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Random-volume scattering: Boundary effects, cross sections, and enhanced backscattering

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Assuming a scalar wave, the Bethe-Salpeter (BS) equation for a system of random medium with rough boundaries is first reviewed, together with the scattering matrices involved. Emphasis is placed on the optical condition of each scattering matrix, as well as that of a random layer with pronounced boundary effect as one scatterer. Their optical expressions are obtained in terms of the cross sections along with the respective optical conditions. The enhanced backscattering can be understood as a natural consequence of requiring coordinate-interchange invariance of the second-order Green's function, and the BS equation is rewritten as an equation for the function of the four coordinates involved, so that the invariance is immediately clear. With the solution, specific expressions of cross sections are obtained for a random layer to the approximation of using the boundary-value solution of the diffusion equation. Nevertheless, the angle distribution in the enhanced backscattering holds sufficient accuracy as long as the optical width of the layer is long enough, although not quite for the background term. Another method of using asymptotic evaluation of the cross sections under the diffusion condition is also discussed. A numerical example is shown for the enhanced backscattering.

I. INTRODUCTION

The scattering of waves by a random volume is an interesting subject, including problems of the boundary effect, the enhanced backscattering by both the medium[1–4] and the boundaries,[5,6] and, when a fixed scatterer is embedded, its shadowing effect as well as its enhanced backscattering. The basic equation is the Bethe-Salpeter (BS) equation for the entire system consisting of the random medium plus the boundaries that can be partially or fully random. For such a composite system, a unified theory of random medium and boundaries has been developed,[7,8] which enables us to construct the solution in several different forms in terms of independent scattering matrices of the medium and boundaries, with the aid of addition formulas of scattering matrices. To obtain specific expressions, the diffusion approximation can be partially utilized with a reasonable boundary condition for the boundaries,[8] which is a generalized version of the condition more previously introduced when the boundaries are prefectly free from reflection.[2–4] Here the enhanced backscattering can be understood as a natural consequence of requiring the coordinate-interchange invariance of the BS equation, based on the reciprocity of the deterministic Green's function for each of the original and complex-conjugate waves, as was emphasized by Vollhardt and Wölfle[9] in connection with an electron-hole wave subject to a random potential (Anderson localization problem). The same idea was later used to investigate the enhanced backscattering of a scalar wave by a medium of independent particles, along with some comparison to experimental results of a light wave,[3,4] and also of an electromagnetic wave by a rough surface.[5,6]

In this paper, a scalar wave is assumed as a model to represent an electromagnetic wave, electron-hole wave, and other quantum-mechanical waves subject to a random potential. The BS equation and scattering matrices are first briefly reviewed based on Ref. 8 to prepare for the following sections, with an emphasis on the optical condition of each scattering matrix as well as that of the entire system as one scatterer, to ensure power conservation at every level of the scattering (Sec. II). Most of the basic equations are written in matrix form with respect to the space coordinates (which can include subscripts to refer to different components of the wave) so that they hold true for the variety of waves, although their specific expressions should differ from one wave to another. The optical expressions are then obtained in terms of the cross sections together with the respective optical conditions (Sec. III). To investigate the enhanced backscattering in unbounded space of a random medium, first, the BS equation for the second-order Green's function is rewritten as an equation for the function of the four coordinates so that the coordinate-interchange invariance of the solution is immediately clear (Sec. IV). Then, to obtain specifically the cross sections of a random layer, the diffusion approximation and boundary condition are examined in some detail, along with the comparison to previous methods (Sec. V and Appendix A).

II. BASIC EQUATIONS AND SCATTERING MATRICES

The coordinate vector in three-dimensional space is denoted by $\hat{\mathbf{x}}=(x_1,x_2,x_3)=(\boldsymbol{\rho},z)$ with $\boldsymbol{\rho}=(x_1,x_2)$ and $z=x_3$, where the z axis is taken in the direction normal to the average boundaries (Fig. 1). The scalar product of two space vectors $\hat{\mathbf{a}}=(\mathbf{a},a_z)$ and $\hat{\mathbf{b}}=(\mathbf{b},b_z)$ is denoted by $\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}=\mathbf{a}\cdot\mathbf{b}+a_z b_z$, where $\mathbf{a}\cdot\mathbf{b}=a_1b_1+a_2b_2$. We first consider two random layers separated by a rough boundary which is planar on average, as illustrated in Fig. 2. A scalar wave function $\psi(\hat{\mathbf{x}})e^{i\omega t}$, where $\omega>0$ and t is time,

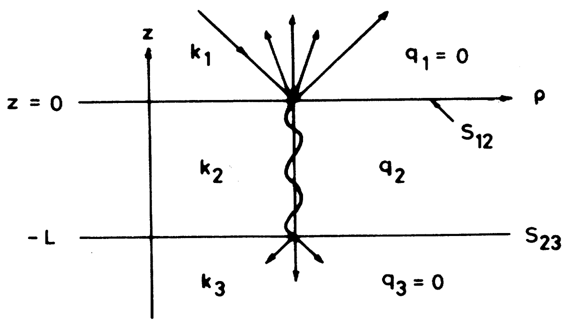

FIG. 1. Geometry and notations of a random layer for Eqs. (2.26) and (2.53).

is considered, and is denoted in each layer by $\psi_a(\hat{\mathbf{x}})$, $a=1,2$, whose wave equation is

$$[\mathcal{L}_a - q_a(\hat{\mathbf{x}})]\psi_a(\hat{\mathbf{x}}) = j_a(\hat{\mathbf{x}}) , \tag{2.1a}$$

$$\mathcal{L}_a = -\left[\frac{\partial}{\partial \hat{\mathbf{x}}}\right]^2 - k_a^2, \quad \mathrm{Im}(k_a) < 0 . \tag{2.1b}$$

Here $q_a(\hat{\mathbf{x}}) = q_a^*(\hat{\mathbf{x}})$ is the random part of the medium, and $j_a(\hat{\mathbf{x}})$ is a source term; k_a is the propagation constant when the medium is free from the random part, and the medium is assumed to be nondissipative for the time being. The boundary condition is first assumed to be the continuity of ψ_a and its gradient normal to the (real) boundary surface, and consistently with this, the power flux vector $\hat{\mathbf{w}}_a(\hat{\mathbf{x}})$ in the space k_a is defined by

$$\hat{\mathbf{w}}_a(\hat{\mathbf{x}}) = \psi_a^* \hat{\boldsymbol{\alpha}} \psi_a(\hat{\mathbf{x}}) , \tag{2.2a}$$

with a vector operator $\hat{\boldsymbol{\alpha}}$, defined by

$$\hat{\boldsymbol{\alpha}} = (2i)^{-1}\left[\frac{\overleftarrow{\partial}}{\partial \hat{\mathbf{x}}} - \frac{\overrightarrow{\partial}}{\partial \hat{\mathbf{x}}}\right] , \tag{2.2b}$$

where the left and right overarrows mean the operation on the left- and right-hand sides, respectively. Hence the power equation is

$$\frac{\partial}{\partial \hat{\mathbf{x}}} \cdot \sum_a \hat{\mathbf{w}}_a(\hat{\mathbf{x}}) = \sum_a (2i)^{-1}[\psi_a^* j_a(\hat{\mathbf{x}}) - j_a^* \psi_a(\hat{\mathbf{x}})] , \tag{2.3}$$

except on the boundary. Here $\hat{\mathbf{w}}_a(\hat{\mathbf{x}})=0$ for $\hat{\mathbf{x}}$ in space $k_b \neq k_a$.

The boundary condition can be transferred from the real boundary onto two reference boundary planes, say, S_1 and S_2 at $z=0$ and $z=-d_2$, respectively, chosen such that the change of the boundary height is ranged between S_1 and S_2 (Fig. 2); hence, with the notation $\partial_n^{(a)} = \hat{\mathbf{n}}^{(a)} \cdot (\partial/\partial \hat{\mathbf{x}})$, where $\hat{\mathbf{n}}^{(a)}$ is the unit vector directed outward normally to S_a, the boundary equation can be written as[7,10]

$$-\partial_n^{(a)} \psi_a(\boldsymbol{\rho}) = \sum_{b=1}^{2} \int d\boldsymbol{\rho}' B_{ab}^{(12)}(\boldsymbol{\rho}|\boldsymbol{\rho}')\psi_b(\boldsymbol{\rho}') . \tag{2.4}$$

Here $\psi_a(\boldsymbol{\rho})$ denotes $\psi_a(\hat{\mathbf{x}})$ bounded on S_a, and, when the boundary is nondissipative

$$(B_{ab}^{(12)})^\dagger(\boldsymbol{\rho}|\boldsymbol{\rho}') \equiv (B_{ba}^{(12)})^*(\boldsymbol{\rho}'|\boldsymbol{\rho}) = B_{ab}^{(12)}(\boldsymbol{\rho}|\boldsymbol{\rho}') , \tag{2.5}$$

i.e., the matrix defined by the elements $B_{ab}^{(12)}(\boldsymbol{\rho}|\boldsymbol{\rho}')$ is Hermitian with respect to both the coordinates and the subscripts. This means that $B^{(12)}$ is a real symmetrical matrix in view of having symmetrical matrix elements, as can be shown by applying Green's theorem to the boundary space enclosed by S_1 and S_2 and using Eq. (2.4), with the vanishing contour surface integral over both sides of S. Hereafter, the boundary space will be neglected, on letting $d_2 \to 0$, unless otherwise noted; so that $S_{12} = S_1 + S_2$ at $z=0$ represents the two reference boundary planes, together.

The wave equations (2.1) and the boundary equation (2.4) can be written by one wave equation of the form

$$(\mathcal{L}_a - q_a)\psi_a - \sum_{b=1}^{2} B_{ab}^{(12)}\psi_b = j_a . \tag{2.6}$$

Here both $B_{ab}^{(12)}$ and q_a are regarded as $\hat{\mathbf{x}}$-coordinate matrices, defined by the elements

$$B_{ab}^{(12)}(\hat{\mathbf{x}}|\hat{\mathbf{x}}') = \delta(z+d_a) B_{ab}^{(12)}(\boldsymbol{\rho}|\boldsymbol{\rho}')\delta(z'+d_b), \quad d_1 = 0 \tag{2.7}$$

and $q_a(\hat{\mathbf{x}}|\hat{\mathbf{x}}') = q_a(\hat{\mathbf{x}})\delta(\hat{\mathbf{x}}-\hat{\mathbf{x}}')$, and the solution is subject to the new boundary condition that $\partial_n^{(a)}\psi_a = 0$, $a=1,2$, inside the boundary space $0 > z > -d_2$. The proof can be given by integrating Eq. (2.6) with respect to z over two infinitesimal regions enclosing S_1 and S_2, separately; hence Eq. (2.4) is reproduced.

With a new matrix v_{ab}, defined by the elements

$$v_{ab} = q_a \delta_{ab} + B_{ab}^{(12)} , \tag{2.8}$$

the equation of the deterministic Green's function for the new wave equation (2.6), say, $g_{ab}(\hat{\mathbf{x}}|\hat{\mathbf{x}}')$, can be written as

$$\sum_c (\mathcal{L}_a \delta_{ac} - v_{ac}) g_{cb}(\hat{\mathbf{x}}|\hat{\mathbf{x}}') = \delta_{ab}\delta(\hat{\mathbf{x}}-\hat{\mathbf{x}}') \tag{2.9a}$$

or in matrix form as

$$(\mathcal{L} - v) g = 1, \quad v = q + B^{(12)} . \tag{2.9b}$$

Here v may be regarded as an effective medium representing both the medium and the boundary on an equal basis. The unified wave equation (2.9b) shows that v is a symmetrical matrix with respect to both the coordinates and the subscripts, $v^T = v$, hence the Green's function is also symmetrical, i.e.,

$$g^T = g, \quad v^T = v , \tag{2.10}$$

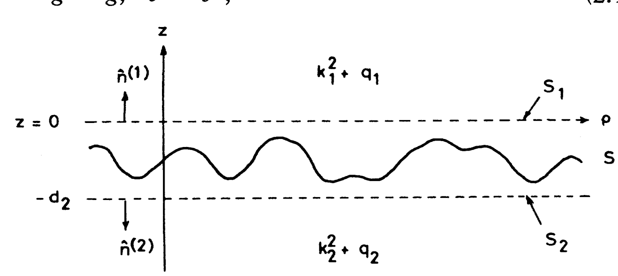

FIG. 2. Geometry of a rough boundary for Eq. (2.4). The real boundary S is distributed within the range $0 > z > -d_2$.

being subject to the reciprocity.

For a general class of scalar waves, continuity conditions at the (real) boundary can be reduced to those of ψ_a and $\eta_a^{-1}\partial_n\psi_a$, with some constant η_a depending on the ath medium, and the equations can be similarly formulated without changing the basic form.[10] Also for electromagnetic waves (having six components), boundary condition and wave equation can be unified to be written in a form similar to Eqs. (2.9), so that the following equations are also obtained in the same form as in this paper.[10]

A. Statistical Green's functions

Equation (2.9b) enables us to obtain the statistical Green's functions in exactly the same form as those in an inhomogeneous random medium v, and the results are summarized as follows.[7,8] The averaged version of Eq. (2.9b) can be written as

$$(\mathcal{L}-M)G=1, \quad G=\langle g\rangle , \tag{2.11}$$

in terms of an effective medium M of v, defined by

$$MG=\langle vg\rangle, \quad M=M^{(q)}+M^{(12)} . \tag{2.12}$$

Here $M^{(q)}$ and $M^{(12)}$ are also defined in the same fashion, by

$$M^{(q)}G=\langle qg\rangle, \quad M^{(12)}G=\langle B^{(12)}g\rangle , \tag{2.13}$$

and are approximately equal to the independent contributions from the medium and the boundary, respectively, with the elements $M_a^{(q)}\delta_{ab}$ and $M_{ab}^{(12)}$.

For the statistical Green's function of second order, defined by

$$I_{ab;cd}(\hat{\mathbf{x}}_1;\hat{\mathbf{x}}_2|\hat{\mathbf{x}}'_1;\hat{\mathbf{x}}'_2)=\langle g^*_{ac}(\hat{\mathbf{x}}_1|\hat{\mathbf{x}}'_1)g_{bd}(\hat{\mathbf{x}}_2|\hat{\mathbf{x}}'_2)\rangle \tag{2.14a}$$

or in matrix form by

$$I(1;2)=\langle g^*(1)g(2)\rangle \tag{2.14b}$$

(here and also hereafter, the subscript 1 is attached to the coordinates of quantities of the complex-conjugate wave function, and the subscript 2 is attached to those of the original wave function), we first introduce a matrix Δv, defined by

$$\Delta v=v-M=\Delta q+\Delta B^{(12)} , \tag{2.15a}$$

where

$$\Delta q=q-M^{(q)}, \quad \Delta B^{(12)}=B^{(12)}-M^{(12)} , \tag{2.15b}$$

and employ the expression

$$g=G(1+\Delta vg), \quad \langle \Delta vg\rangle=0 , \tag{2.16}$$

for both $g^*(1)$ and $g(2)$ in the right-hand side of Eq. (2.14b). Hence we obtain an expression

$$I(1;2)=G^*(1)G(2)[1+K(1;2)I(1;2)] \tag{2.17}$$

of the form of the Bethe-Salpeter equation, with a matrix $K(1;2)$, defined by

$$K(1;2)I(1;2)=\langle \Delta v^*(1)\Delta v(2)g^*(1)g(2)\rangle \tag{2.18}$$

in the same fashion as M is by Eq. (2.12). Here $K(1;2)$ can also be approximated by the independent sum of $K^{(q)}$ from the medium and $K^{(12)}$ from the boundary, as

$$K(1;2)\simeq K^{(q)}(1;2)+K^{(12)}(1;2) . \tag{2.19}$$

Here $K^{(q)}$ is a diagonal matrix with respect to the subscripts, having only the elements $K_a^{(q)}\equiv K_{aa;aa}^{(q)}$, while the important elements of $K^{(12)}$ are $K_{ab}^{(12)}\equiv K_{aa;bb}^{(12)}$. Hence, in terms of the notation $I_{ab}^{(q+12)}=I_{aa;bb}$ and

$$U_{ab}^{(C)}(1;2)=G^*_{ab}(1)G_{ab}(2) , \tag{2.20}$$

the BS equation (2.17) can be written, in 2×2 matrix form, as

$$I^{(q+12)}=U^{(C)}[1+(K^{(q)}+K^{(12)})I^{(q+12)}] . \tag{2.21}$$

The matrices M and K, as defined by Eqs. (2.12) and (2.18), respectively, are not quite independent of each other, subjected to a local (optical) relation of the form

$$\frac{\partial}{\partial\boldsymbol{\rho}}\cdot\boldsymbol{\beta}(\hat{\mathbf{x}}|1;2) =\delta(\hat{\mathbf{x}}|1;2)(2i)^{-1}\{M^*(1)-M(2) -[G^*(1)-G(2)]K(1;2)\} . \tag{2.22a}$$

Here the matrix $\delta(\hat{\mathbf{x}}|1;2)$ is defined by the elements

$$\delta_{ab}(\hat{\mathbf{x}}|\hat{\mathbf{x}}_1;\hat{\mathbf{x}}_2)=\delta_{ab}\delta(\hat{\mathbf{x}}-\hat{\mathbf{x}}_1)\delta(\hat{\mathbf{x}}-\hat{\mathbf{x}}_2) , \tag{2.22b}$$

such that, for any matrices $A^*(1)$ and $B(2)$, the product

$$\delta(\hat{\mathbf{x}}|1;2)A^*(1)B(2)\equiv A^*B(\hat{\mathbf{x}}|1;2)\equiv A^*B(\hat{\mathbf{x}}) \tag{2.22c}$$

represents

$$\sum_{a,b}\int d\hat{\mathbf{x}}_1 d\hat{\mathbf{x}}_2\delta_{ab}(\hat{\mathbf{x}}|\hat{\mathbf{x}}_1;\hat{\mathbf{x}}_2)A^*_{ac}(\hat{\mathbf{x}}_1|\hat{\mathbf{x}}'_1)B_{bd}(\hat{\mathbf{x}}_2|\hat{\mathbf{x}}'_2) =\sum_a A^*_{ac}(\hat{\mathbf{x}}|\hat{\mathbf{x}}'_1)B_{ad}(\hat{\mathbf{x}}|\hat{\mathbf{x}}'_2) ,$$

and the left-hand side of (2.22a) is nonzero only on S_{12}, being a contribution purely from the boundary with the elements of form

$$\boldsymbol{\beta}_{cd}(\hat{\mathbf{x}}|\hat{\mathbf{x}}_1;\hat{\mathbf{x}}_2)=\sum_a\delta(z+d_a)\boldsymbol{\beta}_{a;cd}(\boldsymbol{\rho}|\hat{\mathbf{x}}_1;\hat{\mathbf{x}}_2) . \tag{2.22d}$$

The derivation of (2.22a) is straightforward by making a product of $\delta(\hat{\mathbf{x}}|1;2)$ and difference of the two relations from (2.18) as

$$\langle v(1)g^*(1)g(2)\rangle=[M^*(1)+G(2)K(1;2)]I(1;2) , \tag{2.22e}$$

$$\langle v(2)g^*(1)g(2)\rangle=[M(2)+G^*(1)K(1;2)]I(1;2) ,$$

so that the product is zero in the medium region, in view of the left-hand sides of (2.22e), while it is nonzero on the boundary, yielding the divergence term on the left-hand side as a contribution from the part $B^{(12)}$ of v having nondiagonal symmetrical matrix elements (spatially dispersive).[11] The relation (2.22a) ensures power conservation at every point in the space and on the boundary, and can be rewritten, in terms of the matrixes $\Delta G(\hat{\mathbf{x}}|1;2)$ and $k\gamma(\hat{\mathbf{x}}|1;2)$ defined according to the notation (2.22c), by

$$\Delta G(\hat{\mathbf{x}}|1;2)=\delta(\hat{\mathbf{x}}|1;2)(2i)^{-1}[G^*(1)-G(2)] , \quad (2.23a)$$

$$k\gamma(\hat{\mathbf{x}}|1;2)=\delta(\hat{\mathbf{x}}|1;2)(2i)^{-1}[M^*(1)-M(2)] , \quad (2.23b)$$

respectively, as

$$\frac{\partial}{\partial \boldsymbol{\rho}}\cdot\boldsymbol{\beta}(\hat{\mathbf{x}})=k\gamma(\hat{\mathbf{x}})-\Delta G(\hat{\mathbf{x}})K . \quad (2.23c)$$

Hence, with the operator $\hat{\boldsymbol{\alpha}}$ of (2.2b) and the matrix $\hat{\boldsymbol{\alpha}}(\hat{\mathbf{x}})$ with the elements

$$\hat{\boldsymbol{\alpha}}_{ab}(\hat{\mathbf{x}}|\hat{\mathbf{x}}_1;\hat{\mathbf{x}}_2)=\delta_{ab}(\hat{\mathbf{x}}|\hat{\mathbf{x}}_1;\hat{\mathbf{x}}_2)(2i)^{-1}\left[\frac{\partial}{\partial\hat{\mathbf{x}}_1}-\frac{\partial}{\partial\hat{\mathbf{x}}_2}\right] , \quad (2.23d)$$

the BS equation (2.17) leads to [see also Eqs. (2.58) and (2.59)]

$$\frac{\partial}{\partial\hat{\mathbf{x}}}\cdot(\hat{\boldsymbol{\alpha}}+\boldsymbol{\beta})I(\hat{\mathbf{x}})=\Delta G(\hat{\mathbf{x}}) \quad (2.23e)$$

equivalent to the averaged version of power equation (2.3), except the $\boldsymbol{\beta}$ term meaning an additional power flux by a surface wave propagating along the boundary.

A relation similar to (2.23c) holds true also for each $M^{(\,)}$ and $K^{(\,)}$, approximately,[11] providing the optical condition of each constituent; e.g., for $M^{(q)}$ and $K^{(q)}$, Eq. (2.23c) is replaced by

$$k\gamma^{(q)}(\hat{\mathbf{x}})-\Delta G^{(0)}(\hat{\mathbf{x}})K^{(q)}=0 , \quad (2.24)$$

where $\hat{\boldsymbol{\beta}}^{(q)}(\hat{\mathbf{x}})=0$ in the present case of a nondispersive medium. Here $G^{(0)}$ is the Green's function in an unbounded space of $M^{(q)}$ and the replacement of $M\to M^{(q)}$ and $K\to K^{(q)}$ has been made in view of the negligible boundary effect on $M^{(q)}$ and $K^{(q)}$. By the $\hat{\mathbf{x}}$ integration, Eq. (2.24) leads to optical relation (3.20) for the medium cross section; for the boundary, the counterpart relation is given by Eq. (3.16).

B. Case of three random layers

The situation is the same also for the case of a random layer, as illustrated in Fig. 1, and various equations formally remain unchanged with the setting

$$M=M^{(q)}+M^{(12)}+M^{(23)} , \quad (2.25a)$$

$$K\simeq K^{(q)}+K^{(12)}+K^{(23)} . \quad (2.25b)$$

Thus, using the notation $I_{ab}^{(q+12+23)}$, $a,b=1,2,3$, for the second-order Green's function in this case, we obtain the BS equation in 3×3 matrix form, as

$$I^{(q+12+23)}=U^{(C)}[1+(K^{(q)}+K^{(12)}+K^{(23)})I^{(q+12+23)}] . \quad (2.26)$$

Here $K^{(q)}$ is a diagonal matrix with the elements $K_{ab}^{(q)}=K_a^{(q)}\delta_{ab}$, and $K^{(12)}$ and $K^{(23)}$ are the contributions purely from the boundaries S_{12} and S_{23}, with the nonvanishing elements $K_{ab}^{(12)}$, $a,b=1,2$, and $K_{ab}^{(23)}$, $a,b=2,3$.

C. Solutions and scattering matrices

To obtain the solution of the BS equation (2.21), we first introduce the solution in the special case $K^{(q)}=0$ (on keeping $M^{(q)}\neq 0$), say, $I^{(12)}$; so that

$$I^{(12)}=U^{(C)}(1+K^{(12)}I^{(12)}) , \quad (2.27)$$

with the solution

$$I^{(12)}=U^{(C)}+U^{(C)}S^{(12)}U^{(C)} \quad (2.28)$$

in terms of an (incoherent) scattering matrix $S^{(12)}$ of $K^{(12)}$, defined by

$$S^{(12)}=K^{(12)}(1+U^{(C)}S^{(12)}) \quad (2.29a)$$

$$=(1-K^{(12)}U^{(C)})^{-1}K^{(12)} , \quad (2.29b)$$

$$S_{ab}^{(12)}=K_{ab}^{(12)}+\sum_{c,d}K_{ac}^{(12)}U_{cd}^{(C)}K_{db}^{(12)}+\cdots . \quad (2.29c)$$

Here the last series shows that the multiple scattering on both sides of the boundary is involved in $S^{(12)}$ in a complicated way, including the coherent scattering by the boundary.

The Green's function G^{ab} can also be written in the same form,

$$G_{ab}=G_a^{(0)}\delta_{ab}+G_a^{(0)}T_{ab}^{(12)}G_b^{(0)} , \quad (2.30)$$

in terms of the Green's function $G_a^{(0)}$ in an unbounded medium of $M_a^{(q)}$, whose Fourier representation in the $\boldsymbol{\rho}$ space is therefore

$$G_a^{(0)}(\hat{\mathbf{x}}-\hat{\mathbf{x}}')=(2\pi)^{-2}\int d\boldsymbol{\lambda}\exp[-i\boldsymbol{\lambda}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}')]G_a^{(0)}(z-z') , \quad (2.31a)$$

with

$$G_a^{(0)}(z-z')=[2i\tilde{h}_a(\boldsymbol{\lambda})]^{-1}\exp[-i\tilde{h}_a(\boldsymbol{\lambda})|z-z'|] . \quad (2.31b)$$

Here

$$\tilde{h}_a(\boldsymbol{\lambda})=[(k_a^{(M)})^2-\boldsymbol{\lambda}^2]^{1/2} ,$$
$$k_a^{(M)}=(k_a^2+\tilde{M}_a^{(q)})^{1/2}\simeq k_a,\quad \mathrm{Im}(k_a^{(M)})<0 , \quad (2.31c)$$

where $\tilde{M}_a^{(q)}(\hat{\boldsymbol{\lambda}})$, $\hat{\boldsymbol{\lambda}}=(\boldsymbol{\lambda},\tilde{h}_a)$, is the Fourier transform of $M_a^{(q)}$, and $\mathrm{Im}(\tilde{h}_a)<0$. Hence $G_{ab}(\hat{\mathbf{x}}|\hat{\mathbf{x}}')$ has the Fourier transform $G_{ab}(z|z')$ from Eq. (2.30), as

$$G_{ab}(z|z')=G_a^{(0)}(z-z')\delta_{ab}+G_a^{(0)}(z)\tilde{T}_{ab}^{(12)}G_b^{(0)}(-z') . \quad (2.32)$$

Here

$$\tilde{T}_{ab}^{(12)}=2i\tilde{h}_a\langle R_{ab}^{(12)}\rangle=\tilde{T}_{ba}^{(12)} , \quad (2.33a)$$

where $\langle R_{ab}^{(12)}\rangle\neq\langle R_{ba}^{(12)}\rangle$ is the reflection-transmission coefficient of the boundary and, when it is perfectly smooth,

$$\langle R_{11}^{(12)}\rangle=\frac{\tilde{h}_1-\tilde{h}_2}{\tilde{h}_1+\tilde{h}_2},\quad \langle R_{21}^{(12)}\rangle=\frac{2\tilde{h}_1}{\tilde{h}_1+\tilde{h}_2} . \quad (2.33b)$$

It is generally given in terms of the 2×2 matrix $\widetilde{M}^{(12)}$, by[10]

$$\langle R^{(12)}\rangle=(i\widetilde{h}-\widetilde{M}^{(12)})^{-1}(i\widetilde{h}+\widetilde{M}^{(12)}) , \quad (2.34a)$$

and hence

$$1+\langle R^{(12)}\rangle=(i\widetilde{h}-\widetilde{M}^{(12)})^{-1}2i\widetilde{h} , \quad (2.34b)$$

where $\widetilde{h}$ is a diagonal matrix with the elements $\widetilde{h}_{ab}=\widetilde{h}_a\delta_{ab}$, and $\widetilde{M}^{(12)}$ is the (two-dimensional) Fourier transform of $M^{(12)}$. Hence, setting $z=z'=0$ in Eq. (2.32), use of Eq. (2.34b) leads to the surface Green's function

$$G(z=0|z'=0)=(i\widetilde{h}-\widetilde{M}^{(12)})^{-1} , \quad (2.34c)$$

in a form similar to the original given by Eq. (2.11).

Both Eqs. (2.30) and (2.32) can be written in matrix form by

$$G=G^{(0)}+G^{(0)}T^{(12)}G^{(0)} . \quad (2.35)$$

Therefore, by introducing a diagonal matrix $U_{ab}=U_a\delta_{ab}$, defined by

$$U_a(1;2)=[G_a^{(0)}(1)]^*G_a^{(0)}(2) , \quad (2.36)$$

$U^{(C)}(1;2)$ of Eq. (2.20) can also be written in the same form,

$$U^{(C)}(1;2)=U(1;2)+U(1;2)V^{(12)}(1;2)U(1;2) . \quad (2.37a)$$

Here

$$V^{(12)}(1;2)=T^{(12)*}(1)T^{(12)}(2)+T^{(12)*}(1)[G^{(0)}(2)]^{-1} + T^{(12)}(2)[G^{(0)*}(1)]^{-1} , \quad (2.37b)$$

wherein the interference terms are negligible when the source and the observer are both separated enough from the boundary, whereas they are not negligible otherwise [see, e.g., Eq. (2.39b)].

Thus, with (2.37a), Eq. (2.28) can be written in the form

$$I^{(12)}=U+U\sigma^{(12)}U . \quad (2.38)$$

Here $\sigma^{(12)}$ means a resultant scattering matrix of the boundary and is given by

$$\sigma^{(12)}=V^{(12)}+\bar{F}^{(C)}S^{(12)}F^{(C)} , \quad (2.39a)$$

where, from (2.33a) and (2.31b),

$$F^{(C)}\equiv 1+UV^{(12)} = [1+\langle R^{(12)*}(1)\rangle][1+\langle R^{(12)}(2)\rangle] , \quad (2.39b)$$

and can be made more specific by using Eq. (2.34b); $\bar{F}^{(C)}=1+V^{(12)}U$ is obtained from $F^{(C)}$ by the transposition.

The introduction of $I^{(12)}$ by Eq. (2.27) enables the BS equation (2.21) to be rewritten as

$$I^{(q+12)}=I^{(12)}(1+K^{(q)}I^{(q+12)}) \quad (2.40)$$

and hence the solution as

$$I^{(q+12)}=I^{(12)}+I^{(12)}S^{(q/12)}I^{(12)} , \quad (2.41)$$

in terms of a scattering matrix $S^{(q/12)}$ of $K^{(q)}$, defined by

$$S^{(q/12)}=K^{(q)}(1+I^{(12)}S^{(q/12)}) , \quad (2.42)$$

with the superscript $(q/12)$ to mean the dependence on $\sigma^{(12)}$ through $I^{(12)}$. Here the effect of $\sigma^{(12)}$ can be made explicit by introducing a solution of Eq. (2.42) in the case $\sigma^{(12)}=0$, say, $S^{(0q)}$, governed by

$$S^{(0q)}=K^{(q)}(1+US^{(0q)})=(1-K^{(q)}U)^{-1}K^{(q)} , \quad (2.43)$$

so that Eq. (2.42) becomes written, on using (2.38), as

$$S^{(q/12)}=S^{(0q)}(1+U\sigma^{(12)}US^{(q/12)}) \quad (2.44a)$$

$$=(1-S^{(0q)}U\sigma^{(12)}U)^{-1}S^{(0q)} . \quad (2.44b)$$

Hence Eq. (2.41) is written finally in the form

$$I^{(q+12)}=I^{(12)}+(1+U\sigma^{(12)})\mathcal{J}^{(q/12)}(\sigma^{(12)}U+1) . \quad (2.45)$$

Here the entire effect of the random medium appears only through a new matrix $\mathcal{J}^{(q/12)}$, defined by

$$\mathcal{J}^{(q/12)}=US^{(q/12)}U \quad (2.46)$$

and given as the solution of

$$\mathcal{J}^{(q/12)}=\mathcal{J}^{(0q)}(1+\sigma^{(12)}\mathcal{J}^{(q/12)}) , \quad (2.47)$$

where

$$\mathcal{J}^{(0q)}=US^{(0q)}U \quad (2.48a)$$

$$=UK^{(q)}(U+\mathcal{J}^{(0q)}) , \quad (2.48b)$$

which is a diagonal matrix with respect to the subscripts, and tends to zero as $K^{(q)}\to 0$.

Also for the case of three random layers, as illustrated in Fig. 1, the situation becomes exactly the same by introducing a solution of when $K_a^{(q)}=0$, $a=1,2,3$, say, $I^{(12+23)}$, and letting $I^{(12+23)}$ do all the roles of $I^{(12)}$ in Eq. (2.45); that is, the basic equations (2.45)–(2.48) remain unchanged with the replacement of the superscript (12) by (12+23) and using the expression

$$I_{ab}^{(12+23)}=U_a\delta_{ab}+U_a\sigma_{ab}^{(12+23)}U_b . \quad (2.49)$$

Here, when the distance between the two boundaries, L, is sufficiently large compared with the wave coherence distance, say, γ_2^{-1}, so that $\gamma_2L\gg 1$, $\sigma_{ab}^{(12+23)}$ can be approximated by

$$\sigma_{ab}^{(12+23)}\simeq\sigma_{ab}^{(12)}+\sigma_{ab}^{(23)}, \quad a,b=1,2,3 \quad (2.50)$$

being the independent sum of the two boundary scattering matrices, $\sigma_{ab}^{(12)}$ of S_{12} and $\sigma_{ab}^{(23)}$ of S_{23}. Thus Eq. (2.45) is replaced by a 3×3 matrix equation, as

$$I^{(q+12+23)}=I^{(12+23)}+(1+U\sigma^{(12+23)})\mathcal{J}^{(q/12+23)} \times(\sigma^{(12+23)}U+1) , \quad (2.51)$$

and, with (2.50), Eq. (2.47) is changed to

$$\mathcal{J}^{(q/12+23)}=\mathcal{J}^{(0q)}[1+(\sigma^{(12)}+\sigma^{(23)})\mathcal{J}^{(q/12+23)}] . \quad (2.52)$$

Here $\mathcal{J}^{(0q)}$ is still governed by Eq. (2.48b).

Example: case of a random layer ($q_1=q_3=0$, $q_2\neq 0$). The only nonvanishing element of $\mathcal{J}^{(q/12+23)}$ in (2.51) is $\mathcal{J}_{22}^{(q/12+23)}$ in this case. Hence, when the source is in the space k_1 and the layer width L is large enough so that $\gamma_2 L \gg 1$, $I^{(q+12+23)}$ within the same space is given, with $I_{11}^{(12+23)} \simeq I_{11}^{(12)}$, by

$$I_{11}^{(q+12+23)} = I_{11}^{(12)} + U_1 \sigma_{12}^{(12)} \mathcal{J}_{22}^{(q/12+23)} \sigma_{21}^{(12)} U_1 , \qquad (2.53a)$$

and the transmitted wave into the space k_3 is given by

$$I_{31}^{(q+12+23)} = U_3 \sigma_{32}^{(23)} \mathcal{J}_{22}^{(q/12+23)} \sigma_{21}^{(12)} U_1 , \qquad (2.53b)$$

where the contribution from $I_{31}^{(12+23)}$ is presently negligible. Here the random medium is involved only through $\mathcal{J}_{22}^{(q/12+23)}$, which is the solution of

$$\mathcal{J}_{22}^{(q/12+23)} = \mathcal{J}_2^{(0q)} [1 + (\sigma_{22}^{(12)} + \sigma_{22}^{(23)}) \mathcal{J}_{22}^{(q/12+23)}] , \qquad (2.54a)$$

with the solution $\mathcal{J}_2^{(0q)}$ of

$$\mathcal{J}_2^{(0q)} = U_2 K_2^{(q)} (U_2 + \mathcal{J}_2^{(0q)}), \quad 0 \geq z \geq -L . \qquad (2.54b)$$

Here, as for $\sigma^{(12)}$ and $\sigma^{(23)}$, we may utilize experimental values of the boundaries, instead of the theoretical ones.

Thus the problem is reduced to finding the solution of Eqs. (2.54) and the resulting optical cross sections of the random layer, defined, on rewriting Eqs. (2.53) in the form

$$I_{11}^{(q+12+23)} = U_1 + U_1 \sigma_{11}^{(q+12+23)} U_1 , \qquad (2.55a)$$

$$I_{31}^{(q+12+23)} = U_3 \sigma_{31}^{(q+12+23)} U_1 , \qquad (2.55b)$$

by the asymptotic expressions of the factors $\sigma_{a,1}^{(q+12+23)}$, $a = 1,3$, when the source and the observer are both separated enough from the boundaries (Sec. III).

D. Optical conditions of random volume

As we show below, the local relation (2.24) for $K^{(q)}$ can be written, in terms of the scattering matrix $S^{(0q)}$ of $K^{(q)}$ and a matrix $\gamma^{(q)}(1;2)$ with the diagonal elements [Eq. (2.23b)]

$$\gamma_a^{(q)}(1;2) = (2ik_a)^{-1} \{ [M_a^{(q)}(1)]^* - M_a^{(q)}(2) \} , \qquad (2.56)$$

by

$$\frac{\partial}{\partial \hat{\mathbf{x}}} \cdot \hat{\boldsymbol{\alpha}}(\hat{\mathbf{x}}) U S^{(0q)} = k \gamma^{(q)}(\hat{\mathbf{x}}) , \qquad (2.57)$$

and hence, in terms of $\mathcal{J}^{(0q)}$ of (2.48a), by

$$\frac{\partial}{\partial \hat{\mathbf{x}}} \cdot \hat{\boldsymbol{\alpha}} \mathcal{J}^{(0q)}(\hat{\mathbf{x}}) = k \gamma^{(q)} U(\hat{\mathbf{x}}) , \qquad (2.58)$$

meaning the power equation for the incoherent wave $\mathcal{J}^{(0q)}$. The proof of Eq. (2.57) is given by eliminating the factor $\Delta G^{(0)}(\hat{\mathbf{x}})$ in Eq. (2.24) by using the power equation for U in unbounded space, which is

$$\left[\frac{\partial}{\partial \hat{\mathbf{x}}} \cdot \hat{\boldsymbol{\alpha}} + k \gamma^{(q)} \right] U(\hat{\mathbf{x}}) = \Delta G^{(0)}(\hat{\mathbf{x}}) , \qquad (2.59)$$

in view of (2.36); hence

$$\frac{\partial}{\partial \hat{\mathbf{x}}} \cdot \hat{\boldsymbol{\alpha}}(\hat{\mathbf{x}}) U K^{(q)} = k \gamma^{(q)}(\hat{\mathbf{x}}) (1 - U K^{(q)}) , \qquad (2.60)$$

which leads directly to Eq. (2.57) in consequence of Eq. (2.43). The sum of Eqs. (2.58) and (2.59) reproduces power equation (2.23e) when the boundary is absent.

We now write $I^{(q+12)}$ of Eq. (2.45) by

$$I^{(q+12)} = U + U \sigma^{(q+12)} U \qquad (2.61)$$

in the same form as Eqs. (2.55), with

$$\sigma^{(q+12)} = \sigma^{(12)} + (1 + \sigma^{(12)} U) S^{(q/12)} (U \sigma^{(12)} + 1) , \qquad (2.62)$$

and consider the case of a semi-infinite random layer where $q_1 = 0$ and $q_2 \neq 0$. Hence

$$\sigma_{11}^{(q+12)} = \sigma_{11}^{(12)} + \sigma_{12}^{(12)} \mathcal{J}_{22}^{(q/12)} \sigma_{21}^{(12)} , \qquad (2.63)$$

and means a resultant scattering matrix for the entire random volume (making both boundary and medium scattering) of when the source and the observer are both in the space k_1. To find the optical condition for $\sigma_{11}^{(q+12)}$, we only observe that the integrated power of $I^{(q+12)}$ away from the boundary S_1 is always zero because all the waves propagated into the space k_2 are finally scattered back to the space k_1. Hence

$$\int_{S_1} d\rho \, \alpha_n^{(1)} I_{11}^{(q+12)}(\hat{\mathbf{x}}) = 0 , \qquad (2.64)$$

where

$$\alpha_n^{(j)} = \hat{\mathbf{n}}^{(j)} \cdot \hat{\boldsymbol{\alpha}} . \qquad (2.65)$$

For short, we express Eq. (2.64) by the notation

$$\langle S_1 | I_{11}^{(q+12)}(1;2) = 0 , \qquad (2.66)$$

and hence, using expression (2.61),

$$\langle S_1 | (U_1 + U_1 \sigma_{11}^{(q+12)} U_1) = 0 . \qquad (2.67)$$

With the same notation, we can write the corresponding conditions for $I^{(12)}$ by

$$\langle S_1 | I_{1j}^{(12)} + \langle S_2 | I_{2j}^{(12)} = 0, \quad j = 1,2 \qquad (2.68)$$

which is the surface integral over $S_{12} = S_1 + S_2$ enclosing the whole boundary space (Fig. 2), and $j = 1$ or 2 depending on whether the source is in the space k_1 or k_2, respectively. Thus, using (2.38) in (2.68), we obtain [see also Eqs. (3.43) and (3.44)]

$$\langle S_1 | (U_1 + U_1 \sigma_{11}^{(12)} U_1) + \langle S_2 | U_2 \sigma_{21}^{(12)} U_1 = 0 , \qquad (2.69a)$$

$$\langle S_1 | U_1 \sigma_{12}^{(12)} U_2 + \langle S_2 | (U_2 + U_2 \sigma_{22}^{(12)} U_2) = 0 , \qquad (2.69b)$$

as basic conditions for $\sigma^{(12)}$.

To find the corresponding equation for the medium counterpart $S_{22}^{(q/12)}$, we substitute expression (2.63) in (2.67) and use Eq. (2.69a) to eliminate the $\sigma_{11}^{(12)}$ term; hence, as is proved below,

$$\langle S_1 | U_1 \sigma_{12}^{(12)} \mathcal{J}_{22}^{(q/12)} \sigma_{21}^{(12)} U_1 = \langle S_2 | U_2 \sigma_{21}^{(12)} U_1 , \qquad (2.70)$$

where involved in the left-hand side are only the boundary values of $\mathcal{J}_{22}^{(q/12)}$. This simply says that the entire incident power through the boundary, as given by the right-hand side, is the same as the integrated power of the waves scattered back to the space k_1 by the random medium. The direct proof of Eq. (2.70) by using the basic equation (2.47) for $\mathcal{J}^{(q/12)}$ is straightforward, first by making the space integration of Eq. (2.58) over the space $z \leq 0$

$(q_2 \neq 0)$ for the source located outside $(z>0)$ to obtain

$$\langle S_2 | \mathcal{J}_{22}^{(0q)} = -\langle S_2 | U_2 \tag{2.71}$$

with the aid of Eq. (2.59); hence, using Eq. (2.47),

$$\langle S_2 | \mathcal{J}_{22}^{(q/12)} = -\langle S_2 | U_2 (1 + \sigma_{22}^{(12)} \mathcal{J}_{22}^{(q/12)}) , \tag{2.72}$$

which can be rewritten, on using Eq. (2.69b), as

$$\langle S_1 | U_1 \sigma_{12}^{(12)} \mathcal{J}_{22}^{(q/12)} = \langle S_2 | U_2 , \tag{2.73}$$

being the same equation as (2.70) except for the common factor $\sigma_{21}^{(12)} U_1$ on both sides.

In the case of a random layer where $q_1 = q_3 = 0$, $q_2 \neq 0$ (Fig. 1), Eq. (2.73) is replaced by

$$[\langle S_1 | U_1 \sigma_{12}^{(12)} + \langle S_3 | U_3 \sigma_{32}^{(23)}] \mathcal{J}_{22}^{(q/12+23)} = \langle S_2 | U_2 . \tag{2.74}$$

Here the boundary S_2 on the right-hand side refers to either S_{+2} at $z=0$ or S_{-2} at $z=-L$. As to the scattering matrix for the entire system, $\sigma_{ab}^{(q+12+23)}$, defined by Eqs. (2.55), the optical conditions are the same as given by Eqs. (2.69) with the replacement of $\sigma^{(12)} \rightarrow \sigma^{(q+12+23)}$ and $S_2 \rightarrow S_3$, e.g.,

$$\langle S_1 | (U_1 + U_1 \sigma_{11}^{(q+12+23)} U_1) + \langle S_3 | U_3 \sigma_{31}^{(q+12+23)} U_1 = 0 , \tag{2.75}$$

which can be shown also directly by using Eqs. (2.53), (2.74), and (2.69).

Here it may be noticed that, in the case of a random volume, the optical conditions of scattering matrices are described exclusively by the boundary values, as given by Eqs. (2.73) and (2.74), independent of the details of the inside [see Eq. (3.45) with (3.39b) and (3.41b)].

III. OPTICAL EXPRESSIONS AND CROSS SECTIONS

A specific expression of Eqs. (2.54) is obtained in optical form by partially making the Fourier transformation. We first introduce relative coordinates $\hat{\mathbf{r}}$ and $\hat{\rho}$, defined by

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}_2 - \hat{\mathbf{x}}_1, \quad \hat{\rho} = \tfrac{1}{2}(\hat{\mathbf{x}}_2 + \hat{\mathbf{x}}_1) , \tag{3.1}$$

and the corresponding Fourier variables $\hat{\mathbf{u}}$ and $\hat{\boldsymbol{\lambda}}$, defined by

$$\hat{\mathbf{u}} = \tfrac{1}{2}(\hat{\boldsymbol{\lambda}}_2 + \hat{\boldsymbol{\lambda}}_1), \quad \hat{\boldsymbol{\lambda}} = \hat{\boldsymbol{\lambda}}_2 - \hat{\boldsymbol{\lambda}}_1 , \tag{3.2}$$

so that

$$-\hat{\boldsymbol{\lambda}}_1 \cdot \hat{\mathbf{x}}_1 + \hat{\boldsymbol{\lambda}}_2 \cdot \hat{\mathbf{x}}_2 = \hat{\mathbf{u}} \cdot \hat{\mathbf{r}} + \hat{\boldsymbol{\lambda}} \cdot \hat{\rho} . \tag{3.3}$$

Then the matrix elements of $K^{(q)}$ can be written in the form

$$K^{(q)}(\hat{\mathbf{x}}_1 ; \hat{\mathbf{x}}_2 | \hat{\mathbf{x}}_1' ; \hat{\mathbf{x}}_2') = K^{(q)}(\hat{\mathbf{r}} | \hat{\rho} - \hat{\rho}' | \hat{\mathbf{r}}') , \tag{3.4}$$

in view of the translational invariance, approximately in the vertical direction, and, therefore, the Fourier transform has the form

$$\tilde{K}^{(q)}(\hat{\boldsymbol{\lambda}}_1 ; \hat{\boldsymbol{\lambda}}_2 | \hat{\boldsymbol{\lambda}}_1' ; \hat{\boldsymbol{\lambda}}_2') = (2\pi)^3 \delta(\hat{\boldsymbol{\lambda}} - \hat{\boldsymbol{\lambda}}') \tilde{K}^{(q)}(\hat{\mathbf{u}} | \hat{\boldsymbol{\lambda}} | \hat{\mathbf{u}}') . \tag{3.5}$$

On the other hand, the corresponding Fourier transforms of the wave quantities, e.g., $\tilde{\mathcal{J}}^{(q/12)}(\hat{\mathbf{u}}, \hat{\boldsymbol{\lambda}} | \hat{\mathbf{u}}', \hat{\boldsymbol{\lambda}}')$ of $\mathcal{J}^{(q/12)}$ where $\hat{\boldsymbol{\lambda}} = (\boldsymbol{\lambda}, \lambda_z)$, cannot be written in the same form, and, therefore, on suppressing $\boldsymbol{\lambda}$ and dropping the factor $(2\pi)^2 \delta(\boldsymbol{\lambda} - \boldsymbol{\lambda}')$, we will hereafter use a composite expression, e.g., $\mathcal{J}^{(q/12)}(\hat{\mathbf{u}}, z | \hat{\mathbf{u}}', z')$, by making the Fourier inversion only with respect to λ_z and λ_z'.

As to the transform $\tilde{U}_a$ of U_a, we obtain

$$\begin{aligned} \tilde{U}_a(\hat{\mathbf{u}}, \hat{\boldsymbol{\lambda}}) &= \tilde{G}_a^*(\hat{\mathbf{u}} - \tfrac{1}{2}\hat{\boldsymbol{\lambda}}) \tilde{G}_a(\hat{\mathbf{u}} + \tfrac{1}{2}\hat{\boldsymbol{\lambda}}) \\ &\simeq \pi \delta(\hat{\mathbf{u}}^2 - k_a^2)(k_a \gamma_a - i \hat{\mathbf{u}} \cdot \hat{\boldsymbol{\lambda}})^{-1} , \end{aligned} \tag{3.6}$$

with the approximation $k_a^{(M)} \simeq k_a$, $k_a \gg |\hat{\boldsymbol{\lambda}}| \gtrsim \gamma_a$, and the constant

$$\gamma_a = (2ik_a)^{-1} (\tilde{M}_a^{(q)*} - \tilde{M}_a^{(q)})(\hat{\mathbf{u}}) , \tag{3.7}$$

excluding the case when the medium is intrinsically dispersive.[11]

Hence, on changing the variable $\hat{\mathbf{u}}$ by $\hat{\mathbf{u}} = u \hat{\boldsymbol{\Omega}}$, $d\hat{\mathbf{u}} = u^2 du \, d\hat{\boldsymbol{\Omega}}$, where $u = |\hat{\mathbf{u}}|$ and $\hat{\boldsymbol{\Omega}} = (\boldsymbol{\Omega}, \Omega_z)$, $\hat{\boldsymbol{\Omega}}^2 = 1$, is the unit vector, we obtain an important relation that, for any slowly changing function $f(\hat{\mathbf{u}})$,

$$(2\pi)^{-3} \int d\hat{\mathbf{u}} \, \tilde{U}_a(\hat{\mathbf{u}}, \hat{\boldsymbol{\lambda}}) f(\hat{\mathbf{u}}) = \int_{4\pi} d\hat{\boldsymbol{\Omega}} \, \tilde{U}_a(\hat{\boldsymbol{\Omega}}, \hat{\boldsymbol{\lambda}}) f(\hat{\boldsymbol{\Omega}}) , \tag{3.8}$$

where

$$\tilde{U}_a(\hat{\boldsymbol{\Omega}}, \hat{\boldsymbol{\lambda}}) = (\gamma_a - i \hat{\boldsymbol{\Omega}} \cdot \hat{\boldsymbol{\lambda}})^{-1} = \tilde{U}_a(-\hat{\boldsymbol{\Omega}}, -\hat{\boldsymbol{\lambda}}) , \tag{3.9}$$

$$f(\hat{\boldsymbol{\Omega}}) = (4\pi)^{-2} f(\hat{\mathbf{u}} = k_a \hat{\boldsymbol{\Omega}}) . \tag{3.10}$$

Here the λ_z Fourier inversion of $\tilde{U}_a(\hat{\boldsymbol{\Omega}}, \hat{\boldsymbol{\lambda}})$ is

$$\begin{aligned} U_a(\hat{\boldsymbol{\Omega}}, z) &\equiv \frac{1}{2\pi} \int d\lambda_z \exp(-i\lambda_z z) \tilde{U}_a(\hat{\boldsymbol{\Omega}}, \hat{\boldsymbol{\lambda}}) \\ &= \begin{cases} |\Omega_z|^{-1} \exp[-\Omega_z^{-1}(\gamma_a - i \boldsymbol{\Omega} \cdot \boldsymbol{\lambda}) z], & \Omega_z z > 0 \\ 0, & \Omega_z z < 0 \end{cases} \end{aligned} \tag{3.11}$$

while the three-dimensional inversion $U_a(\hat{\boldsymbol{\Omega}}, \hat{\rho})$ is given by

$$U_a(\hat{\boldsymbol{\Omega}}, \hat{\rho}) = |\hat{\rho}|^{-2} \exp(-\gamma_a |\hat{\rho}|) \delta^2(\hat{\boldsymbol{\Omega}} - \hat{\rho}/|\hat{\rho}|) . \tag{3.12}$$

Hence, with the rule (3.8), Eq. (2.38), for example, leads to the expression

$$\begin{aligned} I_{ab}^{(12)}(\hat{\boldsymbol{\Omega}}, z | \hat{\boldsymbol{\Omega}}', z') &= U_a(\hat{\boldsymbol{\Omega}}, z - z') \delta_{ab} \delta^2(\hat{\boldsymbol{\Omega}} - \hat{\boldsymbol{\Omega}}') \\ &\quad + U_a(\hat{\boldsymbol{\Omega}}, z) \sigma_{ab}^{(12)}(\hat{\boldsymbol{\Omega}} | \hat{\boldsymbol{\Omega}}') U_b(\hat{\boldsymbol{\Omega}}', -z') . \end{aligned} \tag{3.13}$$

Here, from Eqs. (2.39),[7,8]

$$\begin{aligned} \sigma_{ab}^{(12)}(\hat{\boldsymbol{\Omega}} | \hat{\boldsymbol{\Omega}}') = {} & |\Omega_z^{(a)} \langle R_{ab}^{(12)}(\hat{\boldsymbol{\Omega}}) \rangle|^2 \delta_S^2(\hat{\boldsymbol{\Omega}}^{(a)} - \hat{\boldsymbol{\Omega}}^{(a)\prime}) \\ & + \sigma_{ab}^{(I12)}(\hat{\boldsymbol{\Omega}} | \hat{\boldsymbol{\Omega}}') , \end{aligned} \tag{3.14}$$

where $\hat{\boldsymbol{\Omega}}^{(a)} = (\boldsymbol{\Omega}^{(a)}, \Omega_z^{(a)})$ denotes the unit vector in space k_a, defined by $\mathbf{u} = k_a \boldsymbol{\Omega}^{(a)}$ and $\Omega_z^{(a)} = \pm[1 - (\boldsymbol{\Omega}^{(a)})^2]^{1/2}$, and δ_S^2 is a specular δ function of $\hat{\boldsymbol{\Omega}}^{(a)}$ which is not zero only when the scattering is made in the specular direction, regardless of the sign of $\Omega_z^{(a)}$; $\sigma_{ab}^{(I12)}$ is the incoherent cross section, given in terms of the Fourier transform $\tilde{S}_{ij;kl}^{(12)}$ (with the full subscripts), by

$$\sigma_{ab}^{(I12)}(\hat{\Omega}|\hat{\Omega}')=\sum_{i,j,k,l}(4\pi)^{-2}\bar{F}_{aa;ij}(\hat{\mathbf{u}})\tilde{S}_{ij;kl}^{(12)}(\hat{\mathbf{u}}|\boldsymbol{\lambda}=0|\hat{\mathbf{u}}')F_{kl;bb}(\hat{\mathbf{u}}')|_{\Omega} \quad (3.15)$$

where the notation $|_{\Omega}$ means setting $\mathbf{u}=k_a\boldsymbol{\Omega}^{(a)}$ and $\mathbf{u}'=k_b\boldsymbol{\Omega}^{(b)}$, and the $\boldsymbol{\lambda}$ dependence has been neglected.

The transform $\sigma_{ab}^{(12)}(\hat{\Omega}|\hat{\Omega}')$ provides the resultant (including both coherent and incoherent) cross section per unit area of the boundary and is subject to an optical relation resulting from relations (2.69) [see Eq. (3.44)], as

$$\sum_{a=1}^{2}\int_{2\pi}d\hat{\Omega}'k_a\sigma_{ab}^{(12)}(\hat{\Omega}'|\hat{\Omega})=-k_b\Omega_n^{(b)}>0\ , \quad (3.16)$$

where $\Omega_n^{(b)}=\hat{\mathbf{n}}^{(b)}\cdot\hat{\Omega}$. The expression (3.13) should be regarded as the matrix elements of $I^{(12)}$ with respect to $\hat{\Omega}$ and z, to be multiplied with a $\hat{\Omega}$-z vector $f(\hat{\mathbf{n}},z)$, defined by Eq. (3.10).

In the same way, the $\hat{\Omega}$-z expression of Eq. (2.52) gives

$$\mathcal{J}_{22}^{(q/12+23)}(\hat{\Omega},z|\hat{\Omega}',z')=\mathcal{J}_2^{(0q)}(\hat{\Omega},z|\hat{\Omega}',z')+\int d\hat{\Omega}''\int d\hat{\Omega}'''\mathcal{J}_2^{(0q)}(\hat{\Omega},z|\hat{\Omega}'',0)\sigma_{22}^{(12)}(\hat{\Omega}''|\hat{\Omega}''')\mathcal{J}_{22}^{(q/12+23)}(\hat{\Omega}''',0|\hat{\Omega}',z')$$
$$+\int d\hat{\Omega}''\int d\hat{\Omega}'''\mathcal{J}_2^{(0q)}(\hat{\Omega},z|\hat{\Omega}'',-L)\sigma_{22}^{(23)}(\hat{\Omega}''|\hat{\Omega}''')\mathcal{J}_{22}^{(q/12+23)}(\hat{\Omega}''',-L|\hat{\Omega}',z')\ . \quad (3.17)$$

Here the original $\mathcal{J}_2^{(0q)}$ is the solution of Eq. (2.54b), and its $\hat{\Omega}$-z version can be obtained as the solution of the transport equation

$$(\gamma_2+\Omega_z\partial_z)\mathcal{J}_2^{(0q)}(\hat{\Omega},z|\hat{\Omega}',z')=K_2^{(q)}(\hat{\Omega}|\hat{\Omega}')U_2(\hat{\Omega}',z-z')+\int d\hat{\Omega}''K_2^{(q)}(\hat{\Omega}|\hat{\Omega}'')\mathcal{J}_2^{(0q)}(\hat{\Omega}'',z|\hat{\Omega}',z')\ , \quad (3.18)$$

wherein $\partial_z=\partial/\partial z$ and the term $-i\boldsymbol{\Omega}\cdot\boldsymbol{\lambda}$ has been suppressed. Here $z,z'<0$ in the present case, and, from Eq. (3.10),

$$K_2^{(q)}(\hat{\Omega}|\hat{\Omega}')=(4\pi)^{-2}\tilde{K}_2^{(q)}(\hat{\mathbf{u}}=k_2\hat{\Omega}|\hat{\mathbf{u}}'=k_2\hat{\Omega}') \quad (3.19a)$$
$$=K_2^{(q)}(-\hat{\Omega}'|-\hat{\Omega}) \quad (3.19b)$$

is the cross section per unit volume subject to the optical relation

$$\gamma_2=\int_{4\pi}d\hat{\Omega}'K_2^{(q)}(\hat{\Omega}'|\hat{\Omega}) \quad (3.20)$$

from the $\hat{\mathbf{x}}$-integrated (2.24); the solution is subject to the condition of no reflection at the boundary, in view of the factor $U_2(\hat{\Omega},z)$ of (3.11), i.e.,

$$\mathcal{J}_2^{(0q)}(\hat{\Omega},z=0|\hat{\Omega}',z'<0)=0,\quad \Omega_z<0\ , \quad (3.21a)$$

and fulfills the reciprocity

$$\mathcal{J}_2^{(0q)}(-\hat{\Omega}',z'|-\hat{\Omega},z)=\mathcal{J}_2^{(0q)}(\hat{\Omega},z|\hat{\Omega}',z')\ . \quad (3.21b)$$

With known $\mathcal{J}_2^{(0q)}$, $\mathcal{J}_{22}^{(q/12+23)}$ is obtained as the solution of integral equation (3.17), and thereby $I_{ab}^{(q+12+23)}$ is given from Eqs. (2.53), by

$$I_{11}^{(q+12+23)}(\hat{\Omega},z|\hat{\Omega}',z')=I_{11}^{(12)}(\hat{\Omega},z|\hat{\Omega}',z')+\int d\hat{\Omega}''\int d\hat{\Omega}'''U_1(\hat{\Omega},z)\sigma_{12}^{(12)}(\hat{\Omega}|\hat{\Omega}'')\mathcal{J}_{22}^{(q/12+23)}(\hat{\Omega}'',0|\hat{\Omega}''',0)$$
$$\times\sigma_{21}^{(12)}(\hat{\Omega}'''|\hat{\Omega}')U_1(\hat{\Omega}',-z')\ , \quad (3.22)$$

and a similar equation for $I_{31}^{(q+12+23)}$, while the three-dimensional expression of Eq. (3.22) is

$$I_{11}^{(q+12+23)}(\hat{\Omega},\hat{\rho}|\hat{\Omega}',\hat{\rho}')=I_{11}^{(12)}(\hat{\Omega},\hat{\rho}|\hat{\Omega}',\hat{\rho}')+\int d\rho'd\rho'''\int d\hat{\Omega}''d\hat{\Omega}'''U_1(\hat{\Omega},\hat{\rho}-\rho'')\sigma_{12}^{(12)}(\hat{\Omega}|\hat{\Omega}'')$$
$$\times\mathcal{J}_{22}^{(q/12+23)}(\hat{\Omega}'',z''=0|\rho''-\rho'''|\hat{\Omega}''',z'''=0)$$
$$\times\sigma_{21}^{(12)}(\hat{\Omega}'''|\hat{\Omega}')U_1(\hat{\Omega}',\rho'''-\hat{\rho}')\ . \quad (3.23a)$$

Here $U_a(\hat{\Omega},\hat{\rho})$ is given by Eq. (3.12), $\mathcal{J}_{22}^{(q/12+23)}(\hat{\Omega},z|\rho|\hat{\Omega}',z')$ is the $\boldsymbol{\lambda}$ Fourier inversion of $\mathcal{J}_{22}^{(q/12+23)}(\hat{\Omega},z|\boldsymbol{\lambda}|\hat{\Omega}',z')$, and from (3.13),

$$I_{11}^{(12)}(\hat{\Omega},\hat{\rho}|\hat{\Omega}',\hat{\rho}')=U_1(\hat{\Omega},\hat{\rho}-\hat{\rho}')\delta^2(\hat{\Omega}-\hat{\Omega}')+\int d\rho''U_1(\hat{\Omega},\hat{\rho}-\rho'')\sigma_{11}^{(12)}(\hat{\Omega}|\hat{\Omega}')U_1(\hat{\Omega}',\rho''-\hat{\rho}')\ . \quad (3.23b)$$

The average power at a point $\hat{\rho}$ in space k_a for the wave from a point source at $\hat{\rho}'$ in space k_b, say $\langle\hat{\mathbf{w}}_{ab}(\hat{\rho}|\hat{\rho}')\rangle$, is given according to the definitions (2.2), by

$$\langle\hat{\mathbf{w}}_{ab}(\hat{\rho}|\hat{\rho}')\rangle=i\frac{\partial}{\partial\hat{\mathbf{r}}}I_{ab}(\hat{\mathbf{r}},\hat{\rho}|\hat{\mathbf{r}}',\hat{\rho}')|_{\hat{\mathbf{r}}=\hat{\mathbf{r}}'=0}=(4\pi)^{-2}\int d\hat{\Omega}\int d\hat{\Omega}'k_a\hat{\Omega}^{(a)}I_{ab}(\hat{\Omega},\hat{\rho}|\hat{\Omega}',\hat{\rho}') \quad (3.24)$$

in terms of the relative coordinates $\hat{\mathbf{r}}$ of (3.1) and the optical expression of I_{ab} according to Eq. (3.10).

A. Cross section of a random layer with no boundary scattering

When the boundaries make no scattering so that

$$\sigma_{22}^{(12)}=\sigma_{22}^{(23)}=0\ ,\qquad \sigma_{12}^{(12)}(\hat{\Omega}|\hat{\Omega}')=|\Omega_z|\delta^2(\hat{\Omega}-\hat{\Omega}')\ , \tag{3.25}$$

etc., $\mathcal{J}_{22}^{(q/12+23)}$ is reduced to $\mathcal{J}_2^{(0q)}$, in view of (3.17), and hence Eq. (3.23a) is reduced to

$$I_{11}^{(q+12+23)}(\hat{\Omega},\hat{\rho}|\hat{\Omega}',\hat{\rho}')=U_1(\hat{\Omega},\hat{\rho}-\hat{\rho}')\delta^2(\hat{\Omega}-\hat{\Omega}')+\mathcal{J}_2^{(0q)}(\hat{\Omega},\hat{\rho}|\hat{\Omega}',\hat{\rho}')\ . \tag{3.26}$$

Here the last term is a space continuation of $\mathcal{J}_2^{(0q)}$ (originally defined in the space of $q_2\neq 0$) to the outside space k_1, and is obtained in view of (3.25) by replacing U in (2.48a) with

$$U_{12}(\hat{\rho}|\hat{\Omega}|\hat{\rho}')\equiv\int_{-\infty}^{\infty}d\rho''|\Omega_z|U_1(\hat{\Omega},\hat{\rho}-\rho'')\times U_2(\hat{\Omega},\rho''-\hat{\rho}')\ , \tag{3.27}$$

which, by using (3.11), can be shown to be the same as given by (3.12) except for the exponential factor which is replaced by $\exp(\gamma_2|\Omega_z|^{-1}z')$, $z'<0$, yielding a dissipation only in the range of $q_2\neq 0$. Hence, from (2.48a),

$$\mathcal{J}_2^{(0q)}(\hat{\Omega},\hat{\rho}|\hat{\Omega}',\hat{\rho}')=\int d\hat{\rho}''d\hat{\rho}'''U_{12}(\hat{\rho}|\hat{\Omega}|\hat{\rho}'')\times S_2^{(0q)}(\hat{\Omega},\hat{\rho}''|\hat{\Omega}',\hat{\rho}''')U_{21}(\hat{\rho}'''|\hat{\Omega}'|\hat{\rho}')\ , \tag{3.28}$$

where the integration is made over the entire space of the medium $q_2\neq 0$.

Here we suppose that the points $\hat{\rho}$ and $\hat{\rho}'$ are both separated enough from the boundary so that the total incoherent intensity, $\mathcal{J}_2^{(0q)}(\hat{\rho}|\hat{\rho}')$, say, given by

$$\mathcal{J}_2^{(0q)}(\hat{\rho}|\hat{\rho}')=\int d\hat{\Omega}\,d\hat{\Omega}'\mathcal{J}_2^{(0q)}(\hat{\Omega},\hat{\rho}|\hat{\Omega}',\hat{\rho}')\ , \tag{3.29}$$

can be written, on using the relative coordinates $\Delta\rho$ and $\bar{\rho}$ defined by

$$\Delta\rho=\rho''-\rho'''\ ,\qquad \bar{\rho}=\tfrac{1}{2}(\rho''+\rho''')\ , \tag{3.30a}$$

$$d\rho''d\rho'''=d(\Delta\rho)d\bar{\rho}\ ,\qquad \hat{\rho}''=(\bar{\rho}+\tfrac{1}{2}\Delta\rho,z'')\ ,\qquad \hat{\rho}'''=(\bar{\rho}-\tfrac{1}{2}\Delta\rho,z''')\ , \tag{3.30b}$$

in the asymptotic form

$$\mathcal{J}_2^{(0q)}(\hat{\rho}|\hat{\rho}')=\int_{-\infty}^{\infty}d\bar{\rho}|\hat{\rho}-\bar{\rho}|^{-2}S_{+2,+2}^{(0q)}(\hat{\Omega}|\bar{\rho}|\hat{\Omega}')\times|\bar{\rho}-\hat{\rho}'|^{-2}\ . \tag{3.31}$$

Here $\hat{\Omega}$ and $\hat{\Omega}'$ are in the directions of $\hat{\rho}-\bar{\rho}$ and $\bar{\rho}-\hat{\rho}'$, respectively, and $S_{+2,+2}^{(0q)}(\hat{\Omega}|\bar{\rho}|\hat{\Omega}')$ means the scattering cross section per unit area of the layer boundary at $\bar{\rho}$ for scattering of the wave in direction $\hat{\Omega}'$ to $\hat{\Omega}$, and is given, upon using

$$\int_{-\infty}^{\infty}d(\Delta\rho)S_2^{(0q)}(\hat{\Omega},\hat{\rho}''|\hat{\Omega}',\hat{\rho}''')=S_2^{(0q)}(\hat{\Omega},z''|\boldsymbol{\lambda}=0|\hat{\Omega}',z''')\ ,$$

by

$$S_{+2,+2}^{(0q)}(\hat{\Omega}|\bar{\rho}|\hat{\Omega}')=\tilde{S}_{+2,+2}^{(0q)}(\hat{\Omega},\lambda_z|\boldsymbol{\lambda}=0|\hat{\Omega}',\lambda_z')\ , \tag{3.32}$$

where the right-hand side is a Fourier transform defined by

$$\tilde{S}_{+2,+2}^{(0q)}(\hat{\Omega},\lambda_z|\boldsymbol{\lambda}|\hat{\Omega}',\lambda_z')=\int_{-L}^{0}dz\,dz'\exp(i\lambda_z z-i\lambda_z'z')\times S_2^{(0q)}(\hat{\Omega},z|\boldsymbol{\lambda}|\hat{\Omega}',z')\ , \tag{3.33}$$

with

$$\lambda_z=-i(\gamma_2-i\boldsymbol{\Omega}\cdot\boldsymbol{\lambda})\Omega_z^{-1},\quad \Omega_z>0 \tag{3.34a}$$

$$\lambda_z'=-i(\gamma_2-i\boldsymbol{\Omega}'\cdot\boldsymbol{\lambda})(\Omega_z')^{-1},\quad \Omega_z'<0\ , \tag{3.34b}$$

so that Eq. (3.33) is a function of $\hat{\Omega}$, $\hat{\Omega}'$, and $\boldsymbol{\lambda}$.

Equations (3.34) and (3.2) indicate that the Fourier variables $\hat{\boldsymbol{\lambda}}_1$ and $\hat{\boldsymbol{\lambda}}_2$ are given, when $\boldsymbol{\lambda}=0$, by

$$\hat{\boldsymbol{\lambda}}_1=k_2\hat{\Omega}+\tfrac{1}{2}i\gamma_2(\Omega_z)^{-1}\hat{\mathbf{n}}_z\ , \tag{3.35a}$$

$$\hat{\boldsymbol{\lambda}}_2=k_2\hat{\Omega}-\tfrac{1}{2}i\gamma_2(\Omega_z)^{-1}\hat{\mathbf{n}}_z\ , \tag{3.35b}$$

and, similarly,

$$\hat{\boldsymbol{\lambda}}_1'\equiv-\hat{\boldsymbol{\lambda}}_3=k_2\hat{\Omega}'+\tfrac{1}{2}i\gamma_2(\Omega_z')^{-1}\hat{\mathbf{n}}_z\ , \tag{3.35c}$$

$$\hat{\boldsymbol{\lambda}}_2'\equiv-\hat{\boldsymbol{\lambda}}_4=k_2\hat{\Omega}'-\tfrac{1}{2}i\gamma_2(\Omega_z')^{-1}\ , \tag{3.35d}$$

and Eq. (3.32) says that the cross section $S_{+2,+2}^{(0q)}(\hat{\Omega}|\hat{\Omega}')\equiv S_{+2,+2}^{(0q)}(\hat{\Omega}|\bar{\rho}|\hat{\Omega}')$ is the same as given by the full Fourier transform $\tilde{S}_2^{(0q)}(\hat{\boldsymbol{\lambda}}_1;\hat{\boldsymbol{\lambda}}_2|\hat{\boldsymbol{\lambda}}_1';\hat{\boldsymbol{\lambda}}_2')$ except for the factor $(2\pi)^2\delta(\boldsymbol{\lambda}-\boldsymbol{\lambda}')$.

B. Case involving the boundary scattering

The optical expression of $I_{11}^{(q+12+23)}$ can be written, on using Eqs. (2.55a) and (3.23a), in the form

$$I_{11}^{(q+12+23)}(\hat{\Omega},\hat{\rho}|\hat{\Omega}',\hat{\rho}')=U_1(\hat{\Omega},\hat{\rho}-\hat{\rho}')\delta^2(\hat{\Omega}-\hat{\Omega}')+\int d\rho''d\rho'''U_1(\hat{\Omega},\hat{\rho}-\rho'')\sigma_{11}^{(q+12+23)}(\hat{\Omega}|\rho''-\rho'''|\hat{\Omega}')\times U_1(\hat{\Omega}',\rho'''-\hat{\rho}')\ . \tag{3.36}$$

Here the second term is a surface integral on S_1, with

$$\sigma_{11}^{(q+12+23)}(\hat{\Omega}|\rho''-\rho'''|\hat{\Omega}')=\sigma_{11}^{(12)}(\hat{\Omega}|\hat{\Omega}')\delta(\rho''-\rho''')+\int d\hat{\Omega}''d\hat{\Omega}'''\sigma_{12}^{(12)}(\hat{\Omega}|\hat{\Omega}'')\mathcal{I}_{22}^{(q/12+23)}(\hat{\Omega}'',z''=0|\rho''-\rho'''|\hat{\Omega}''',z'''=0)\sigma_{21}^{(12)}(\hat{\Omega}'''|\hat{\Omega}') . \quad (3.37)$$

Hence, when the points $\hat{\rho}$ and $\hat{\rho}'$ are both separated enough from the boundary, the total intensity from Eq. (3.36), say $I_{11}^{(q+12+23)}(\hat{\rho}|\hat{\rho}')$, is obtained in the asymptotic form similar to (3.31), as

$$I_{11}^{(q+12+23)}(\hat{\rho}|\hat{\rho}')=|\hat{\rho}-\hat{\rho}'|^{-2}+\int d\bar{\rho}|\hat{\rho}-\bar{\rho}|^{-2}\sigma_{11}^{(q+12+23)}(\hat{\Omega}|\hat{\Omega}')|\bar{\rho}-\hat{\rho}'|^{-2} . \quad (3.38)$$

Here $\sigma_{11}^{(q+12+23)}(\hat{\Omega}|\hat{\Omega}')$ means the resultant cross section per unit area of the boundary at $\bar{\rho}$ due to the scattering by both the medium and the boundaries, and is given by

$$\sigma_{11}^{(q+12+23)}(\hat{\Omega}|\hat{\Omega}')\equiv\int_{-\infty}^{\infty}d(\Delta\rho)\sigma_{11}^{(q+12+23)}(\hat{\Omega}|\Delta\rho|\hat{\Omega}') \quad (3.39a)$$

$$=\sigma_{11}^{(12)}(\hat{\Omega}|\hat{\Omega}')+\int d\hat{\Omega}''d\hat{\Omega}'''\sigma_{12}^{(12)}(\hat{\Omega}|\hat{\Omega}'')|\Omega_z''|^{-1}S_{+2,+2}^{(q/12+23)}(\hat{\Omega}''|\hat{\Omega}''')|\Omega_z'''|^{-1}\sigma_{21}^{(12)}(\hat{\Omega}'''|\hat{\Omega}') , \quad (3.39b)$$

where, in the last term, use has been made of

$$\mathcal{I}_{22}^{(q/12+23)}(\hat{\Omega}'',z''=0|\hat{\Omega}''',z'''=0)=|\Omega_z''\Omega_z'''|^{-1}S_{+2,+2}^{(q/12+23)}(\hat{\Omega}''|\hat{\Omega}''') \quad (3.40a)$$

with

$$S_{+2,+2}^{(q/12+23)}(\hat{\Omega}''|\hat{\Omega}''')\equiv\tilde{S}_{+2,+2}^{(q/12+23)}(\hat{\Omega}'',\lambda_z''|\lambda=0|\hat{\Omega}''',\lambda_z''') , \quad (3.40b)$$

whose right-hand side is a Fourier transform defined by

$$\tilde{S}_{+2,+2}^{(q/12+23)}(\hat{\Omega},\lambda_z|\lambda|\hat{\Omega}',\lambda_z')=\int_{-L}^{0}dz\,dz'\exp(i\lambda_z z-i\lambda_z'z')\times S_{22}^{(q/12+23)}(\hat{\Omega},z|\lambda|\hat{\Omega}',z') \quad (3.40c)$$

similar to $\tilde{S}_{+2,+2}^{(0q)}$ by Eq. (3.33); these equations are obtained from an expression of $\mathcal{I}^{(q/12+23)}$ similar to (2.46) with the same procedure as that leading to Eqs. (3.32)–(3.34).

Also for the transmitted wave, we similarly obtain the asymptotic expression

$$I_{31}^{(q+12+23)}(\hat{\rho}|\hat{\rho}')=\int d\bar{\rho}|\hat{\rho}-\bar{\rho}|^{-2}\sigma_{31}^{(q+12+23)}(\hat{\Omega}|\hat{\Omega}')\times|\bar{\rho}-\hat{\rho}'|^{-2} , \quad (3.41a)$$

with

$$\sigma_{31}^{(q+12+23)}(\hat{\Omega}|\hat{\Omega}')=\int d\hat{\Omega}''d\hat{\Omega}'''\sigma_{32}^{(23)}(\hat{\Omega}|\hat{\Omega}'')|\Omega_z''|^{-1}S_{-2,+2}^{(q/12+23)}(\hat{\Omega}''|\hat{\Omega}''')|\Omega_z'''|^{-1}\sigma_{21}^{(12)}(\hat{\Omega}'''|\hat{\Omega}') , \quad (3.41b)$$

Here $\Omega_z'',\Omega_z'''<0$ and

$$S_{-2,+2}^{(q/12+23)}(\hat{\Omega}|\hat{\Omega}')=\int_{-L}^{0}dz\,dz'\exp[i\lambda_z(z+L)-i\lambda_z'z']S_{22}^{(q/12+23)}(\hat{\Omega},z|\lambda|\hat{\Omega}',z') , \quad (3.42)$$

where $\lambda=0$, and λ_z and λ_z' are defined by Eqs. (3.34) except that $\Omega_z<0$.

Summarizing, the scattered waves are described by the asymptotic expressions (3.38) and (3.41a) for the reflected and transmitted waves, respectively, in terms of the cross sections per unit area of the boundary surfaces, $\sigma_{ab}^{(q+12+23)}(\hat{\Omega}|\hat{\Omega}')$, $a,b=1,3$, which are composed of the entire contributions from both the medium and the boundaries. Here the medium scattering is manifested only through the $\hat{\Omega}$ matrices $S_{ab}^{(q/12+23)}(\hat{\Omega}|\hat{\Omega}')$, $a,b=\pm2$, and the resultant cross sections including the boundaries' can be constructed according to Eqs. (3.39b) and (3.41b), by the successive $\hat{\Omega}$-matrix multiplication of the boundary and medium matrices (that are involved in the equations on an equal basis) on following the order of the scatterings and with a weighting function $|\Omega_z|^{-1}$ when making the $\hat{\Omega}$ integration.

C. Optical conditions and reciprocity

The optical relation (3.16) for $\sigma^{(12)}$ is obtained from Eqs. (2.69) by rewriting them in the optical form, according to the definition of $\langle S_j|$ by Eqs. (2.64) and (2.66) and using Eq. (3.11); i.e., since the optical expression of $\langle S_1|U_1$ is

$$\int_{\Omega_\zeta<0}d\hat{\Omega}\,k_1\Omega_\zeta U_1(\hat{\Omega},\zeta=0)\delta^2(\hat{\Omega}-\hat{\Omega}')|_{\lambda=0}=-k_1 , \quad (3.43)$$

Eq. (2.69a) leads directly to

$$\int_{\Omega_z>0} d\hat{\Omega}\, k_1 \sigma_{11}^{(12)}(\hat{\Omega}|\hat{\Omega}') + \int_{\Omega_z<0} d\hat{\Omega}\, k_2 \sigma_{21}^{(12)}(\hat{\Omega}|\hat{\Omega}') = k_1 |\Omega_z'| \ . \quad (3.44)$$

In the same way, Eq. (2.75) leads to the optical relation for $\sigma_{ab}^{(q+12+23)}$, as

$$\int d\hat{\Omega}\, k_1 \sigma_{11}^{(q+12+23)}(\hat{\Omega}|\hat{\Omega}') + \int d\hat{\Omega}\, k_3 \sigma_{31}^{(q+12+23)}(\hat{\Omega}|\hat{\Omega}') = k_1 |\Omega_z'| \ . \quad (3.45)$$

The more detailed version of the relation is obtained from Eq. (2.74), which is expressed by

$$\int d\hat{\Omega}\, d\hat{\Omega}'' k_1 \sigma_{12}^{(12)}(\hat{\Omega}|\hat{\Omega}'')|\Omega_z''|^{-1} S_{+2,+2}^{(q/12+23)}(\hat{\Omega}''|\hat{\Omega}') + \int d\hat{\Omega}\, d\hat{\Omega}'' k_3 \sigma_{32}^{(23)}(\hat{\Omega}|\hat{\Omega}'')|\Omega_z''|^{-1} S_{-2,+2}^{(q/12+23)}(\hat{\Omega}''|\hat{\Omega}') = k_2 |\Omega_z^{(2)\prime}| \ , \quad (3.46)$$

in terms of the medium cross sections $S_{\pm 2,+2}^{(q/12+23)}(\hat{\Omega}|\hat{\Omega}')$ defined by Eqs. (3.40) and (3.42); hence the relation (3.45) is reproduced in consequence of Eqs. (3.39b), (3.41b), and (3.44). As for $S_{\pm 2,+2}^{(0q)}(\hat{\Omega}|\hat{\Omega}')$ where no boundary scattering is involved, we obtain

$$\int d\hat{\Omega}\, S_{+2,+2}^{(0q)}(\hat{\Omega}|\hat{\Omega}') + \int d\hat{\Omega}\, S_{-2,+2}^{(0q)}(\hat{\Omega}|\hat{\Omega}') = |\Omega_z'| \quad (3.47)$$

as a special case of (3.46).

Here the following reciprocity relations hold:

$$S_{+2,+2}^{(q/12+23)}(\hat{\Omega}|\hat{\Omega}') = S_{+2,+2}^{(q/12+23)}(-\hat{\Omega}'|-\hat{\Omega}) \ , \quad (3.48a)$$

$$S_{-2,+2}^{(q/12+23)}(\hat{\Omega}|\hat{\Omega}') = S_{+2,-2}^{(q/12+23)}(-\hat{\Omega}'|-\hat{\Omega}) \ , \quad (3.48b)$$

in consequence of the reciprocities

$$\sigma_{ab}^{(\,)}(\hat{\Omega}|\hat{\Omega}') = \sigma_{ba}^{(\,)}(-\hat{\Omega}'|-\hat{\Omega}) \ , \qquad K_2^{(q)}(\hat{\Omega}|\hat{\Omega}') = K_2^{(q)}(-\hat{\Omega}'|-\hat{\Omega}) \ , \quad (3.49a)$$

and

$$S_2^{(0q)}(-\hat{\Omega}',z'|-\lambda|-\hat{\Omega},z) = S_2^{(0q)}(\hat{\Omega},z|\lambda|\hat{\Omega}',z') \quad (3.49b)$$

from the definitions (3.32)–(3.34); and thereby the reciprocity of the cross section $\sigma_{ab}^{(q+12+23)}(\hat{\Omega}|\hat{\Omega}')$ for the entire system is ensured.

IV. STRUCTURE OF K

The deterministic Green's function is subject to the reciprocity $g(\hat{\mathbf{x}}|\hat{\mathbf{x}}') = g(\hat{\mathbf{x}}'|\hat{\mathbf{x}})$ or $g^T = g$, in view of $v^T = v$ in the governing equation (2.9b), and, therefore, not only the first-order Green's function subject to $G(\hat{\mathbf{x}}|\hat{\mathbf{x}}') = G(\hat{\mathbf{x}}'|\hat{\mathbf{x}})$, but also the second-order Green's function $I(\hat{\mathbf{x}}_1;\hat{\mathbf{x}}_2|\hat{\mathbf{x}}_1';\hat{\mathbf{x}}_2')$ should be invariant for each of the interchanges $\hat{\mathbf{x}}_1 \leftrightarrow \hat{\mathbf{x}}_1'$ and $\hat{\mathbf{x}}_2 \leftrightarrow \hat{\mathbf{x}}_2'$, independently.[9,3,4] It may be remarked that, with this simple symmetry alone, we can find a fundamental structure of the basic matrix K to a considerable extent without knowing the details of the specific medium involved.

We first consider the case of a homogeneous random medium q so that $K = K^{(q)}$, and introduce a four-coordinate function $\check{I}(\hat{\mathbf{x}}_1,\hat{\mathbf{x}}_2,\hat{\mathbf{x}}_3,\hat{\mathbf{x}}_4)$ defined by

$$\check{I}(\hat{\mathbf{x}}_1,\hat{\mathbf{x}}_2,\hat{\mathbf{x}}_3,\hat{\mathbf{x}}_4) = I(\hat{\mathbf{x}}_1;\hat{\mathbf{x}}_2|\hat{\mathbf{x}}_1' \to \hat{\mathbf{x}}_3;\hat{\mathbf{x}}_2' \to \hat{\mathbf{x}}_4) \ , \quad (4.1)$$

and similarly a four-coordinate function $\check{U}$ by

$$\check{U}(\hat{\mathbf{x}}_1,\hat{\mathbf{x}}_2,\hat{\mathbf{x}}_3,\hat{\mathbf{x}}_4) = G^*(\hat{\mathbf{x}}_1|\hat{\mathbf{x}}_3) G(\hat{\mathbf{x}}_2|\hat{\mathbf{x}}_4) \ . \quad (4.2)$$

Also we rewrite the matrix K in the BS equation (2.17) by K_{12} to make sure that it is a two-coordinate matrix with respect to $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ with the elements $K(\hat{\mathbf{x}}_1;\hat{\mathbf{x}}_2|\hat{\mathbf{x}}_1';\hat{\mathbf{x}}_2')$, on using the primed coordinates for the row; so that $K_{12}\check{I}$ represents

$$\int d\hat{\mathbf{x}}_1' d\hat{\mathbf{x}}_2' K(\hat{\mathbf{x}}_1;\hat{\mathbf{x}}_2|\hat{\mathbf{x}}_1';\hat{\mathbf{x}}_2') \check{I}(\hat{\mathbf{x}}_1',\hat{\mathbf{x}}_2',\hat{\mathbf{x}}_3,\hat{\mathbf{x}}_4) \ . \quad (4.3)$$

In the same way, U will be rewritten by U_{12} when using it in the original meaning, and the original I will likewise be rewritten by I_{12} whenever confusing. On the other hand, the matrix K_{12} also can be regarded as the four-coordinate function $\check{K}_{12}(\hat{\mathbf{x}}_1,\hat{\mathbf{x}}_2,\hat{\mathbf{x}}_3,\hat{\mathbf{x}}_4)$, defined in the same way as $\check{I}$ has been defined in terms of $I = I_{12}$ by Eq. (4.1).

Thus, BS equation (2.17) can be written by

$$\check{I} = \check{U} + U_{12} K_{12} \check{I} \ , \quad (4.4)$$

with the solution

$$\check{I} = \check{U} + U_{12} U_{34} \check{S} \ , \quad (4.5)$$

which represents

$$I = U + USU \quad (4.6)$$

in terms of the scattering matrix S of K, defined by

$$KI = SU, \quad IK = US \ , \quad (4.7)$$

and given as the solution of

$$S = K(1+US) = (1-KU)^{-1} K \quad (4.8a)$$

$$= K + KIK \ . \quad (4.8b)$$

Here U_{34} is the matrix when U is regarded as a matrix with respect to $\hat{\mathbf{x}}_3$ and $\hat{\mathbf{x}}_4$, say the $\hat{\mathbf{x}}_3$-$\hat{\mathbf{x}}_4$ matrix, with the elements

$$U(\hat{\mathbf{x}}_3;\hat{\mathbf{x}}_4|\hat{\mathbf{x}}_3';\hat{\mathbf{x}}_4') = G^*(\hat{\mathbf{x}}_3|\hat{\mathbf{x}}_3') G(\hat{\mathbf{x}}_4|\hat{\mathbf{x}}_4') \ , \quad (4.9)$$

and therefore commutable with U_{12}, i.e., $U_{12}U_{34} = U_{34}U_{12}$; a function $\check{U}_{34}$ also is defined by (4.9) with $\hat{\mathbf{x}}_3' \to \hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_4' \to \hat{\mathbf{x}}_2$, so that

$$\check{U} = \check{U}_{12} = \check{U}_{34} \ . \quad (4.10)$$

Hence Eqs. (4.8a) and (4.8b) can also be written by

$$\check{S} = \check{K}_{12} + K_{12} U_{12} \check{S} \quad (4.11a)$$

$$= \check{K}_{12} + K_{12} K_{34} \check{I} \ , \quad (4.11b)$$

in a form similar to Eq. (4.5).

Here we observe that the function $\check{I}$ is invariant against the interchange of $\hat{\mathbf{x}}_2$ and $\hat{\mathbf{x}}_4$, as we already remarked based on the reciprocity, and therefore $\check{I}\equiv\check{I}_{12}=\check{I}_{14}$, in view of the elements of the matrix I_{14} given by those of I_{12} with $\hat{\mathbf{x}}_2\to\hat{\mathbf{x}}_4$ and $\hat{\mathbf{x}}_2'\to\hat{\mathbf{x}}_4'$. Similarly, $\check{U}=\check{U}_{14}$ and

$$U_{12}U_{34}=U_{14}U_{32}=U^{(4)}\equiv G^*(1)G(2)G^*(3)G(4)\ . \tag{4.12}$$

Thus, making the same interchange in expression (4.5) of $\check{I}$, we learn that

$$\check{S}\equiv\check{S}_{12}=\check{S}_{14}\ , \tag{4.13}$$

and hence, by the same interchange of Eqs. (4.11),

$$\check{S}=\check{K}_{14}+K_{14}U_{14}\check{S}=\check{K}_{14}+K_{14}K_{32}\check{I}\ , \tag{4.14}$$

which, upon comparing with the original (4.11b), shows that $\check{K}_{14}\neq\check{K}_{12}$, and that $\check{K}_{12}$ can be written in the form[9]

$$\check{K}_{12}=\check{K}^0+K_{14}U_{14}\check{S}\ , \tag{4.15}$$

with a symmetrical (and irreducible as defined below) matrix K^0 subject to

$$\check{K}^0\equiv\check{K}^0_{12}=\check{K}^0_{14}=\check{K}^0_{34}=\check{K}^0_{32}\ . \tag{4.16}$$

In fact, the second term of (4.15), $\check{K}^{(\times)}$, say, is "U_{12} irreducible" in the sense of having no part that can be written in the form $A_{12}U_{12}B_{12}$, so that its diagram is inseparable into two parts A_{12} and B_{12} by cutting two lines of $U_{12}=G^*(1)G(2)$ (see Fig. 3). The substitution of Eq. (4.15) in the first term of (4.11a) yields a symmetrical expression of $\check{S}$, as

$$\check{S}=\check{K}^0+K_{12}U_{12}\check{S}+K_{14}U_{14}\check{S}\ , \tag{4.17}$$

which, from Eq. (4.14), shows that

$$\check{K}_{14}=\check{K}^0+K_{12}U_{12}\check{S}\ , \tag{4.18}$$

being the same equation as that obtained from Eq. (4.15) by interchanging $\hat{\mathbf{x}}_2$ and $\hat{\mathbf{x}}_4$, as it should be to be consistent. Equation (4.18) shows that $\check{K}_{14}$ is U_{14} irreducible with the irreducible $\check{K}^0$ with respect to both U_{12} and U_{14}.

Equations (4.7) can be written by the function equations as

$$K_{12}\check{I}=U_{34}\check{S},\quad K_{34}\check{I}=U_{12}\check{S}\ , \tag{4.19a}$$

which, by the x_2-x_4 interchange, yield

$$K_{14}\check{I}=U_{32}\check{S},\quad K_{32}\check{I}=U_{14}\check{S}\ , \tag{4.19b}$$

showing therefore that

$$K_{12}U_{12}\check{S}=K_{34}U_{34}\check{S}=K_{12}K_{34}\check{I}\ , \tag{4.20a}$$

$$K_{14}U_{14}\check{S}=K_{32}U_{32}\check{S}=K_{14}K_{32}\check{I}\ . \tag{4.20b}$$

Hence Eqs. (4.15) and (4.18) can be written by

$$\check{K}_{12}=\check{K}^0+K_{14}K_{32}\check{I}\ , \tag{4.21a}$$

$$\check{K}_{14}=\check{K}^0+K_{12}K_{34}\check{I}\ , \tag{4.21b}$$

and Eq. (4.17) in a symmetrical form, by

$$\check{S}=\check{K}^0+K^{(4)}\check{I}\ , \tag{4.22a}$$

in terms of a coordinate-interchange-invariant matrix $K^{(4)}$ defined by

$$K^{(4)}=K_{12}K_{34}+K_{14}K_{32}\ . \tag{4.22b}$$

Here, from Eq. (4.5) with (4.12),

$$\check{I}=\check{U}+U^{(4)}\check{S}\ . \tag{4.23}$$

Hence, upon the substitution in (4.22a), we obtain a governing equation for $\check{S}$, as

$$\check{S}=\check{K}^{(1)}+K^{(4)}U^{(4)}\check{S}\ , \tag{4.24a}$$

where

$$\check{K}^{(1)}=\check{K}^0+K^{(4)}\check{U}\ , \tag{4.24b}$$

with the formal solution

$$\check{S}=(1-K^{(4)}U^{(4)})^{-1}\check{K}^{(1)}\ . \tag{4.24c}$$

Similarly, substituting (4.22a) in (4.23), $\check{I}$ is found to be the solution of

$$\check{I}=\check{U}^{(1)}+U^{(4)}K^{(4)}\check{I}\ , \tag{4.25a}$$

where

$$\check{U}^{(1)}=\check{U}+U^{(4)}\check{K}^0\ , \tag{4.25b}$$

which is the same function of $\check{U}$, $U^{(4)}$, and $\check{K}^0$ as $\check{K}^{(1)}$ of (4.24b) is of $\check{K}^0$, $K^{(4)}$, and $\check{U}$.

Here we introduce the incoherent part of $\check{I}$, $\check{\mathcal{J}}$, defined by

$$\check{\mathcal{J}}=U^{(4)}\check{S}=I^{(4)}\check{K}^{(1)}\ , \tag{4.26a}$$

to write

$$\check{I}=\check{U}+\check{\mathcal{J}}\ . \tag{4.26b}$$

Here, in view of Eq. (4.24c),

$$I^{(4)}=U^{(4)}(1-K^{(4)}U^{(4)})^{-1}\ , \tag{4.27}$$

which is the solution of an equation similar in form to the BS equation (2.21), as

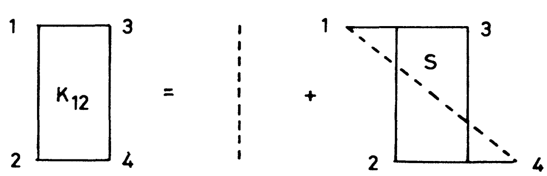

FIG. 3. The matrix K_{12} by Eq. (4.15) is diagrammatically shown with the approximation $K_{14}\simeq K^0_{14}$ on the right-hand side. The dashed lines represent the matrix K^0, and the (horizontal) solid lines represent G^* or G.

$$I^{(4)}=U^{(4)}(1+K^{(4)}I^{(4)}) , \qquad (4.28)$$

with the solution

$$I^{(4)}=U^{(4)}+U^{(4)}S^{(4)}U^{(4)} , \qquad (4.29)$$

in terms of a scattering matrix $S^{(4)}$ of $K^{(4)}$, defined by

$$K^{(4)}I^{(4)}=S^{(4)}U^{(4)} \qquad (4.30a)$$

and governed by

$$S^{(4)}=K^{(4)}(1+U^{(4)}S^{(4)}) = K^{(4)}+K^{(4)}I^{(4)}K^{(4)} \qquad (4.30b)$$

similar to Eqs. (4.7) and (4.8) for S.

The function $\check{S}$ of (4.24c) can also be written in the form

$$\check{S}=\check{K}^{0}+S^{(4)}\check{U}^{(1)} \qquad (4.31)$$

similar to (4.26b) for $\check{I}$. $I^{(4)}$ is a four-coordinate matrix of $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$, $\hat{\mathbf{x}}_3$, and $\hat{\mathbf{x}}_4$, and is the solution of Eq. (4.28) which ensures $I^{(4)}$ to be invariant for each of the interchanges $\hat{\mathbf{x}}_2 \leftrightarrow \hat{\mathbf{x}}_4$ and $\hat{\mathbf{x}}_1 \leftrightarrow \hat{\mathbf{x}}_3$, independently. It can be regarded as a Green's function of fourth order, in view of $U^{(4)}$ and $K^{(4)}$ defined by Eqs. (4.12) and (4.22b), respectively.

A. Approximation

We first rewrite Eq. (4.17) in the original matrix form

$$S=K^{0}+\grave{S}^{(L)}+\grave{S}^{(\times)} , \qquad (4.32)$$

by introducing scattering matrices $\grave{S}^{(L)}$ and $\grave{S}^{(\times)}$, defined in terms of the four-coordinate functions, by

$$\grave{\check{S}}^{(L)}=K_{12}U_{12}\check{S} \simeq K_{12}^{0}U_{12}\check{S} , \qquad (4.33a)$$

$$\grave{\check{S}}^{(\times)}=K_{14}U_{14}\check{S} \simeq K_{14}^{0}U_{14}\check{S} , \qquad (4.33b)$$

which can be obtained from each other by interchanging $\hat{\mathbf{x}}_2$ and $\hat{\mathbf{x}}_4$, in view of the coordinate-interchange invariance of $\check{S}$ being ensured.

Here, to evaluate $\grave{S}^{(L)}$, we substitute (4.32) in (4.33a) and neglect the term from $\grave{S}^{(\times)}$ to obtain

$$\grave{S}_{12}^{(L)}=K_{12}^{0}U_{12}(K_{12}^{0}+\grave{S}_{12}^{(L)}) , \qquad (4.34a)$$

with the solution

$$\grave{S}_{12}^{(L)}=K_{12}^{0}I_{12}^{(L)}K_{12}^{0} . \qquad (4.34b)$$

Here

$$I_{12}^{(L)}=(1-U_{12}K_{12}^{0})^{-1}U_{12}=U_{12}+U_{12}S_{12}^{(L)}U_{12} , \qquad (4.35)$$

which is the solution of the original BS equation to the approximation $K \simeq K^{0}$, and Eq. (4.34b) shows in view of relation (4.8b) that

$$S^{(L)}=K^{0}+\grave{S}^{(L)} . \qquad (4.36)$$

On the other hand, Eqs. (4.33) indicate that

$$\grave{\check{S}}^{(\times)} \equiv \grave{\check{S}}_{12}^{(\times)}=\grave{\check{S}}_{14}^{(L)} , \qquad (4.37a)$$

where, from Eq. (4.34b),

$$\grave{S}_{14}^{(L)}=K_{14}^{0}I_{14}^{(L)}K_{14}^{0} . \qquad (4.37b)$$

Thus the resulting I is obtained, on substituting (4.32) in (4.6) and using (4.36), as[9,1,3,4]

$$I=U+UK^{0}U+U(\grave{S}^{(L)}+\grave{S}^{(\times)})U \qquad (4.38a)$$

$$=I^{(L)}+U\grave{S}^{(\times)}U . \qquad (4.38b)$$

An exact version of the solution is obtained by rewriting (4.15) in the original matrix form

$$K=K^{0}+K^{(\times)} , \qquad (4.39a)$$

with the matrix $K^{(\times)}$ defined by the function equation

$$\check{K}^{(\times)}=K_{14}U_{14}\check{S} . \qquad (4.39b)$$

The BS equation with K of (4.39a) has the same form as Eq. (2.21), and the same procedure as leading to the solution (2.41) leads to

$$I=I^{(L)}+I^{(L)}S^{(\times/L)}I^{(L)} , \qquad (4.40)$$

with a scattering matrix $S^{(\times/L)}$ of $K^{(\times)}$, defined by

$$S^{(\times/L)}=K^{(\times)}(1+I^{(L)}S^{(\times/L)}) \qquad (4.41a)$$

$$=S^{(0\times)}(1+US^{(L)}US^{(\times/L)}) , \qquad (4.41b)$$

where

$$S^{(0\times)}=K^{(\times)}(1+US^{(0\times)})=(1-K^{(\times)}U)^{-1}K^{(\times)} . \qquad (4.42)$$

Here it may be remarked that $K^{(\times)}$ is the necessary part of K for the solution I to be consistent with the generalized reciprocity condition; and that the anti-Hermitian part of M has a contribution, not only from K^{0}, but also from $K^{(\times)}$ consistent with optical condition (2.24), so that the extinction coefficient is given according to (3.20) by the total cross section including $K^{(\times)}$.

B. Optical cross section from $\grave{S}^{(\times)}$ for a bounded layer (Fig. 1)

The optical cross sections of the random layer are given in terms of the cross sections per unit area of the layer surfaces, $\sigma_{ab}^{(q+12+23)}$, $a,b=1,3$ by Eqs. (3.39b) and (3.41b), wherein $S_{\pm 2,\pm 2}^{(q/12+23)}$ are defined by Eqs. (3.40c) and (3.42). Here, in the present notations, the medium scattering matrix is

$$S_{2}^{(0q)}=K^{0}+\grave{S}^{(L)}+\grave{S}^{(\times)} \qquad (4.43)$$

in the case of a reflection-free boundary, and is $S_{22}^{(q/12+23)}$ in the general case, which is given in terms of $S_{2}^{(0q)}$ as the solution of Eq. (2.44a) with $\sigma^{(12)} \rightarrow \sigma^{(12)}+\sigma^{(23)}$. Here the cross section from $\grave{S}^{(\times)}$ (or $\grave{\check{S}}^{(\times)}$), say $S^{(\times)}(\hat{\mathbf{\Omega}}|\hat{\mathbf{\Omega}}')$, can be obtained from $\grave{S}^{(L)}$ (or $\grave{\check{S}}^{(L)}$) by interchanging the roles of $\hat{\mathbf{x}}_2$ and $\hat{\mathbf{x}}_2'$ (or $\hat{\mathbf{x}}_2$ and $\hat{\mathbf{x}}_4$). Hence, from Eqs. (3.32)–(3.34),

$$S_{+2,+2}^{(\times)}(\hat{\mathbf{\Omega}}|\hat{\mathbf{\Omega}}')=\grave{\check{S}}_{+2,+2}^{(\times)}(\hat{\mathbf{\Omega}},\lambda_z|\boldsymbol{\lambda}=0|\hat{\mathbf{\Omega}}',\lambda_z') , \qquad (4.44)$$

whose right-hand side can be given in view of relation (4.37a) by

$$\grave{\check{S}}_{+2,+2}^{(L)}(\hat{\mathbf{\Omega}},\lambda_z|\boldsymbol{\lambda}|\hat{\mathbf{\Omega}}',\lambda_z') ,$$

with the interchange of $\hat{\lambda}_2$ and $\hat{\lambda}_4$ so that

$$\hat{\mathbf{u}}=\hat{\mathbf{u}}_{12}\to\hat{\mathbf{u}}_{14}=\tfrac{1}{2}(\hat{\lambda}_1+\hat{\lambda}_4)\equiv k_2\hat{\Omega}_{14} , \quad (4.45a)$$

$$\hat{\mathbf{u}}'=-\hat{\mathbf{u}}_{34}\to-\hat{\mathbf{u}}_{32}=-\tfrac{1}{2}(\hat{\lambda}_3+\hat{\lambda}_2)\equiv k_2\hat{\Omega}_{32} , \quad (4.45b)$$

or, using Eqs. (3.35),

$$\hat{\Omega}\to\hat{\Omega}_{14}\simeq\tfrac{1}{2}(\hat{\Omega}-\hat{\Omega}') , \quad (4.46a)$$

$$\hat{\Omega}'\to\hat{\Omega}_{32}\simeq\tfrac{1}{2}(\hat{\Omega}'-\hat{\Omega}) . \quad (4.46b)$$

Here, the γ_2 terms have been neglected, which are of the order of $|\gamma_2/k_2|$ and negligible therefore unless $|\hat{\Omega}-\hat{\Omega}'|\lesssim|\gamma_2/k_2|\ll1$. In the same way, $\hat{\lambda}=\hat{\lambda}_2-\hat{\lambda}_1$ and $\hat{\lambda}'=-\hat{\lambda}_4+\hat{\lambda}_3$ are replaced by $\hat{\lambda}_{14}=\hat{\lambda}_4-\hat{\lambda}_1$ and $-\hat{\lambda}_{32}=-\hat{\lambda}_2+\hat{\lambda}_3$, respectively, or

$$\hat{\lambda}_{14}=-k_2(\hat{\Omega}+\hat{\Omega}')-\tfrac{1}{2}i\gamma_2[(\Omega_z)^{-1}-(\Omega_z')^{-1}]\hat{\mathbf{n}}_z , \quad (4.47a)$$

$$-\hat{\lambda}_{32}=-k_2(\hat{\Omega}+\hat{\Omega}')+\tfrac{1}{2}i\gamma_2[(\Omega_z)^{-1}-(\Omega_z')^{-1}]\hat{\mathbf{n}}_z , \quad (4.47b)$$

which show that the horizontal components λ_{14} and λ_{32} are both nonzero, being given by

$$\lambda_{14}=-\lambda_{32}=\lambda=-k_2(\hat{\Omega}+\hat{\Omega}') , \quad (4.48)$$

in contrast to the case of $\grave{S}^{(L)}$ wherein $\lambda=0$.

On the other hand, the transforms $\tilde{S}^{(L)}_{\pm2,\pm2}$ are appreciable only within the range $|\hat{\lambda}|,|\hat{\lambda}'|\lesssim\gamma_2$, in view of $\tilde{U}_2(\hat{\mathbf{u}},\hat{\lambda})$ given by Eq. (3.6) and involved in all the equations as a weighting function of when making the $\hat{\mathbf{u}}$ integration [e.g., Eqs. (2.44)]. Hence it follows from (4.47) that the contribution from $\grave{S}^{(\times)}$ can be appreciable only within the range

$$|\hat{\Omega}+\hat{\Omega}'|\lesssim\gamma_2/k_2 , \quad (4.49a)$$

and, under this condition, Eqs. (4.47) lead to

$$(\lambda_{14})_z=-i\gamma_2\Omega_z^{-1}(1-i\gamma_2^{-1}\lambda\cdot\Omega) , \quad (4.49b)$$

$$-(\lambda_{32})_z=-i\gamma_2(\Omega_z')^{-1}(1-i\gamma_2^{-1}\lambda\cdot\Omega') , \quad (4.49c)$$

with the λ given by (4.48); Eq. (4.49b) is proved by observing that $\Omega_z'\sim-\Omega_z$ in Eq. (4.47a) and hence

$$(\lambda_{14})_z\simeq-i\gamma_2\Omega_z^{-1}\{1-i(k_2/\gamma_2)\tfrac{1}{2}[\Omega_z^2-(\Omega_z')^2]\} ,$$

wherein $\Omega_z^2-(\Omega_z')^2=\Omega'^2-\Omega^2\simeq-2\Omega\cdot(\Omega+\Omega')$.

Note that Eqs. (4.49b) and (4.49c) are exactly the same as (3.34) for λ_z and λ_z', and hold true even when $\hat{\Omega}$ and $\hat{\Omega}'$ are replaced by $\hat{\Omega}_{14}$ and $\hat{\Omega}_{32}$ of (4.46a) and (4.46b), respectively, to the present approximation.

Thus Eq. (4.44) gives

$$\begin{aligned}S^{(\times)}_{+2,+2}(\hat{\Omega}|\hat{\Omega}')&=\tilde{S}^{(L)}_{+2,+2}(\hat{\Omega}_{14},(\lambda_{14})_z|\lambda|\hat{\Omega}_{32},-(\lambda_{32})_z)\\&\simeq\tilde{S}^{(L)}_{+2,+2}(\hat{\Omega},(\lambda_{14})_z|-k_2(\Omega+\Omega')|-\hat{\Omega},-(\lambda_{32})_z) ;\end{aligned} \quad (4.50)$$

and hence, when $\hat{\Omega}+\hat{\Omega}'=0$ exactly,

$$S^{(\times)}_{+2,+2}(\hat{\Omega}|-\hat{\Omega})=S^{(L)}_{+2,+2}(\hat{\Omega}|-\hat{\Omega}) , \quad (4.51)$$

as has been known.[2]

The situation is the same also for the boundary-dependent $S^{(q/12+23)}_{+2,+2}(\hat{\Omega}|\hat{\Omega}')$ involved in Eq. (3.39b) and given by Eqs. (3.40) as a function of $\hat{\Omega}$, $\hat{\Omega}'$, and λ; so that, once it is found to the ladder approximation, say $S^{(L/12+23)}_{+2,+2}(\hat{\Omega}|\lambda|\hat{\Omega}')$, the cross section from the crossed diagrams, say $S^{(\times/12+23)}_{+2,+2}(\hat{\Omega}|\lambda|\hat{\Omega}')$, can be obtained by the replacement of $\hat{\Omega}$, $\hat{\Omega}'$, and λ, according to Eqs. (4.46a), (4.46b), and (4.48), respectively. To this end, we only need in view of Eq. (3.37) the quantity $\mathcal{J}^{(q/12+23)}_{22}(\hat{\Omega},z=0|\lambda|\hat{\Omega}',z'=0)$ that is the boundary value at $z=0$ of the solution of the integral equation (3.17).

V. APPLICATION OF THE DIFFUSION APPROXIMATION

When the source and the observer are both separated enough from the layer boundaries, $I^{(q+12+23)}_{11}$ and $I^{(q+12+23)}_{31}$ are given by the asymptotic expressions (3.38) and (3.41), respectively, in terms of the cross sections per unit area of the layer surfaces, $\sigma^{(q+12+23)}_{11}$ and $\sigma^{(q+12+23)}_{31}$, of the form

$$\sigma^{(q+12+23)}_{11}(\hat{\Omega}|\hat{\Omega}')=\sigma^{(12)}_{11}(\hat{\Omega}|\hat{\Omega}')+\sigma^{(q/12+23)}_{11}(\hat{\Omega}|\hat{\Omega}') , \quad (5.1a)$$

$$\sigma^{(q+12+23)}_{31}(\hat{\Omega}|\hat{\Omega}')=\sigma^{(q/12+23)}_{31}(\hat{\Omega}|\hat{\Omega}') . \quad (5.1b)$$

Here, from (3.39) and (3.40),

$$\sigma^{(q/12+23)}_{11}(\hat{\Omega}|\hat{\Omega}')=\int d\hat{\Omega}''\int d\hat{\Omega}'''\sigma^{(12)}_{12}(\hat{\Omega}|\hat{\Omega}'')\mathcal{J}^{(q/12+23)}_{22}(\hat{\Omega}'',z''=0|\hat{\Omega}''',z'''=0)\sigma^{(12)}_{21}(\hat{\Omega}'''|\hat{\Omega}') , \quad (5.2a)$$

$$\sigma^{(q/12+23)}_{31}(\hat{\Omega}|\hat{\Omega}')=\int d\hat{\Omega}''\int d\hat{\Omega}'''\sigma^{(23)}_{32}(\hat{\Omega}|\hat{\Omega}'')\mathcal{J}^{(q/12+23)}_{22}(\hat{\Omega}'',z''=-L|\hat{\Omega}''',z'''=0)\sigma^{(12)}_{21}(\hat{\Omega}'''|\hat{\Omega}') , \quad (5.2b)$$

which both tend to zero as $q_2\to0$ and hence can be regarded as additional boundary cross sections caused by the medium fluctuation. Here, $\mathcal{J}^{(q/12+23)}_{22}$ is the solution of Eqs. (2.54), and can be obtained as a boundary-value solution of the diffusion equation, approximately, as has been tried to investigate the enhanced backscattering in a bounded layer space,[2-4] as well as unbounded, on assuming reflection-free boundaries.

To make the diffusion approximation,[8] we introduce a set of eigenfunctions $f_A(\hat{\Omega},\hat{\lambda})$ and $\bar{f}_A(\hat{\Omega},\hat{\lambda})$ of the cross section $K^{(q)}_2(\hat{\Omega}|\hat{\Omega}')$, defined by the eigenvalue equations

$$\int d\hat{\Omega}' K_2^{(q)}(\hat{\Omega}|\hat{\Omega}')\tilde{U}_2(\hat{\Omega}',\hat{\lambda})f_A(\hat{\Omega}',\hat{\lambda}) = A(\hat{\lambda})f_A(\hat{\Omega},\hat{\lambda}) , \tag{5.3a}$$

$$\int d\hat{\Omega}' \bar{f}_A(\hat{\Omega}',\hat{\lambda})\tilde{U}_2(\hat{\Omega}',\hat{\lambda})K_2^{(q)}(\hat{\Omega}'|\hat{\Omega}) = A(\hat{\lambda})\bar{f}_A(\hat{\Omega},\hat{\lambda}) , \tag{5.3b}$$

with the normalization

$$\int d\hat{\Omega}\, \bar{f}_A(\hat{\Omega},\hat{\lambda})\tilde{U}_2(\hat{\Omega},\hat{\lambda})f_B(\hat{\Omega},\hat{\lambda}) = \delta_{AB} . \tag{5.4}$$

Here $\tilde{U}_2(\hat{\Omega},\hat{\lambda})$ is regarded as a weighting function when making the $\hat{\Omega}$ integration, and $A(\hat{\lambda})$ is the eigenvalue with branch points at $|\hat{\lambda}| = \pm i\gamma_2$ on the $|\hat{\lambda}|$ plane, caused by the pole of $\tilde{U}_2$ of (3.9), and tends to zero as $|\hat{\lambda}| \to \infty$. In terms of the eigenfunctions and the eigenvalues, the cross section can be exhibited by the series

$$K_2^{(q)}(\hat{\Omega}|\hat{\Omega}') = \sum_A A(\hat{\lambda})f_A(\hat{\Omega},\hat{\lambda})\bar{f}_A(\hat{\Omega}',\hat{\lambda}) , \tag{5.5}$$

and hence, generally,

$$(K_2^{(q)}\tilde{U}_2)^n K_2^{(q)}(\hat{\Omega}|\hat{\lambda}|\hat{\Omega}') = \sum_A A^{n+1}(\hat{\lambda})f_A(\hat{\Omega},\hat{\lambda})\bar{f}_A(\hat{\Omega}',\hat{\lambda}) . \tag{5.6}$$

To find a similar expansion for $\tilde{S}_2^{(0q)}$ from (2.43) we observe that

$$\tilde{S}_2^{(0q)} = (1-K_2^{(q)}\tilde{U}_2)^{-1}K_2^{(q)} = K_2^{(q)} + K_2^{(q)}\tilde{U}_2K_2^{(q)} + \cdots + (K_2^{(q)}\tilde{U}_2)^{N-1}K_2^{(q)} + \tilde{S}_2^{(0q,N)} , \tag{5.7}$$

where

$$\tilde{S}_2^{(oq,N)} = (1-K_2^{(q)}\tilde{U}_2)^{-1}(K_2^{(q)}\tilde{U}_2)^N K_2^{(q)} \tag{5.8}$$

and hence

$$\tilde{S}_2^{(0q,N)}(\hat{\Omega}|\hat{\lambda}|\hat{\Omega}') = \sum_A (1-A)^{-1}A^{N+1}(\hat{\lambda})f_A(\hat{\Omega},\hat{\lambda})\bar{f}_A(\hat{\Omega}',\hat{\lambda}) , \tag{5.9}$$

in view of Eq. (5.6); the original $S_2^{(0q)}$ is given by $\tilde{S}_2^{(0q,N)}$ when $N=0$. The series (5.7) consists of the scattering terms up to the Nth order of short-range functions and the remainder $\tilde{S}_2^{(0q,N)}$ of a long-range function. The diffusion term is the first term with the eigenvalue of the form

$$A(\hat{\lambda}) = 1-\gamma_2^{-1}D_2\hat{\lambda}^2 + O[(\hat{\lambda}/\gamma_2)^4] \tag{5.10}$$

in the range $|\hat{\lambda}/\gamma_2| \ll 1$, when the medium cross section is rotationally invariant in space with the form $K_2^{(q)}(\hat{\Omega}\cdot\hat{\Omega}')$. Here

$$D_2 = (3\gamma_2)^{-1}(1-a_1)^{-1} = (3\gamma_2)^{-1} + \bar{D}_2 , \tag{5.11a}$$

$$\bar{D}_2 = (3\gamma_2)^{-1}a_1(1-a_1)^{-1} , \tag{5.11b}$$

where a_1 is the average of the cosine of the scattering angle, defined by

$$a_1 = \gamma_2^{-1}\int d\hat{\Omega}(\hat{\Omega}\cdot\hat{\Omega}')K_2^{(q)}(\hat{\Omega}\cdot\hat{\Omega}') . \tag{5.12}$$

Hence, discarding all the terms other than the diffusion in Eq. (5.9), we obtain the expression

$$\tilde{S}_2^{(0q,N)}(\hat{\Omega}|\hat{\lambda}|\hat{\Omega}') = \tilde{S}_A^{(N)}(\hat{\lambda})f_A(\hat{\Omega},\hat{\lambda})\bar{f}_A(\hat{\Omega}',\hat{\lambda}) , \tag{5.13}$$

$$\tilde{S}_A^{(N)}(\hat{\lambda}) = (1-A)^{-1}A^{N+1}(\hat{\lambda}) = \gamma_2(D_2\hat{\lambda}^2)^{-1} + \sum_{n=0}^{\infty} d_n^{(N)}\hat{\lambda}^{2n} , \tag{5.14}$$

where the last is the series expansion with respect to $\hat{\lambda}^2$ within a range of $|\hat{\lambda}/\gamma_2| < 1$. Here the first term, say $\tilde{S}_A(\hat{\lambda})$, is the same independent of the order N, and its Fourier inversion with respect to z, say, $S_A(z|\lambda|z') \equiv S_A(z|z')$, is a solution of the diffusion equation

$$\gamma_2^{-1}[\gamma^{(ab)} + D_2(\lambda^2 - \partial_z^2)]S_A(z|\lambda|z') = \delta(z-z') , \tag{5.15}$$

with a new parameter $\gamma^{(ab)}$ to represent an intrinsic dissipation by the medium.

To investigate the asymptotic form of $\tilde{S}_A^{(N)}$ at $|\hat{\lambda}| \to \infty$, on the other hand, we observe on using Eq. (5.3) and the normalization (5.4) that

$$A = \bar{f}_A\tilde{U}_2K_2^{(q)}\tilde{U}_2f_A \sim O(|\hat{\lambda}|^{-1}) , \tag{5.16}$$

and hence $\tilde{S}_A^{(N)} \sim O(|\hat{\lambda}|^{-N-1})$ and $\tilde{S}_2^{(0q,N)} \sim O(|\hat{\lambda}|^{-N})$, confirming therefore that the short-range behavior of $S_2^{(0q,N)}$ is not the same, depending on N.

When the q_2 space is bounded by the boundaries at $z=0$ and $-L$, the diffusion equation (5.15) is subjected to boundary conditions to be described below, and, with the solution $S_A^{(12+23)}$, say, we obtain a diffusion expression of $S_{22}^{(q/12+23)}$ as

$$S_{22}^{(q/12+23)}(\hat{\Omega},z|\hat{\Omega}',z') \sim f_A(\hat{\Omega},i\vec{\partial}_z)S_A^{(12+23)}(z|z')\bar{f}_A(\hat{\Omega}',-i\overleftarrow{\partial}_z') , \tag{5.17}$$

and hence also

$$\mathcal{I}_{22}^{(q/12+23)}(\hat{\Omega},z|\hat{\Omega}',z') \sim \phi_A(\hat{\Omega},i\vec{\partial}_z)S_A^{(12+23)}(z|z')\bar{\phi}_A(\hat{\Omega}',-i\overleftarrow{\partial}_z') \tag{5.18a}$$

where

$$\phi_A(\hat{\Omega},i\partial_z) = \tilde{U}_2f_A(\hat{\Omega},i\partial_z) , \quad \bar{\phi}_A(\hat{\Omega},-i\partial_z) = \bar{f}_A\tilde{U}_2(\hat{\Omega},-i\partial_z) , \tag{5.18b}$$

which is a crude approximation when $z=z'=0$, neglecting all the other terms in the series (5.7) and (5.9). Here, to the first order of $\hat{\lambda} = (\lambda, i\partial_z)$, we obtain, with $\bar{D}_2$ of (5.11b),

$$f_A(\hat{\Omega},i\partial_z) = 4\pi\gamma_2^{-1}\bar{f}_A(\hat{\Omega},i\partial_z) = 1 + 3\bar{D}_2(i\Omega\cdot\lambda - \Omega_z\partial_z) , \tag{5.19a}$$

$$\gamma_2\phi_A(\hat{\Omega},i\partial_z) = 4\pi\bar{\phi}_A(\hat{\Omega},i\partial_z) \simeq 1 + 3D_2(i\Omega\cdot\lambda - \Omega_z\partial_z) , \tag{5.19b}$$

$$\tilde{U}_2(\hat{\Omega},i\partial_z) \simeq \gamma_2^{-1}[1+\gamma_2^{-1}(i\Omega\cdot\lambda - \Omega_z\partial_z)] , \tag{5.19c}$$

and $S_A^{(12+23)}$ is subject to the boundary condition at $z=0$ of the form (Appendix A)

$$(2\gamma_2)^{-1}(\tfrac{1}{2}+D_2\partial_z)S_A^{(12+23)}[z(=0)|z'] = \langle\sigma_{22}^{(12)}(\partial_z)\rangle S_A^{(12+23)}[z(=0)|z'], \quad z'<0 . \tag{5.20}$$

Here

$$\langle\sigma_{ab}^{(12)}(\partial_z)\rangle = (4\pi)^{-1}\int_{2\pi} d\hat{\Omega}\int_{2\pi} d\hat{\Omega}'\sigma_{ab}^{(12)}(\hat{\Omega}|\hat{\Omega}')\phi_A(\hat{\Omega}',i\partial_z) \tag{5.21}$$

$(a,b=1,2)$, and is subject in view of Eq. (3.16) to the relation

$$\sum_{a=1}^{2} k_a\langle\sigma_{a2}^{(12)}(\partial_z)\rangle = k_2(2\gamma_2)^{-1}(\tfrac{1}{2}-D_2\partial_z) . \tag{5.22}$$

To apply the boundary condition (5.20), it is convenient to rewrite it, on using relation (5.22), as

$$[k_2\gamma_2^{-1}D_2\partial_z + k_1\langle\sigma_{12}^{(12)}(\partial_z)\rangle]S_A^{(12+23)}[z(=0|z']=0 \tag{5.23}$$

in terms of $\langle\sigma_{12}^{(12)}(\partial_z)\rangle$, and further as

$$(D_2\partial_z+Z^{(12)})S_A^{(12+23)}[z(=0)|z']=0 , \tag{5.24}$$

with a constant $Z^{(12)}$ given by the root of the equation

$$k_2\gamma_2^{-1}Z^{(12)} = k_1\langle\sigma_{12}^{(12)}(-D_2^{-1}Z^{(12)})\rangle \tag{5.25a}$$

$$= k_1(2\gamma_2)^{-1}(\langle\sigma_{12}^{(12)}\rangle_0+\langle\sigma_{12}^{(12)}\rangle_1 Z^{(12)}) , \tag{5.25b}$$

where the last expression is obtained by using (5.19b), in terms of the notations

$$\langle\sigma_{ab}^{(\,)}\rangle_0 = \frac{1}{2\pi}\int_{2\pi} d\hat{\Omega}\,d\hat{\Omega}'\sigma_{ab}^{(\,)}(\hat{\Omega}|\hat{\Omega}')(1+i3D_2\boldsymbol{\lambda}\cdot\boldsymbol{\Omega}^{(b)\prime}) , \tag{5.26a}$$

$$\langle\sigma_{ab}^{(\,)}\rangle_1 = \frac{3}{2\pi}\int_{2\pi} d\hat{\Omega}\,d\hat{\Omega}'\sigma_{ab}^{(\,)}(\hat{\Omega}|\hat{\Omega}')|\Omega_z^{(b)\prime}| . \tag{5.26b}$$

Hence

$$Z^{(12)} = [2(k_2/k_1)-\langle\sigma_{12}^{(12)}\rangle_1]^{-1}\langle\sigma_{12}^{(12)}\rangle_0 , \tag{5.27}$$

and becomes $\frac{1}{2}$ when the boundary is free from reflection by giving $\langle\sigma_{12}^{(12)}\rangle_1=1$ and $\langle\sigma_{12}^{(12)}\rangle_0=\frac{1}{2}$, consistent with Eq. (5.20). When the matrix $\sigma^{(12)}$ is rotationally invariant around the z axis, $Z^{(12)}$ is independent of $\boldsymbol{\lambda}$, in view of no contribution from the responsible integrand in Eq. (5.26a). Another expression of $Z^{(12)}$ is given, in terms of $\sigma_{22}^{(12)}$, by (A12).

Another boundary equation at $z=-L$ is similarly given by

$$(-D_2\partial_z+Z^{(23)})S_A^{(12+23)}[z(=-L)|z']=0 , \tag{5.28}$$

where $Z^{(23)}$ is determined by $\sigma^{(23)}$ as $Z^{(12)}$ is by $\sigma^{(12)}$, according to Eqs. (5.25) or (5.27).

To express $\sigma_{11}^{(q/12+23)}$ of (5.2a) specifically by using (5.18a), (5.19b), and boundary condition (5.24), we introduce two new functions $\sigma_{12}^{(12)}(\hat{\Omega}|Z)$ and $\sigma_{21}^{(12)}(Z|\hat{\Omega})$, defined by

$$\sigma_{12}^{(12)}(\hat{\Omega}|Z) = \int d\hat{\Omega}'\sigma_{12}^{(12)}(\hat{\Omega}|\hat{\Omega}')\phi_A(\hat{\Omega}',i\partial_z)|_Z , \tag{5.29a}$$

$$\sigma_{21}^{(12)}(Z|\hat{\Omega}) = \int d\hat{\Omega}'\bar{\phi}_A(\hat{\Omega}',-i\partial_z)\sigma_{21}^{(12)}(\hat{\Omega}'|\hat{\Omega})|_Z , \tag{5.29b}$$

where $|_Z$ designates the setting $\partial_z=-D_2^{-1}Z^{(12)}$. Hence

$$\sigma_{12}^{(12)}(\hat{\Omega}|Z) = (4\pi/\gamma_2)\sigma_{21}^{(12)}(Z|-\hat{\Omega})|_{\lambda\to-\lambda} , \tag{5.30}$$

and let Eq. (5.25a) and a similar equation for $Z^{(23)}$ be written by

$$k_1\int d\hat{\Omega}\,\sigma_{12}^{(12)}(\hat{\Omega}|Z) = (4\pi k_2/\gamma_2)Z^{(12)} , \tag{5.31a}$$

$$k_3\int d\hat{\Omega}\,\sigma_{32}^{(23)}(\hat{\Omega}|Z) = (4\pi k_2/\gamma_2)Z^{(23)} . \tag{5.31b}$$

Thus Eqs. (5.2) become written, on using (5.18a) and (B7)–(B16), as

$$\sigma_{11}^{(q/12+23)}(\hat{\Omega}|\hat{\Omega}') = (\gamma_2/Z^{(12)})\sigma_{12}^{(12)}(\hat{\Omega}|Z)\{\sigma_{21}^{(12)}(Z=0|\hat{\Omega}')+[(Z^{(12)}/\gamma_2)S_A^{(12+23)}(0|0)-1]\sigma_{21}^{(12)}(Z|\hat{\Omega}')\} \tag{5.32a}$$

$$= (\gamma_2/Z^{(12)})\sigma_{12}^{(12)}(\hat{\Omega}|Z)[\sigma_{21}^{(12)}(Z=0|\hat{\Omega}')-Y(Z^{(12)}+Y)^{-1}\sigma_{21}^{(12)}(Z|\hat{\Omega}')] , \tag{5.32b}$$

$$\sigma_{31}^{(q/12+23)}(\hat{\Omega}|\hat{\Omega}') = \sigma_{32}^{(23)}(\hat{\Omega}|Z)S_A^{(12+23)}(-L|0)\sigma_{21}^{(12)}(Z|\hat{\Omega}') , \tag{5.32c}$$

in terms of the variable Y defined by Eq. (B7) and the $S_A^{(12+23)}$ by Eqs. (B8) and (B9). Here, with $\kappa=(\gamma^{(ab)}D_2^{-1}+\lambda^2)^{1/2}$,

$$Y|_{\kappa=0} \equiv Y_0 = Z^{(23)}[1+(L/D_2)Z^{(23)}]^{-1} , \tag{5.33a}$$

$$Y|_{\kappa L\gg 1} \simeq \kappa D_2 , \tag{5.33b}$$

and, when $\kappa=0$, the relation equivalent to (2.74) or (3.46) holds:

$$\int d\hat{\Omega}k_1\sigma_{11}^{(q/12+23)}(\hat{\Omega}|\hat{\Omega}')+\int d\hat{\Omega}\,k_3\sigma_{31}^{(q/12+23)}(\hat{\Omega}|\hat{\Omega}') = 4\pi k_2\sigma_{21}^{(12)}(Z=0|\hat{\Omega}') = k_2\int d\hat{\Omega}\,\sigma_{21}^{(12)}(\hat{\Omega}|\hat{\Omega}') , \tag{5.34}$$

which is a direct consequence of relations (5.31) and

$$S_A^{(12+23)}(-L|0) = (\gamma_2/Z^{(23)})Y_0(Z^{(12)}+Y_0)^{-1} , \tag{5.35}$$

and, together with the corresponding relation (3.16) for each boundary, ensures the optical relation for the entire system, (3.45).

In the case of a semi-infinite random layer where $L=\infty$ and $\kappa=0$, Eq. (5.32b) is reduced in view of Eq. (5.33a) to

$$\sigma_{11}^{(q/12+\infty)}(\hat{\Omega}|\hat{\Omega}') \equiv (\gamma_2/Z^{(12)})\sigma_{12}^{(12)}(\hat{\Omega}|Z)\sigma_{21}^{(12)}(Z=0|\hat{\Omega}') , \tag{5.36}$$

which depends on the medium characteristics only through $\sigma^{(12)}$ and $Z^{(12)}$. In terms of $\sigma_{11}^{(q/12+\infty)}$, Eq. (5.32b) can be written as

$$\sigma_{11}^{(q/12+23)}(\hat{\Omega}|\hat{\Omega}')-\sigma_{11}^{(q/12+\infty)}(\hat{\Omega}|\hat{\Omega}')=-(Z^{(23)}/Z^{(12)})\sigma_{12}^{(12)}(\hat{\Omega}|Z)S_A^{(12+23)}(-L|0)\sigma_{21}^{(12)}(Z|\hat{\Omega}') \quad (5.37a)$$

$$=-(\gamma_2/Z^{(12)})\sigma_{12}^{(12)}(\hat{\Omega}|Z)Y(Z^{(12)}+Y)^{-1}\sigma_{21}^{(12)}(Z|\hat{\Omega}') , \quad (5.37b)$$

wherein the right-hand sides are symmetrical with respect to $\hat{\Omega}$ and $\hat{\Omega}'$ in both expressions. Here (5.37a) is valid only when $\kappa=0$ or $\gamma^{(ab)}=\lambda=0$, and leads directly to the basic optical relation (5.34) by perfect cancellation of the $\sigma_{31}^{(q/12+23)}$ term by the right-hand side, in virtue of the expression (5.32c) with relations (5.31). This means that, including the case of $\kappa\neq0$, the right-hand-side term of (5.37b) has the same accuracy as $\sigma_{31}^{(q/12+23)}$ for the transmitted wave when $\gamma_2 L \gg 1$, i.e., a sufficient accuracy expected by the diffusion approximation.

A. Case of a random layer with smooth boundaries

From Eq. (3.14), $\sigma_{ab}^{(12)}$ in this case is given by

$$\sigma_{ab}^{(12)}(\hat{\Omega}|\hat{\Omega}')=|\Omega_z^{(a)}\langle R_{ab}^{(12)}(\hat{\Omega})\rangle^2|\delta_S^2(\hat{\Omega}^{(a)}-(\hat{\Omega}^{(a)})')$$

$$=|\Omega_z^{(b)}\langle R_{ba}^{(12)}(\hat{\Omega})\rangle^2|\delta_S^2(\hat{\Omega}^{(b)}-(\hat{\Omega}^{(b)})') . \quad (5.38b)$$

Here the reflection-transmission coefficient $\langle R_{ab}^{(12)}\rangle$ is given, when using Eq. (2.33b), by

$$\langle R_{ab}^{(12)}(\hat{\Omega})\rangle=\frac{2|\Omega_z^{(b)}|}{|\Omega_z^{(b)}|+(k_a/k_b)|\Omega_z^{(a)}|}, \quad a\neq b , \quad (5.39)$$

and the second expression is a consequence of the reciprocity (3.49a) or, more directly,

$$\delta_S^2(\hat{\Omega}^{(a)}-(\hat{\Omega}^{(a)})')=\left|\frac{\partial\hat{\Omega}^{(b)}}{\partial\hat{\Omega}^{(a)}}\right|\delta_S^2(\hat{\Omega}^{(b)}-(\hat{\Omega}^{(b)})') , \quad (5.40a)$$

$$\left|\frac{\partial\hat{\Omega}^{(b)}}{\partial\hat{\Omega}^{(a)}}\right|=(k_a/k_b)^2|\Omega_z^{(a)}/\Omega_z^{(b)}| . \quad (5.40b)$$

Hence

$$\int d\hat{\Omega}'\sigma_{12}^{(12)}(\hat{\Omega}'|\hat{\Omega})=|\Omega_z^{(1)}\langle R_{12}^{(12)}(\hat{\Omega})\rangle^2| , \quad (5.41)$$

and $Z^{(12)}$ is given by Eq. (5.27) with

$$\langle\sigma_{12}^{(12)}\rangle_0=\frac{1}{2\pi}\int d\hat{\Omega}^{(2)}|\Omega_z^{(1)}\langle R_{12}(\hat{\Omega})\rangle^2| \quad (5.42a)$$

from (5.26a), without any contribution from the λ term, and

$$\langle\sigma_{12}^{(12)}\rangle_1=\frac{3}{2\pi}\int d\hat{\Omega}^{(2)}|\Omega_z^{(1)}\Omega_z^{(2)}\langle R_{12}^{(12)}(\hat{\Omega})\rangle^2| . \quad (5.42b)$$

Thus the cross section per unit area of the layer surface is given, upon using Eqs. (5.1a) and (5.32b), by

$$\sigma_{11}^{(q+12+23)}(\hat{\Omega}|\hat{\Omega}')=\sigma_{11}^{(12)}(\hat{\Omega}|\hat{\Omega}')+(\gamma_2/Z^{(12)})\sigma_{12}^{(12)}(\hat{\Omega}|Z)[\sigma_{21}^{(12)}(Z=0|\hat{\Omega}')-Y(Z^{(12)}+Y)^{-1}\sigma_{21}^{(12)}(Z|\hat{\Omega}')] , \quad (5.43)$$

where, using (5.38b) in (5.29),

$$\sigma_{12}^{(12)}(\hat{\Omega}|Z)=(4\pi/\gamma_2)\sigma_{21}^{(12)}(Z|-\hat{\Omega})|_{\lambda\to-\lambda}$$

$$=\gamma_2^{-1}(1+3|\Omega_z^{(2)}|Z^{(12)}+i3D_2\lambda\cdot\Omega^{(2)})|\Omega_z^{(2)}\langle R_{21}^{(12)}(\hat{\Omega})\rangle^2| , \quad (5.44)$$

which is a real function when $\lambda=0$.

B. Reciprocity

From Eqs. (5.32c) and (5.30), the reciprocity

$$\sigma_{31}^{(q/12+23)}(\hat{\Omega}|\hat{\Omega}')=\sigma_{13}^{(q/12+23)}(-\hat{\Omega}'|-\hat{\Omega}) \quad (5.45)$$

follows directly, and the same holds true also for the right-hand-side terms of (5.37) for $\sigma_{11}^{(q/12+23)}$. However, from (5.36), this is not the case of the term $\sigma_{11}^{(q/12+\infty)}$ on the left-hand side unless $Z^{(12)} \ll 1$ [as realized when $k_1/k_2 \ll 1$ in view of (5.54b)] in spite of satisfying the correct optical condition

$$k_1\int d\hat{\Omega}\,\sigma_{11}^{(q/12+\infty)}(\hat{\Omega}|\hat{\Omega}')=k_2\int d\hat{\Omega}\,\sigma_{21}^{(12)}(\hat{\Omega}|\hat{\Omega}') , \quad (5.46)$$

in view of (5.31a) and (5.29b). This implies that the diffusion approximation is not good enough for the term $\sigma_{11}^{(q/12+\infty)}$ in spite of its success for the other terms given by the right-hand sides of the expressions (5.37).

Formally, a symmetrical (reciprocal) expression can be obtained by taking the principal value of $2_z S_A^{(12+\infty)}(z|z')$ at the discontinuity $z=z'=0$, which results in giving $\sigma_{11}^{(q/12+\infty)}$ by the average of the expression (5.36) and the same expression with the interchange of $\hat{\Omega}$ and $-\hat{\Omega}'$. But, the resulting cross section does not satisfy relation (5.46) necessary to satisfy the basic (5.34). Another symmetrical expression is obtained by using an asymptotic expression of the integral representation (3.40c) at $|\lambda_z|,|\lambda_z'|\sim\infty$, upon making use of the diffusion condition that the change of $S_{22}^{(q/12+\infty)}(\hat{\Omega},z|\hat{\Omega}',z')$ be negligibly small within the distance $|\lambda_z|^{-1}\sim\gamma_2^{-1}$. Here, to evaluate the integral by using Eq. (B1), we first disregard the discontinuity at $z=z'$ so that we obtain the expression to the first order of ∂_z ($\partial_z'=0$), as

$$S^{(q/12+\infty)}_{+2,+2}(\hat{\Omega}|\hat{\Omega}')=(\lambda_z\lambda_z')^{-1}[1-(i\lambda_z)^{-1}\partial_z]S^{(q/12+\infty)}_{22}(\hat{\Omega},z(=0)|\hat{\Omega}',z'=-0) \tag{5.47a}$$

$$=|\Omega_z\Omega_z'|\mathcal{J}^{(q/12+\infty)}_{22}(\hat{\Omega},z=0|\hat{\Omega}',z'=-0)\ , \tag{5.47b}$$

where the last expression is required from Eq. (3.40a), to be consistent. Here $S^{(q/12+\infty)}_{22}(\hat{\Omega},z|\hat{\Omega}',z')$ on the right-hand side of (5.47a) is given from (5.17), (5.19a), and (5.11a), by

$$S^{(q/12+\infty)}_{22}(\hat{\Omega},z|\hat{\Omega}',z')=(4\pi)^{-1}\gamma_2[1+(\gamma_2^{-1}-3D_2)\Omega_z\partial_z]S^{(q/12+\infty)}_{A}(z|z'=-0),\quad \lambda=0$$

and λ_z and λ_z' are given by Eqs. (3.34). Hence Eq. (5.47a) agrees in fact with Eq. (5.47b) when using the diffusion expression (B13); this holds true including the case $\lambda\neq 0$, as may be shown by subsituting the λ-dependent expressions (3.34) in the first factor $(\lambda_z\lambda_z')^{-1}$.

On the other hand, if we took into account the discontinuity of the integrand at $z=z'$, the expression (5.47a) would be replaced by

$$S^{(q/12+\infty)}_{+2,+2}(\hat{\Omega}|\hat{\Omega}')=(\lambda_z\lambda_z')^{-1}[1+i(\lambda_z-\lambda_z')^{-1}\partial_z]S^{(q/12+\infty)}_{22}(\hat{\Omega},z(=0)|\hat{\Omega}',z'=-0)\ , \tag{5.47c}$$

which is invariant against the interchange λ_z and $-\lambda_z'$, fulfilling therefore the reciprocity, but is not consistent with the realtion (3.46) necessary to ensure power conservation.[12]

Summarizing, the diffusion equation is an equation for the coefficient $S^{(12+23)}_{A}(z|z')$ in the expression (5.18a) for $\mathcal{J}^{(q/12+23)}_{22}$ which, with (5.19b), is an asymptotic expression in the same sense as Eq. (5.47a) is asymptotic; therefore, whenever using the solution, it gives the asymptotic answer, directly, without any need of further asymptotic evaluation according to Eq. (5.47c).

C. Enhanced backscattering

In the basic expression (3.39b) for $\sigma^{(q+12+23)}_{11}(\hat{\Omega}|\hat{\Omega}')$, the contribution from the random medium is given by the second term in terms of $S^{(q/12+23)}_{+2,+2}(\hat{\Omega}|\hat{\Omega}')$ which, except for the factor $|\Omega_z\Omega_z'|$, is the same as the boundary value of $\mathcal{J}^{(q/12+23)}_{22}$ given to the diffusion approximation by Eqs. (5.18) with the boundary-value solution $S^{(12+23)}_{A}$ of the diffusion equation (5.15); thereby power conservation is ensured strictly. Here, from (4.43) on neglecting the K^0 term, we divide $S^{(q/12+23)}_{+2,+2}$ into two parts by

$$S^{(q/12+23)}_{+2,+2}(\hat{\Omega}|\hat{\Omega}')=S^{(L/12+23)}_{+2,+2}(\hat{\Omega}|\hat{\Omega}')+S^{(\times/12+23)}_{+2,+2}(\hat{\Omega}|\hat{\Omega}')\ , \tag{5.48}$$

as the sum of the contribution from the ladder digrams, $S^{(L/12+23)}_{+2,+2}$, and that from the maximally crossed diagrams, $S^{(\times/12+23)}_{+2,+2}$; and obtain the latter from the former by the replacement of $\hat{\Omega}$, $\hat{\Omega}'$ and λ, according to Eqs. (4.46a), (4.46b) and (4.48), respectively. Hence, in the present case of $\lambda=0$, the contribution from the former to $\sigma^{(q/12+23)}_{11}(\hat{\Omega}|\hat{\Omega}')$, say $\sigma^{(L/12+23)}_{11}$, is the same as given by Eq. (5.37b) with $Y=Y_0$ from (5.33a), while that from the latter, say $\sigma^{(\times/12+23)}_{11}$, is given, when limiting ourselves to the case of smooth boundaries and suppressing the $\Omega\cdot\lambda$ terms, by

$$\sigma^{(\times/12+23)}_{11}(\hat{\Omega}|\hat{\Omega}')\simeq(4\pi Z^{(12)})^{-1}|\Omega_z^{(2)}|^2\langle R_{21}(\hat{\Omega})\rangle^4(1+3|\Omega_z^{(2)}|Z^{(12)})[1-Y(Z^{(12)}+Y)^{-1}(1+3|\Omega_z^{(2)}|Z^{(12)})]\ , \tag{5.49}$$

in consequence of Eq. (5.44). Here Y is defined by Eq. (B7) as a function of $\kappa=\lambda=|k_2(\mathbf{\Omega}+\mathbf{\Omega}')|$, and the right-hand side of (5.49) is appreciable only when $\hat{\Omega}+\hat{\Omega}'\sim 0$ so that $|\lambda/\gamma_2|\ll 1$ [Eq. (5.51)], and independent of the $\mathbf{\Omega}\cdot\boldsymbol{\lambda}$ terms suppressed, when the wave is vertically incident.

1. Case of $\langle R_{21}\rangle\sim 1$ ($k_1\sim k_2$) and vertical incidence of the wave

For later convenience, we here assume a medium of isotropic scatterers ($a_1=0$) and introduce a new parameter τ_0 to write $Z^{(12)}=(3\tau_0)^{-1}$ in Eq. (5.49). Hence

$$\sigma^{(\times/12+23)}_{11}(\hat{\Omega}|\hat{\Omega}')=(4\pi)^{-1}3(1+\tau_0)[1-3Y(1+3Y\tau_0)^{-1}(1+\tau_0)] \tag{5.50a}$$

$$=(4\pi)^{-1}3(1+\tau_0)(1+3Y\tau_0)^{-1}(1-3Y)\ . \tag{5.50b}$$

Here, when $\lambda L\gg 1$, $\lambda=|\boldsymbol{\lambda}|$,

$$3Y\simeq\lambda/\gamma_2=(k_2/\gamma_2)|\mathbf{\Omega}+\mathbf{\Omega}'| \tag{5.51}$$

from (5.33b) and (5.11a), and hence, to the first order of λ, Eq. (5.50a) gives

$$\sigma^{(\times/12+23)}_{11}=(4\pi)^{-1}3(1+\tau_0)\times[1-(1+\tau_0)(k_2/\gamma_2)|\mathbf{\Omega}+\mathbf{\Omega}'|]\ , \tag{5.52a}$$

which is valid as long as $|\lambda/\gamma_2|\ll 1$ and $\lambda L\gg 1$, including the case $L=\infty$. Here it may be remarked that $\sigma_{11}^{(\times/12+23)}$ is dependent on the normalized scattering angle Y of Eq. (5.51) only through the second term in the square brackets of Eq. (5.50a) which is from the right-hand side of Eq. (5.37b), i.e., that term having the same accuracy as the transmitted wave with the same value of Y would have it; and, therefore, that the change with respect to the scattering angle is sufficiently accurate even though this may not be the case of the constant part [see Eqs. (5.37) and the following statements]. On the other hand, a corresponding previous result was given by[3]

$$(4\pi)^{-1}3[(\tfrac{1}{2}+\tau_0)-(1+\tau_0)^2(k_2/\gamma_2)|\mathbf{\Omega}+\mathbf{\Omega}'|] , \tag{5.52b}$$

with Milne's value $\tau_0=0.7104$ ($Z^{(12)}=0.4669\sim0.5$), showing that the their second ($\boldsymbol{\lambda}$-dependent) terms agree with each other, but the first terms do not. Here the difference comes from the fact that the method employed in the latter is equivalent to using the asymptotic expression (5.47c), instead of (5.47a).

Equation (5.50b) shows that the cross section becomes negative for $3Y>1$; but, this range is beyond the available range of the diffusion equation, i.e., $|\hat{\lambda}/\gamma_2|\ll1$. Mathematically, this is a consequence of using the asymptotic expression (5.47a) even in the range where the integrand in (3.40c) changes rapidly with a factor $\exp(\lambda z')$, $|\lambda/\gamma_2|\gtrsim1$, from $S_A^{(12+23)}(0|z')$ [see Eq. (B5)]; hence the difficulty can be overcome by the replacement of $\lambda_z'\to\lambda_z'+i\lambda$, resulting in the replacement of

$$1-3Y\to(1+\lambda/\gamma_2)^{-1}, \quad \lambda L\gg1 . \tag{5.53}$$

2. *Quantitative discussion of the case $k_1/k_2\ll1$*

Assuming the vertical incidence of the wave, we obtain

$$\langle R_{21}\rangle\sim2k_1/k_2 , \tag{5.54a}$$

$$Z^{(12)}\sim\tfrac{2}{3}(k_1/k_2)^3\ll1 , \tag{5.54b}$$

where the last is from Eq. (B20) of Ref. 8. Hence Eq. (5.49) is reduced to

$$\sigma_{11}^{(\times/12+23)}\sim(4\pi)^{-1}\langle R_{21}\rangle^4(Z^{(12)}+Y)^{-1}(1-3Y) , \tag{5.55}$$

which is appreciable only for Y in the range $1\gg Z^{(12)}\lesssim Y$. Here, when $L=\infty$, the maximum value at $Y=0$ is given by

$$\sigma_{11}^{(\times/12+23)}\sim(4\pi Z^{(12)})^{-1}\langle R_{21}\rangle^4 \tag{5.56a}$$

$$\sim(4\pi)^{-1}24(k_1/k_2) , \tag{5.56b}$$

in consequence of Eqs. (5.54).

On the other hand, the contribution purely from the boundary reflection is presently

$$\langle R_{11}\rangle^2\sim[1-2(k_1/k_2)]^2 ,$$

which, when $k_1/k_2=0.2$, is 0.4, being comparable with the value given by (5.56b). In fact, the precise calculation shows that $\sigma_{11}^{(\times/12+23)}=0.2413$ at the peak , with $Z^{(12)}=0.004\,122$. It is thus suggested that, when $k_1/k_2\gtrsim0.2$ and $L=\infty$, the peak value of $\sigma_{11}^{(q/12+23)}$ due to the enhanced backscattering can exceed the value by the specular boundary reflection.

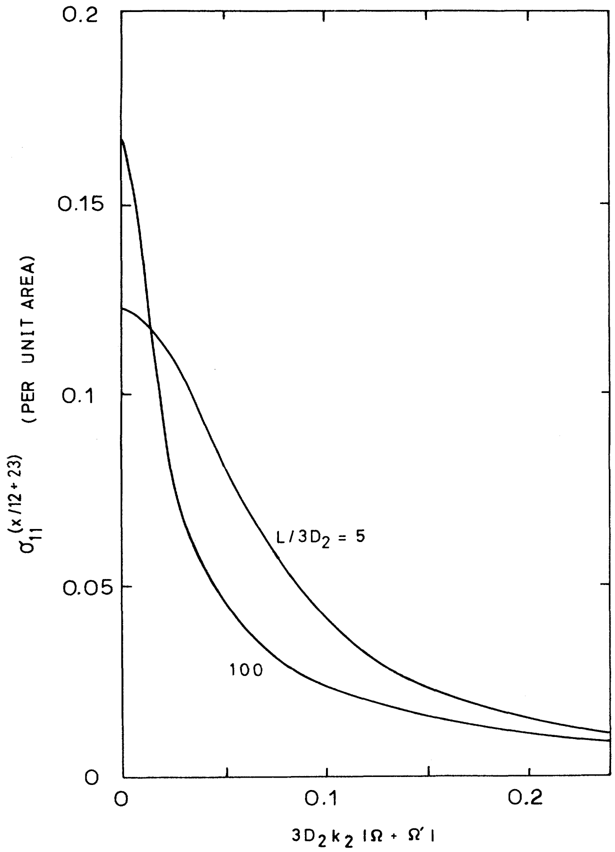

FIG. 4. The cross section per unit area $\sigma_{11}^{(\times/12+23)}$ by Eq. (5.49) is shown as a function of the normalized scattering angle $3D_2k_2|\mathbf{\Omega}+\mathbf{\Omega}'|$. The wave is vertically incident on a nondissipative layer of width L, $k_1/k_2=0.2$, and smooth boundaries. The parameter is a normalized width $L/3D_2$, which is chosen to be 5 and 100. $Z^{(12)}=0.004\,122$.

Illustrated in Fig. 4 by using Eq. (5.49) is the cross section per unit area $\sigma_{11}^{(\times/12+23)}$ as a function of the normalized scattering angle $3D_2k_2|\mathbf{\Omega}+\mathbf{\Omega}'|$ when the wave is vertically incident on a nondissipative layer of width L and $k_1/k_2=0.2$. The parameter is the normalized width $L/3D_2$ and chosen to be 5 and 100.

VI. SUMMARY AND DISCUSSION

The solution of the BS equation for a random layer of q_2 with two rough boundaries $S_{12}+S_{23}$ was obtained in terms of the resultant scattering matrix $\sigma_{ab}^{(q+12+23)}$, $a,b=1,3$ for the entire volume. The latter matrix can be constructed by successive addition of independent scattering matrices of the medium and the boundaries, and fulfills the optical condition to be consistent with power conservation, as so does each of the scattering matrices fulfilling their own. Their optical expressions [Eqs. (3.38)–(3.41)] are obtained therefrom along with the respective optical conditions [Eqs. (3.45) and (3.46)]. The enhanced backscattering can be understood as a natural consequence of requiring the coordinate-interchange invariance of the BS equation, i.e., the invariance of the second-order Green's function based on the independent reciprocity for each of the two deterministic Green's functions involved. It is hence convenient to rewrite the BS equation as an equation for the function of the four coordinates so that the invariance is immediately clear (Sec. IV). Here it may be remarked that the term $\check{K}^0$ in Eq. (4.17) also has a structure and, in consequence of this, the equation can be further rewritten so that the four coordinates are involved in it on exactly the same footing; thereby the fundamental structures of the basic matrices K and M are found.

To obtain specific expression of the cross sections, the diffusion approximation was examined in some detail based on expansions of physical quantities in terms of the eigenfunctions of the medium cross section [Eqs. (5.3) and (5.4)], together with the boundary condition of the diffusion equation [Eqs. (5.24), (5.27), and (A12)]. When the boundaries are smooth, the cross section for the backscattered waves is given by Eq. (5.43) with (5.44) or, more generally, (5.29). The result agrees with what would be obtained by asymptotic evaluation of the integral representation [Eq. (3.40c)] under the diffusion condition. Here, when the integrand is obtained by using the boundary-value solution of the diffusion equation, care is necessary about its discontinuity that could lead to a result not consistent with power conservation. The term of the enhanced backscattering can be obtained from the above results according to the coordinate-interchange principle [Eq. (4.50)]. Even to the diffusion approximation, the angle distribution of the enhanced wave holds a sufficient accuracy (although not quite for the background term) as long as the optical width of the layer is long enough [Eqs. (5.37) and (5.49)].

APPENDIX A: BOUNDARY CONDITION OF THE DIFFUSION EQUATION

To find the boundary condition, we investigate the (power) equation of continuity for $\mathcal{I}_{22}^{(q/12+23)}$ by studying first the equation for $\mathcal{I}_2^{(0q)}$ which is given, upon the $\hat{\Omega}$ integration of the transport equation (3.18), by

$$\partial_z \int_{4\pi} d\hat{\Omega}\, \Omega_z \mathcal{I}_2^{(0q)}(\hat{\Omega},z|\hat{\Omega}',z') = \gamma_2 U_2(\hat{\Omega}',z-z') , \quad \text{(A1)}$$

in consequence of optical relation (3.20). Here the power flux of $\mathcal{I}_2^{(0q)}$ in the z direction, say $\langle w_z^{(0q)} \rangle$, can be written as a sum of the two components $\langle w_z^{(0q)} \rangle^{\pm}$ propagating in the positive and negative directions, respectively, and given by

$$\langle w_z^{(0q)}(z|\hat{\Omega}',z') \rangle^{\pm} \equiv \int_{\Omega_z \gtrless 0} d\hat{\Omega}\, \Omega_z \mathcal{I}_2^{(0q)}(\hat{\Omega},z|\hat{\Omega}',z') , \quad \text{(A2)}$$

except for a numerical factor. Hence

$$\langle w_z^{(0q)} \rangle = \langle w_z^{(0q)} \rangle^{+} + \langle w_z^{(0q)} \rangle^{-} , \quad \text{(A3)}$$

and (A1) is written by

$$\partial_z \langle w_z^{(0q)}(z|\hat{\Omega}',z') \rangle = \gamma_2 U_2(\hat{\Omega},z-z') . \quad \text{(A4)}$$

Here we introduce a distance l subject to $\gamma_2 l \gg 1$, and integrate both sides of (A4) over the range $0 \ge z \ge -l$ to obtain

$$\langle w_z^{(0q)}(z|\hat{\Omega}',z') \rangle |_{z=-l}^{0} = \int_{-l}^{0} dz\, \gamma_2 U_2(\hat{\Omega}',z-z') \simeq \begin{cases} 1, & \Omega_z' < 0, \quad z'=0 \\ 0, & \Omega_z' \gtrless 0, \quad \gamma_2(l+z') \ll -1 , \end{cases} \quad \text{(A5)}$$

where use has been made of (3.11). Here, on the left-hand side, the part $\langle w_z^{(0q)} \rangle^{-}$ is zero at $z=0$ in view of (3.21a), while, for the part $\langle w_z^{(0q)} \rangle^{+}$, we assume that the length $l \gg \gamma_2^{-1}$ can be minimized so that, within the region $0 \ge z \ge -l$, $\langle w_z^{(0q)} \rangle^{+}$ (propagating toward the boundary S_{12}) remains almost unchanged; this is a severe condition not quite realized, though. Thus, on the left-hand side of (A5), $\langle w_z^{(0q)} \rangle^{+}$ makes no contribution, resulting in that

$$-\int_{\Omega_z<0} d\hat{\Omega}\, \Omega_z \mathcal{I}_2^{(0q)}(\hat{\Omega},z=-l|\hat{\Omega}',z') \simeq \begin{cases} 1, & \Omega_z' < 0, \quad z'=0 & \text{(A6)} \\ 0, & \Omega_z' \gtrless 0, \quad \gamma_2(l+z') \ll -1 . & \text{(A7)} \end{cases}$$

It is now straightforward to find the corresponding equation for the boundary-dependent $\mathcal{I}_{22}^{(q/12+23)}$ by applying the above relations to the governing equation (3.17), hence

$$-\int_{\Omega_z<0} d\hat{\Omega}\, \Omega_z \mathcal{I}_{22}^{(q/12+23)}(\hat{\Omega},z=-l|\hat{\Omega}',z') \simeq \int_{\Omega_z<0} d\hat{\Omega} \int_{\Omega_z''>0} d\hat{\Omega}''\, \sigma_{22}^{(12)}(\hat{\Omega}|\hat{\Omega}'') \mathcal{I}_{22}^{(q/12+23)}(\hat{\Omega}'',z''=0|\hat{\Omega}',z') , \quad \text{(A8)}$$

where $\gamma_2(l+z') \ll -1$, $\Omega_z' \gtrless 0$, and the $\sigma_{22}^{(23)}$ term does not make a contribution, in view of (A7). Equation (A8) simply says that the total backscattered power by the boundary is transported without gettting any change to an imaginary plane at $z=-l$ assumed in the diffusion region. Here we observe that the left-hand side of (A8) can therefore be approximated by the diffusion term (5.18a), and also that

$$S_A^{(12+23)}(z=-l|z') \simeq S_A^{(12+23)}(z=0|z') , \quad \text{(A9)}$$

being a slowly changing function of z; while, for the right-hand side, the $\hat{\Omega}''$ integration makes the contribution from the diffusion term dominant, in view of the fact that the $\hat{\Omega}$-integrated $\sigma_{22}^{(12)}(\hat{\Omega}|\hat{\Omega}'')$ provides a slowly changing factor with respect to $\hat{\Omega}''$ [Eq. (3.16)].

Thus, Eq. (A8) is reduced, upon substitution of the expression (5.18a) for $\mathcal{I}_{22}^{(q/12+23)}$, to a boundary equation of the form

$$(2\gamma_2)^{-1}(\tfrac{1}{2}+D_2\partial_z)S_A^{(12+23)}[z(=0)|z'] = \langle \sigma_{22}^{(12)}(\partial_z)\rangle S_A^{(12+23)}[z(=0)|z'] \ . \quad \text{(A10)}$$

Here, with (5.19b), use has been made of

$$(4\pi)^{-1}\int_{\Omega_z \gtrless 0} d\hat{\Omega}\, \Omega_z \phi_A(\hat{\Omega}, i\partial_z) = (2\gamma_2)^{-1}(\pm\tfrac{1}{2} - D_2\partial_z) \ , \quad \text{(A11)}$$

and $\langle \sigma_{22}^{(12)}(\partial_z)\rangle$ is defined by (5.21). Hence we directly obtain an expression of $Z^{(12)}$ defined by the boundary equation (5.24), as[8]

$$Z^{(12)} = (\tfrac{1}{2} - \langle \sigma_{22}^{(12)}\rangle_0)/(1+\langle \sigma_{22}^{(12)}\rangle_1) \ , \quad \text{(A12)}$$

in terms of $\langle \sigma_{22}^{(12)}\rangle_0$ and $\langle \sigma_{22}^{(12)}\rangle_1$ by Eqs. (5.26).

APPENDIX B: SOLUTIONS OF THE DIFFUSION EQUATION

A plane-wave solution of the diffusion equation (5.15) is given by

$$S_A^{(12+23)}(z|z') = C_A \varphi^{(12)}(z_>)\varphi^{(23)}(z_<) \ . \quad \text{(B1)}$$

Here $z_>$ and $z_<$ designate the larger and the smaller of z and z', respectively, and $\varphi^{(12)}(z)$ and $\varphi^{(23)}(z)$ are solutions of the homogeneous diffusion equation subjected to the boundary conditions at $z=0$ and $-L$, respectively; C_A is a constant given by

$$C_A = \frac{\gamma_2}{D_2}\left[\varphi^{(12)}\frac{\partial}{\partial z}\varphi^{(23)} - \varphi^{(23)}\frac{\partial}{\partial z}\varphi^{(12)}\right]^{-1} \ . \quad \text{(B2)}$$

Hence, by using the boundary conditions (5.24), (5.28), and a parameter κ defined by

$$\kappa = (\gamma^{(ab)}D_2^{-1} + \lambda^2)^{1/2} \ , \quad \text{(B3)}$$

we can set

$$\varphi^{(12)}(z) = \cosh(\kappa z) - Z^{(12)}(\kappa D_2)^{-1}\sinh(\kappa z) \ , \quad \text{(B4)}$$

$$\varphi^{(23)}(z) = \cosh[\kappa(z+L)] + Z^{(23)}(\kappa D_2)^{-1}\sinh[\kappa(z+L)] \ , \quad \text{(B5)}$$

and determine C_A at $z=0$, hence

$$C_A = \gamma_2\{Z^{(12)}[\cosh(\kappa L) + Z^{(23)}(\kappa D_2)^{-1}\sinh(\kappa L)] + \kappa D_2 \sinh(\kappa L) + Z^{(23)}\cosh(\kappa L)\}^{-1} \ . \quad \text{(B6)}$$

The solutions can be written, in terms of a new variable Y defined by

$$Y = \kappa D_2 \left[\frac{\kappa D_2 \tanh(\kappa L) + Z^{(23)}}{\kappa D_2 + Z^{(23)}\tanh(\kappa L)}\right] \ , \quad \text{(B7)}$$

as

$$S_A^{(12+23)}(0|0) = \gamma_2(Z^{(12)}+Y)^{-1} \ , \quad \text{(B8)}$$

$$S_A^{(12+23)}(-L|0) = S_A^{(12+23)}(0|0)[\cosh(\kappa L) + (\kappa D_2)^{-1}Z^{(23)}\sinh(\kappa L)]^{-1} \ . \quad \text{(B9)}$$

Hence, in the special case of $\kappa=0$ or $\gamma^{(ab)}=\lambda=0$, we obtain

$$Y|_{\kappa=0} \equiv Y_0 = Z^{(23)}[1+(L/D_2)Z^{(23)}]^{-1} \ , \quad \text{(B10)}$$

$$S_A^{(12+23)}(-L|0) = (\gamma_2/Z^{(23)})Y_0(Z^{(12)}+Y_0)^{-1} \ , \quad \text{(B11)}$$

together with the relation

$$Z^{(12)}S_A^{(12+23)}(0|0) + Z^{(23)}S_A^{(12+23)}(-L|0) = \gamma_2 \ . \quad \text{(B12)}$$

Here the last relation can be generally shown by integrating Eq. (5.15) over the range $0 \ge z \ge -L$ for $z'=-0$ and followed using the boundary conditions.

To derive expression (5.32b) for $\sigma_{11}^{(q/12+23)}$, we substitute (5.19b) in (5.18a) to obtain

$$\mathcal{I}_{22}^{(q/12+23)}(\hat{\Omega}, z|\hat{\Omega}', z') = (4\pi\gamma_2)^{-1}[1+3D_2(i\Omega\cdot\lambda - \Omega_z\partial_z)][1+3D_2(i\Omega'\cdot\lambda + \Omega_z'\partial_z')]S_A^{(12+23)}(z|z') \ . \quad \text{(B13)}$$

Here, using the boundary condition (5.24),

$$D_2\partial_z' S_A^{(12+23)}[z=0|z'(=0)] = [D_2(\partial_z' - \partial_z) + D_2\partial_z]S_A^{(12+23)}[z(=0)|z'(=0)] = \gamma_2 - Z^{(12)}S_A^{(12+23)}(0|0) \ , \quad \text{(B14)}$$

in consequence of Eqs. (B1) and (B2). Hence the substitution in (B13) and use of (B8) lead to the expression

$$\mathcal{I}_{22}^{(q/12+23)}(\Omega, z=0|\hat{\Omega}', z'=0) = (4\pi)^{-1}(Z^{(12)}+Y)^{-1}(1+i3D_2\Omega\cdot\lambda + 3\Omega_z Z^{(12)})(1+i3D_2\Omega'\cdot\lambda + 3\Omega_z' Y) \quad \text{(B15)}$$

$$= (\gamma_2/Z^{(12)})\phi_A(\Omega, i\partial_z)|_Z[\bar{\phi}_A(\hat{\Omega}', i\partial_z=0) - Y(Z^{(12)}+Y)^{-1}\bar{\phi}_A(\hat{\Omega}', -i\partial_z)|_Z] \ , \quad \text{(B16)}$$

which yields expression (5.32b) for $\sigma_{11}^{(q/12+23)}$, directly, according to the definition (5.2a) with Eqs. (5.29), and also expression (5.32a) in consequence of (B8). As for expression (5.32c) for $\sigma_{31}^{(q/12+23)}$, the derivation is straightforward by using the boundary equations (5.24) and (5.28) in Eq. (B13) with $z=-L$ and $z'=0$.

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