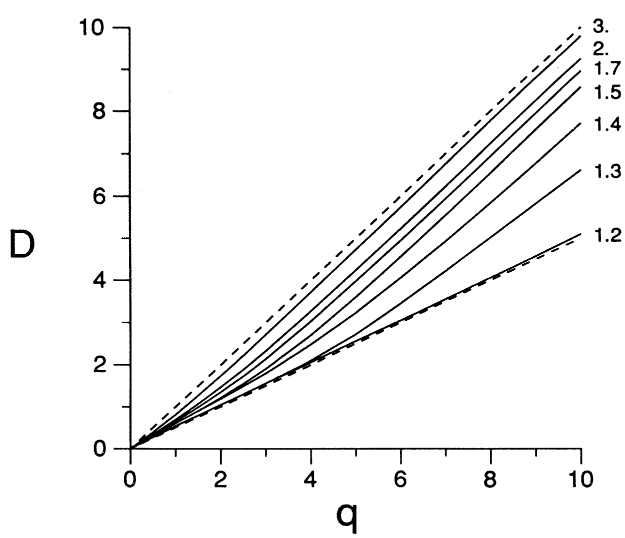

FIG. 2. The diffusion indices obtained for the mapping Fig. 1 for z ranging from 1.2 to 3. The dashed straight lines have slopes $\frac{1}{2}$ and 1.

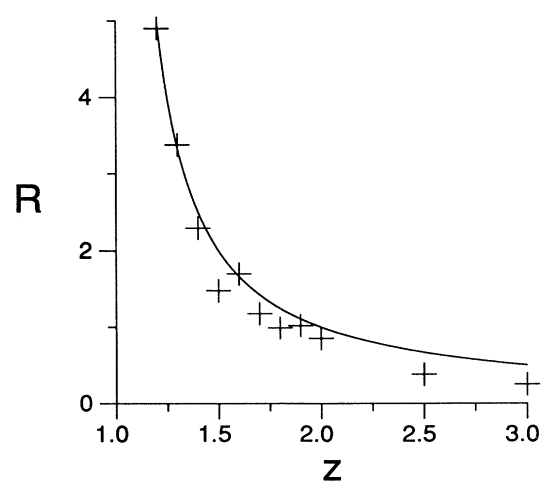

FIG. 4. Comparison of the diffusion indices for $q=10$ with the theoretical prediction (10).

Formula (10) agrees qualitatively with the data of Fig. 2. For quantitative comparison we plotted in Fig. 4 the values $R=10-D_{10}$ versus z ranging from 1.3 to 3. These points are compared with the theoretical prediction $R=(z-1)^{-1}$. It should be emphasized that it follows from (10) that for small z one has $\langle y^2\rangle\sim t$, this means that dynamically generated anomalous diffusion reveals only in the behavior of higher moments of displacement.

The approach described above was also applied for statistics of anomalous diffusion generated by a Hamiltonian system—the so-called separatrix map:[3]

$$\begin{aligned} x_{t+1}&=x_t+\sin\theta_t\ ,\\ \theta_{t+1}&=\theta_t-\lambda\ln|x_{t+1}|\ ,\\ y_{t+1}&=y_t+x_t\ . \end{aligned} \tag{11}$$

In Refs. 3 and 4 it was shown that the correlations in the mapping $(x_t,\theta_t)\rightarrow(x_{t+1},\theta_{t+1})$ decay as $c(t)\sim t^{-0.45}$; consequently we expect that $\langle y^2\rangle\sim t^{1.55}$. The diffusion indices D_q for the system (11) are presented in Fig. 5. One can see that here D_q does not look like a piecewise-linear curve (10). This means that the statistical properties in the separatrix mapping cannot be described by our simple theory.

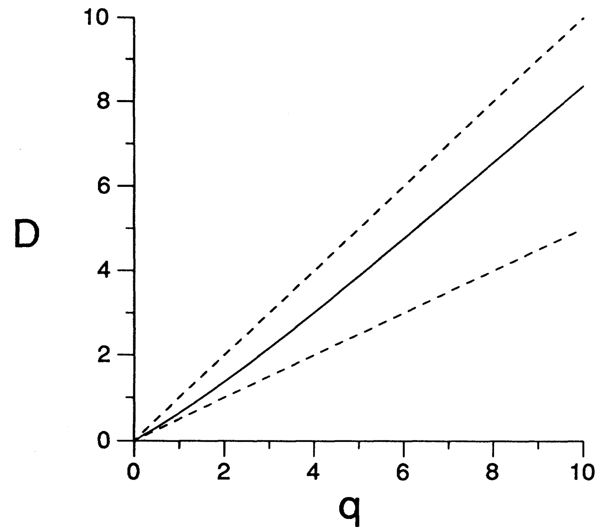

FIG. 5. The diffusion indices for the separatrix mapping (11).

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