Generalized back-action evasion schemes for the detection of weak classical forces

Roberto Onofrio
Dipartimento di Fisica dell'Università di Roma "La Sapienza," Scuola del Dottorato di Ricerca, Piazzale Aldo Moro, 2-00185 Roma, Italy

Franco Bordoni*
Istituto di Fisica dello Spazio Interplanetario, Consiglio Nazionale delle Ricerche, via Galileo Galilei, 00044 Frascati, Italy
(Received 13 August 1990)

After a general review of the analysis already available for back-action evasion schemes a perturbative approach is introduced and applied to a simple case. As a generalization of the back-action evasion schemes the concept of optimal pumping spectrum is introduced: it works for a unified treatment of stroboscopic and continuous quantum nondemolition strategies. Generalized back-action evasion strategies are introduced and studied in simple applications: they open the possibility for developing high-sensitivity schemes for the measurements of small forces. In particular, estimates of the noise figure for generalized back-action evasion transducers are discussed in detail.

I. INTRODUCTION

In the current attempts to test the predictions of the standard model of physics (gauge theories of strong and electroweak interactions and general relativity), a crucial point is the identification of the force carriers, called intermediate bosons, between the particles of which the matter consists, i.e., quarks and leptons. With the recent discovery of the intermediate vector bosons, responsible for charged and neutral weak interactions, the everyday observability of photons and the impossibility of directly observing gluons as suggested by quantum chromodynamics, the only sector in the standard model framework which seems open to new discovery of intermediate bosons is the graviton detection. Due to the weakness of gravitational interaction it is easy to see that the only events available for the search of gravitons [and only in their classical nature as gravitational waves (GW)] are astrophysical events. Indeed the search for bursts of gravitational waves emitted during the collapses of supernovae in the Virgo cluster of galaxies, which are expected to occur with a reasonable rate, is one of the objectives of current research in experimental gravitation.[1] Among the working GW detectors there are the resonant antennae which can be schematized as macroscopic oscillators.[2] The resonant GR antennae now operative work by monitoring the displacements of one collective mode of oscillation of a body, schematized as a macroscopic harmonic oscillator. The impinging GW burst can be detected by measuring the variations in the amplitude and phase induced in the oscillations and properly converted into an electrical signal. However, to achieve such a goal it is necessary to measure displacements of the same order of magnitude as, or even lower than, the de Broglie wavelength associated with the same harmonic oscillator.[3–5] A new technique of measurement which overcomes the limitation of this standard quantum limit has been developed. The idea is that of measuring only one component of the harmonic oscillator's complex amplitudes, instead of measuring both of them as usual. These new measurement strategies are known as quantum nondemolition (QND) methods; in a classical scenario they allow one to overcome the limit imposed on the measurement by amplifier noise. They are also known as back-action evasion (BAE) strategies of measurements.[5,6] It is important to stress that BAE techniques, despite their origin from quantum mechanical considerations, are already useful in a classical regime of sensitivity. Indeed, they may be conceived as a recipe to increase the sensitivity of a displacement detector using particular phase-sensitive parametric transducers. This is the basis of the early developments in many gravitational laboratories, although they are far from reaching the quantum limit of sensitivity. Classical models of BAE methods have been developed in various groups and some of their characteristics have been tested so far with an appropriate experimental setup. At present there is no experimental evidence on their ability to obtain noise reduction.[7–11]

We deal here with the BAE strategies which are derived from a generalization of the usual BAE technique. The paper is organized as follows. In Sec. II a brief review of the models previously developed to treat BAE devices is given. In Sec. III a perturbative approach is introduced and applied to calculate the equation of a parametric transducer under a resonant forcing term. The concept of optical pumping spectrum is then introduced giving the general framework in which it can be fully developed. We show that the stroboscopic measurement scheme may be considered as a limiting case of an optimal pumping spectrum with an infinite number of pumps: this allows the introduction of new generalized BAE schemes. In Sec. IV we report some considerations on the analysis of generalized parametric transduction schemes in two different situations. The first one corresponds to the use of more than two pumps in the parametric transducer. The use of a multipump system allows one to overcome the limit on the sensitivity imposed by means of a low-dissipation electrical circuit. The second deals with the feasibility of BAE stroboscopic measurements which may overcome the use of sinusoidal pump signals and the related limitations imposed by the amplitude and phase noise of such devices. Final remarks on the potentialities of this line of research for the implementation of new high-sensitivity measurement schemes are given in Sec. V.

II. THE BACK-ACTION EVASION EFFECT: A REVIEW

The measurement of mechanical quantities as position or momentum is obtained by correlating them to proper electrical quantities. A device which operates in such a way is called an electromechanical transducer. A simple example of an electromechanical transducer is obtained when a mechanical oscillator (mass m and frequency $\omega_1/2\pi$) and an electrical oscillator (inductance L and frequency $\omega_2/2\pi$) are coupled through an interaction term. When the dissipative effects of the two oscillators are neglected Hamiltonian formalism can be applied, obtaining for the whole system

$$H=\frac{p^2}{2m}+\frac{m\omega_1^2x^2}{2}+\frac{\pi^2}{2L}+\frac{L\omega_2^2q^2}{2}+H_i-xF(t)-qV(t)\ , \tag{1}$$

where (x,p) and (q,π) are coordinates and momenta of the two oscillators, F and V are the external force and voltage acting on them.

If $H_i=0$ the two systems are uncoupled and the relative temporal evolutions will be uncorrelated. An interaction between the two oscillators allows us to know the mechanical quantities via the electrical ones and, on the other hand, perturbs the same quantities: such perturbation is what we mean by back action.

In order to show this effect more explicitly we assume that the interaction Hamiltonian H_i has the form

$$H_i=E(t)xq\ , \tag{2}$$

where $E=E(t)$ is time dependent. By doing so we restrict our analysis to systems in which we measure the mechanical displacement by means of the charge variations in the electrical oscillator, as depicted in Fig. 1.

The Hamiltonian equations take the well-known expressions

$$\dot{x}=\frac{p}{m},\quad \dot{p}=-m\omega_1^2x-E(t)q+F\ , \tag{3}$$

$$\dot{q}=\frac{\pi}{L},\quad \dot{\pi}=-L\omega_2^2q-E(t)x+V\ . \tag{4}$$

In order to have maximum sensitivity we want to minimize the back-action effect maximizing at the same time the information on the status of the mechanical oscillator. This is equivalent to the following detection problem: for a given amount of false alarm probability produced by part of the back-action effect we want to apply strategies that maximize the information on some of the quantities of the mechanical oscillator.[12] We note that

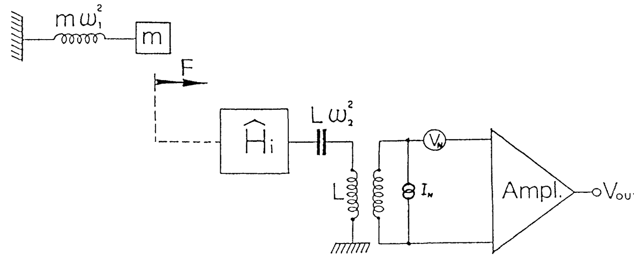

FIG. 1. Transduction of the displacement x of a mechanical oscillator through a parametric coupling to an electrical oscillator.

both the information and the back-action effects depend linearly on the amplitude of the electric field. Therefore this parameter cannot be used in order to obtain the highest ratio between the information on the mechanical oscillator and back-action effects acting on it. Improvements may be obtained only by means of a proper choice of the time dependence of the electric field. It is interesting to note that such choice can also be suggested by studying the same system quantum mechanically. Indeed the analysis of the sensitivity limits of such a detection system in the quantum regime give rise to the introduction of the quantum nondemolition measurement strategies in which a repeated set of high-accuracy measurements of only one observable is possible, in agreement with the Heisenberg uncertainty principle. The quantum uncertainties always present in a measurement will be enhanced in the other noncommuting observables. According to quantum theory of measurement two conditions have to be satisfied in order to perform a QND strategy. An unperturbed measurement of the observables $\hat{X}_\alpha$ is obtained if the interaction Hamiltonian operator $\hat{H}_i$ commutes with $\hat{X}_\alpha$,

$$[\hat{H}_i,\hat{X}_\alpha]=0\ . \tag{5}$$

This may be satisfied if $\hat{H}_i$ depends only on $\hat{X}_\alpha$ and the other simultaneously diagonizable observables $\hat{Y}_\beta$ are such that $[\hat{X}_\alpha,\hat{Y}_\beta]=0$. A particular example is that of linear dependence of the interaction Hamiltonian on $\hat{X}_\alpha$ through a function $O(\hat{Y}_\beta)$,

$$\hat{H}_i=O(\hat{Y}_\beta)\hat{X}_\alpha\ . \tag{6}$$

The repeatability of the measurement of $\hat{X}_\alpha$ in a nondemolition way, i.e., without Heisenberg uncertainties in this observable, in the measurement time set τ: $\tau=t_0,t_1,\ldots,t_n$ is obtained if $\hat{X}_\alpha$ commutes with itself in the time set τ, i.e.,

$$[\hat{X}_\alpha(t_i),\hat{X}_\alpha(t_j)]=0,\quad \forall t_i,t_j\in\tau\ . \tag{7}$$

If τ is a discrete set the strategy is also called stroboscopic, as opposed to the other situation of a continuous set corresponding to a so called continuous QND strategy. These considerations may be applied to the transducers as modeled by Eqs. (1)–(4). It is possible to prove that $\hat{x}$ is an observable commuting with itself each half a period of the motion

$$[\hat{x}(t+\Delta t_n),\hat{x}(t)]=i\frac{\hbar}{m\omega_1}\sin(\omega_1\Delta t_n)\ , \tag{8}$$

where $\Delta t_n = n\pi/\omega_1$, i.e., it is, on the basis of (7), a stroboscopic observable. Thus a stroboscopic measurement of $\hat{x}$ is obtained by means of an interaction Hamiltonian simultaneously diagonizable with $\hat{x}$ according to (5):

$$\hat{H}_i = E(t)\hat{x}\hat{q} = E_0\delta\left[t-\frac{n\pi}{\omega_1}\right]\hat{x}\hat{q}\ . \tag{9}$$

On the other hand the complex amplitude $\hat{X}_1$ defined as

$$\hat{X}_1 = \mathrm{Re}\left[\left[\hat{x}+i\frac{\hat{p}}{m\omega_1}\right]e^{i\omega_1 t}\right] \tag{10}$$

is an example of a continuous QND observable, being

$$[\hat{X}_1(t+\Delta t),\hat{X}_1(t)]=0,\quad \forall\Delta t\ , \tag{11}$$

and a QND continuous measurement of $\hat{X}_1$ may be obtained with the interaction Hamiltonian

$$\hat{H}_i = E(t)\hat{X}_1\hat{q} = E_0\cos(\omega_2 t)\cos(\omega_1 t)\hat{x}\hat{q}\ , \tag{12}$$

provided that a filtering around $\omega_2/2\pi$ is used in order to suppress the effects at the frequencies $\omega_2 \pm 2\omega_1$. A continuous-average measurement of a complex amplitude of the mechanical oscillator is achieved when the electric field depends on time as

$$E(t)=\frac{E_0}{2}\{\cos[\omega_2-\omega_1)t]+\cos[\omega_2+\omega_1)t]\}\ , \tag{13}$$

i.e., by pumping with a coherent superposition at the frequencies $\omega_2-\omega_1$ and $\omega_2+\omega_1$.

It has been suggested that in the classical limit the same strategy allows the limit imposed by a classical amplifier to be overcome, and therefore it is known as back-action evasion (BAE) effect strategy. Now we summarize the classical models developed to describe BAE measurements.

Introducing in Eqs. (3) and (4) the effects of the dissipation the final equation of the motion for x and q can be written as

$$\ddot{x}+\gamma_1\dot{x}+\omega_1^2 x+\frac{E(t)}{m}q=\frac{F}{m}\ , \tag{14}$$

$$\ddot{q}+\gamma_2\dot{q}+\omega_2^2 q+\frac{E(t)}{L}x=\frac{V}{L}\ , \tag{15}$$

where the two dissipation constants γ_1 and γ_2 are related to the mechanical and electrical quality factors Q_1, Q_2 as $\gamma_{1,2}=\omega_{1,2}/2Q_{1,2}$. The external force F acting on the mechanical oscillator has at least two contributions, the known or unknown signal force and a random Langevin force responsible for the Brownian motion of the oscillator. The external voltage V is given by the sum of two contributions, the voltage and current noise of the amplifier and the Johnson noise in the electrical circuit. Dissipation constant and force noise are related between them through the fluctuation-dissipation theorem.

Equations (14) and (15) are a system of two coupled linear differential equations with variable coefficients. They cannot be exactly solved by using the Fourier transform technique due to the time dependence of the interaction parameter. This does not allow an exact solution in term of the Mathieu functions to be obtained due to the presence of a mutual coupling between the two oscillators. The sensitivity of an electromechanical transducer for detecting impulsive forces can be expressed using the burst noise temperature T_b. $E_b = k_B T_b$ (k_B being the Boltzmann constant) is the energy which would be deposited in a mechanical oscillator from an impulse in order to obtain in the filtered output a signal-to-noise ratio of one.[13] The evaluation of the sensitivity dependence on the particular $E(t)$ chosen can be quantified by a merit factor r such that the burst noise temperature T_b can be written as

$$T_b = 2\frac{\omega_1}{\omega_2}T_N\frac{1}{r}\ , \tag{16}$$

where T_N is the amplifier noise temperature. For the back-action evasion devices is $r>1$. An approximate solution for the system (14) and (15) when the electric field (13) is used can be obtained by using the Mathieu functions.[14] The calculated value of the r factor is, for a single pumping at the frequency $\omega_2-\omega_1$,[10]

$$r=\left[1+\frac{\alpha}{\beta}\frac{\omega_2}{\omega_1}\frac{1}{Q_e}\right]^{-1/2}\ , \tag{17}$$

where

$$\alpha=\frac{T}{T_N}\frac{\omega_2}{\omega_1}\frac{1}{Q_1} \tag{18}$$

is a parameter expressing the ratio between the Brownian noise and the amplifier noise, T is the thermodynamical temperature of the system, and

$$\beta=\frac{\omega_2}{\omega_1}\frac{E_0^2}{m\omega_1^2 L\omega_2^2} \tag{19}$$

is the electromechanical coupling factor for the parametric transducer. When a coherent pumping at $\omega_2-\omega_1$ and $\omega_2+\omega_1$ is used we obtain[10]

$$r=\left[\frac{1}{8}\left[\frac{\omega_2}{\omega_1}\right]^2\frac{1}{Q_2^2}+8\frac{\alpha}{\beta}\frac{\omega_2}{\omega_1}\frac{1}{Q_2}\right]^{-1/2}\ . \tag{20}$$

In this calculation the matching between the electrical oscillator impedance and the amplifier noise impedance is not considered. Also, the solutions are meaningful only if the small coupling limit, i.e., $\beta \ll 1$.

A complementary approach has been developed by using the Fourier transform of the equation.[15] Terms at high frequencies and not close to the frequency ω_2 are neglected and this assumption is valid only in the limit of high ω_2/ω_1 ratios. In this hypothesis r has been expressed as

$$r=\left[\left[1+\frac{8\gamma}{\beta}\left[\alpha+\frac{3\beta\gamma}{64}\right]\right]\times\left[1+\frac{\beta}{4\gamma^2}\left\{\left[1+\frac{8\gamma}{\beta}\left[\alpha+\frac{3\beta\gamma}{64}\right]\right]^{1/2}-1\right\}\right]\right]^{-1/2}, \tag{21}$$

for a single pumping at the frequency $\omega_2-\omega_1$ and

$$r=\left[\frac{8\gamma}{\beta}\left[\alpha+\frac{3\beta\gamma}{64}\right]\right.\\ \left.\times\left\{1+\frac{\beta}{4\gamma^2}\left[\frac{8\gamma}{\beta}\left[\alpha+\frac{3\beta\gamma}{64}\right]\right]^{1/2}\right\}\right]^{-1/2} \tag{22}$$

for a coherent pumping at $\omega_2-\omega_1$ and $\omega_2+\omega_1$, where α and β have the same meaning as in the previous case and γ takes into account the matching between the electrical oscillator impedance and the amplifier noise impedance.

By analyzing the previous expression we observe that, despite the very different approximations, both the models confirm that in the limit of negligible Brownian noise ($\alpha\to 0$) the r factor is $\ll 1$ for the single mode pumping of (17) and (21) and can be >1 only for a BAE mode pumping of (20) and (22) giving a quantitative support to the back-action evasion effect.

III. A PERTURBATIVE APPROACH FOR THE EQUATIONS OF PARAMETRIC DEVICES AND THE OPTIMAL PUMPING SPECTRUM

A new method for the solution of the system (14) and (15) can be obtained when the interaction between the two oscillators is considered as a small perturbation on the motion of the two uncoupled oscillators. The system (14) and (15) can be written as

$$\ddot{x}+\gamma_1\dot{x}+\omega_1^2x+\lambda E(t)q=\frac{F}{m}\ , \tag{23}$$

$$\ddot{q}+\gamma_2\dot{q}+\omega_2^2q+\lambda E(T)x=\frac{V}{L}\ , \tag{24}$$

where λ is considered as a perturbative parameter. It is possible to write the solutions of the motion equations as an expansion in the perturbative parameter $\lambda(\lambda\ll 1)$

$$x(t)=\sum_{n=0}^{\infty}\lambda^n x_n(t);\quad q(t)=\sum_{n=0}^{\infty}\lambda^n q_n(t)\ , \tag{25}$$

and by considering only the terms containing the same power of the parameter λ. We have for the zeroth order

$$\ddot{x}_0+\gamma_1\dot{x}_0+\omega_1^2x_0=\frac{F}{m}\ , \tag{26}$$

$$\ddot{q}_0+\gamma_2\dot{q}+\omega_2^2q_0=\frac{V}{L}\ , \tag{27}$$

and

$$\ddot{x}_n+\gamma_1\dot{x}_n+\omega_1^2x_n=-\frac{E}{m}q_{n-1}\ , \tag{28}$$

$$\ddot{q}_n+\gamma_2\dot{q}_n+\omega_2^2q_n=-\frac{E}{L}x_{n-1}\ , \tag{29}$$

for the successive terms. In particular, by taking into account only $n=0$, $n=1$, and $n=2$ we observe that the second and the last terms, respectively, are the first contribution to the transduction effect and to the back-action effect. In other words (x_0,q_0) is the solution of the motion due to the generalized forces while (x_1,q_1) is the correction due to the coupling between the two oscillators and (x_2,q_2) is, in a further approximation, the feedback on each oscillator due to the back-action. On the basis of this approach we can solve the zeroth-order perturbation and, after this, iteratively solve the other equations by means of the known coupling due to the previous order solution for x and q. Obviously the solution is meaningful only if $\lambda\ll 1$, i.e., in a small β regime.

An intuitive image of this approach can be easily given. For a weak coupling the main contribution to the motion of each oscillator derives from the uncoupled evolution itself. If we refine the analysis it is possible to observe the presence of a mutual interaction and, increasing the accuracy, the presence of a back-action effect which acts on each oscillator and due to this interaction.

The equations of the motion can be solved through the perturbative method by supposing the existence of two generalized forces acting on each oscillator respectively at the frequencies $\omega/2\pi$ and $\Omega/2\pi$. The electric field is expressed as

$$E(t)=\frac{E_0}{2}[(1+f)\cos(\omega_2-\omega_1)t+(1-f)\cos(\omega_2+\omega_1)t]\ , \tag{30}$$

where $f=1$, $f=-1$ respectively describe the parametric up-converter and the parametric amplifier pumping; $f=0$ corresponds to a BAE system. The equations for the zeroth-order approximation are

$$\ddot{x}_0+\gamma_1\dot{x}_0+\omega_1^2x_0=\frac{F_0}{m}e^{i\omega t}\ , \tag{31}$$

$$\ddot{q}_0+\gamma_2\dot{q}_0+\omega_2^2q_0=\frac{V_0}{L}e^{i\Omega t}\ . \tag{32}$$

We can write the solutions as

$$x_0(t)=\frac{F_0}{m}\frac{e^{i\delta}e^{i\omega t}}{[(\omega_1^2-\omega^2)^2+\gamma_1^2\omega^2]^{1/2}}\quad\left[\tan\delta=\frac{\gamma_1\omega}{\omega^2-\omega_1^2}\right]\ , \tag{33}$$

$$q_0(t)=\frac{V_0}{L}\frac{e^{i\Delta}e^{i\Omega t}}{[(\omega_2^2-\Omega^2)^2+\gamma_2^2\Omega^2]^{1/2}}\quad\left[\tan\Delta=\frac{\gamma_2\Omega}{\Omega^2-\omega_2^2}\right]\ , \tag{34}$$

and the equations for the first-order correction ($n=1$) can be written as

$$\ddot{x}_1+\gamma_1\dot{x}_1+\omega_1^2x_1=-\frac{E(t)}{m}q_0\ , \tag{35}$$

$$\ddot{q}_1+\gamma_2\dot{q}_1+\omega_2^2q_1=-\frac{E(t)}{L}x_0\ . \tag{36}$$

We note that the forcing term in the first equation represents the main contribution of the back action on the mechanical oscillator, i.e., $-E(t)q_0\simeq F_{\mathrm{BA}}$. By developing the expression for the back-action force due to the resonant term at ω_2 and by considering only the terms at the frequency ω_1 we obtain

$$F_{\mathrm{BA}}(f=1)=\frac{iE_0V_0}{2L\gamma_2\omega_2}e^{i\omega_1 t}\ , \tag{37}$$

$$F_{BA}(f=-1)=\frac{iE_0V_0}{2L\gamma_2\omega_2}e^{-i\omega_1 t}\,, \tag{38}$$

$$F_{BA}(f=0)=\frac{iE_0V_0}{2L\gamma_2\omega_2}\cos\omega_1 t\ . \tag{39}$$

The third expression can be considered as a coherent superposition of the first two "circularly polarized" terms, producing therefore a linear polarization (Fig. 2). Obviously the back action does not affect the force in quadrature respect to F_{BA}.

If we compute the back-action voltage $V_2=-E(t)x_1$ for the BAE pumping without approximations we obtain, after a tedious but straightforward calculation, the term at the frequency $\omega_2/2\pi$,

$$V_2(t)=-\frac{\omega_1}{\omega_2}\beta Q_1Q_2V_0\cos(\omega_2 t)+\frac{i}{16}\left[\frac{\omega_1}{\omega_2}\right]^3\beta Q_2V_0e^{i\delta_{11}}e^{i\omega_2 t}\ . \tag{40}$$

We observe that the first term can be overcome in a phase-sensitive detection by observing the quadrature phase [$\propto\sin(\omega_2 t)$], but the second term is always present coming from the conversion of the upper lateral bands as depicted in Fig. 3. The second term is reduced with respect to the first, by a factor of the order of $(\omega_2/\omega_1)^2Q_1$, but can become significant for large back-action evasion reduction factor r. Indeed in this case the first term on the right side of (40) will be suppressed. In order to obtain a high r, a high value of the electrical quality factor is required as seen for instance in (20) but at the same time this enhances the latter term in (40). Thus an intrinsic limitation to the maximum squeezing achievable exists.

This perturbative method suggests in a natural way the generalization of the BAE detection scheme using a multimode electromagnetic pump. Here we develop a framework for this problem which can be also useful for more complicated systems as the transducers mechanically matched to the antenna or more elaborate interaction Hamiltonians. The burst noise temperature depends on the amplifier noise, on the parametric ratio of electrical and mechanical frequencies, and on the particular time dependence of the electric field. The coupling electric field $E(t)$ may be expanded in Fourier integral as

$$E(t)=\int_{-\infty}^{+\infty}\tilde{E}(\omega)e^{i\omega t}\frac{1}{2\pi}d\omega\ , \tag{41}$$

where

$$\tilde{E}(\omega)=\int_{-\infty}^{+\infty}E(t)e^{-i\omega t}dt \tag{42}$$

is the Fourier transform of the time-dependent electric field, called the pumping spectrum. In this way both the interaction Hamiltonian and the burst noise temperature T_b become functionals of the $\tilde{E}(\omega)$ and the minimum burst noise temperature T_b will be obtained by solving the extremal problem, i.e., determining the optimal pumping spectrum $\tilde{E}(\omega)_{opt}$ which minimizes T_b as

$$T_{b_{min}}=T_b[\tilde{E}(\omega)_{opt}]\ . \tag{43}$$

On the basis of the previous considerations on the BAE effect is clear that $\tilde{E}(\omega)_{opt}$ for some BAE systems is ex-

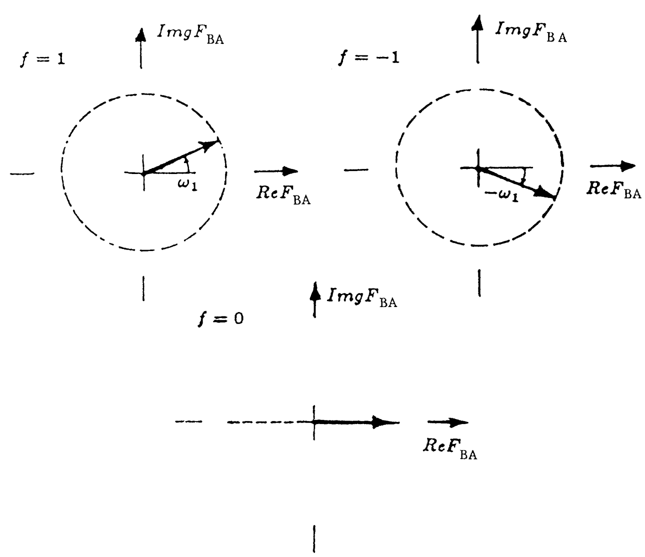

FIG. 2. Motion of the back-action force vectors in the complex plane for the three situations, $f=1,-1,0$. Note that the latter situation can be considered as linearly polarized resulting from the coherent superposition of two circularly polarized terms. In this example no contribution exists to the imaginary component of the back action for $f=0$.

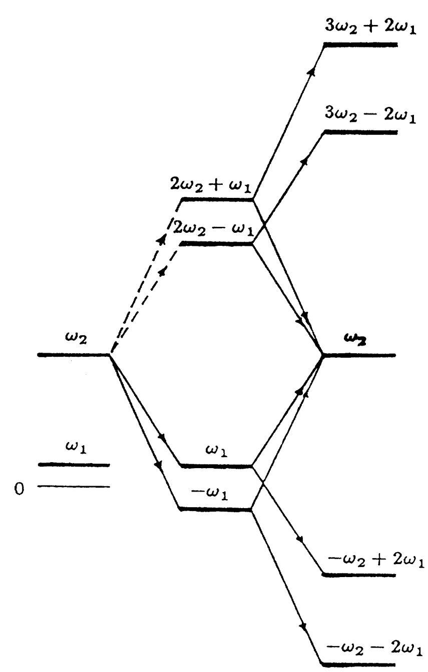

FIG. 3. Up and down conversion spectra of the noise at $\omega_2/2\pi$ frequency according to the model. Note the contributions out of the mechanical resonance (dashed lines) to the noise induced at the second order in the electrical oscillator.

pressed in (13) as a superposition of two Dirac distributions at the sum and at the difference of the electrical and the mechanical frequencies

$$\tilde{E}(\omega)_{\rm opt} \propto [\delta(\omega \pm (\omega_2-\omega_1)) + \delta(\omega \pm (\omega_2+\omega_1))] \ . \tag{44}$$

We note that this definition of the optimal pumping spectrum includes, but is not limited to, the pumping suggested by the classical limit of a QND measure. In fact, a general experimental setup will not be optimized by choosing a QND strategy, but certainly will be optimized by using the optimal pumping spectrum. We note that in order to define the optimal pumping spectrum we have to consider equations of motion in the frequency domain, so the approach outlined in Sec. III seems the best for a full development of this concept. For practical reasons we are interested at a transduction scheme which uses a finite number of monochromatic pumps

$$E_n(t) = \sum_{k=1}^{n} \tilde{E}_k \frac{e^{i\omega_k t} + e^{-i\omega_k t}}{2} = \sum_{k=1}^{n} \tilde{E}_k \cos(\omega_k t) \ . \tag{45}$$

The pumping spectrum will be written as

$$\tilde{E}_n(\omega) = \sum_{k=1}^{n} \alpha_k [\delta(\omega-\omega_k) + \delta(\omega+\omega_k)] \ . \tag{46}$$

The concept of generalized pumping allows us to link stroboscopic QND measurement schemes and continuous BAE measurements.[16] We recall that a stroboscopic QND scheme for monitoring the position x is available if the observable commutes with itself at different discrete times as given by (7) and if, at the same instants of time, the measuring apparatus is prepared in such a way that the interaction Hamiltonian has the form given by (6). This corresponds to a train of Dirac distributions having a period equal to $T=T_1/2=\pi/\omega_1$; using this distribution the commutator relation (8) for the corresponding operators is satisfied.

Considering now a more realistic $E(t)$ of the type

$$E(t) = E_0, \quad -T \le t \le -T+\tau, \quad 0 \le t \le \tau \ , \tag{47}$$

$$E(t) = 0, \quad -T+\tau \le t \le 0, \quad \tau \le t \le T \ , \tag{48}$$

which simulates, when $\tau \ll T$, a pulse train with a Dirac distribution having period T, we have the complex Fourier expansion

$$E(t) = \sum_{n=-\infty}^{+\infty} \tilde{E}_{2n} e^{i2n\omega_1 t} \ , \tag{49}$$

where

$$\tilde{E}_{2n} = E_0 \tau \frac{\omega_1}{\pi} \frac{\sin(n\omega_1\tau)}{n\omega_1\tau} \ . \tag{50}$$

The spectrum for this pumping is centered around zero frequency and contains an infinite number of multiples of $2\omega_1$ with the same weight in the limit of $\tau \to 0, E_0\tau = \text{const}$. By considering a modulation at a frequency $\omega_2/2\pi$ the spectrum is simply upconverted as shown in Fig. 4(a).

If the above procedure is repeated for a stroboscopic measurement of the complex amplitude X_1 with a new interaction Hamiltonian

$$H_i = E(t) X_1 q = E_0 \delta\left(t - \frac{n\pi}{\omega_1}\right) X_1 q$$

$$= E_0 (-1)^n \delta\left(t - \frac{n\pi}{\omega_1}\right) x q \ , \tag{51}$$

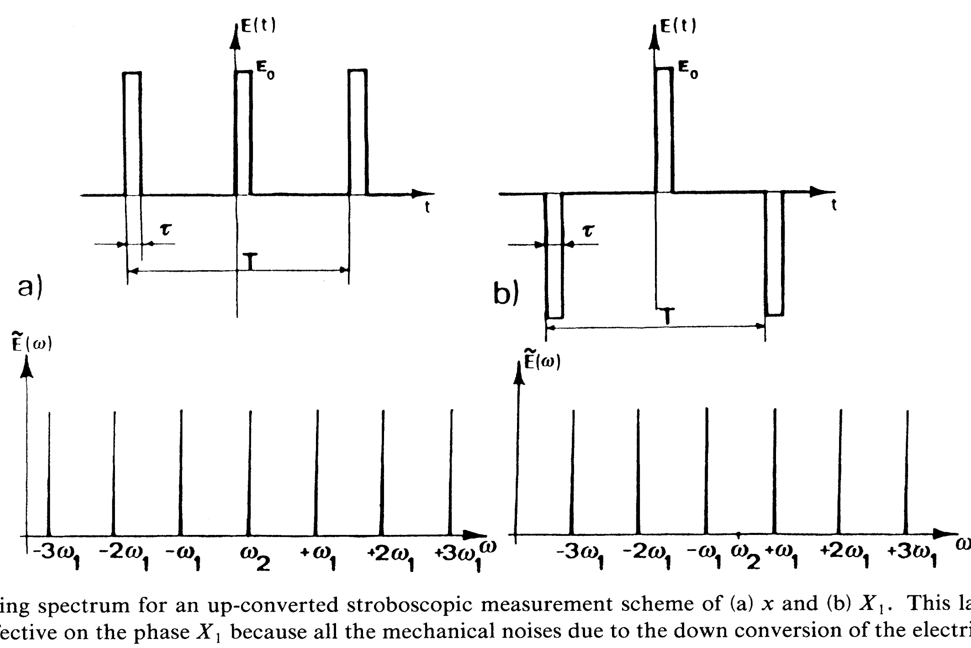

FIG. 4. Pumping spectrum for an up-converted stroboscopic measurement scheme of (a) x and (b) X_1. This last scheme is back-action evasion effective on the phase X_1 because all the mechanical noises due to the down conversion of the electrical noise by means of each pair of pumps $[\omega_2 \pm (2n+1)\omega_1, n=0,1,\ldots]$ are fully correlated.

we obtain a complex Fourier expansion

$$E(t)=\sum_{n=-\infty}^{+\infty} \tilde{E}_{2n+1}e^{i(2n+1)\omega_1 t} \tag{52}$$

where, by using the same discretization used for the Dirac distributions, we have

$$\tilde{E}_{2n+1}=E_0\tau\frac{\omega_1}{\pi}\frac{\sin\left[\dfrac{(2n+1)\omega_1\tau}{2}\right]}{\left[\dfrac{(2n+1)\omega_1\tau}{2}\right]} . \tag{53}$$

Only odd harmonics are present. By upconverting at ω_2 the spectrum one has the Fourier expansion as shown in Fig. 4(b). A physical interpretation of the spectral pumping obtained is possible by extending previous considerations.[17,18] The back-action evasion capability for a two-frequency pumping of BAE type can be seen as due to the simultaneous conversion of the electrical noise which lies in a band frequency extending from $\omega_2-\omega_1$ to $\omega_2+\omega_1$ into a force noise acting on the mechanical oscillator. Because one pump is phase inverting and the other one is phase preserving the coherent sum of the back-action forces is cancelled on one mechanical phase and added on the other one in quadrature, producing a squeezing of the total back-action force. The electrical noise lying outside the previous band is converted with the same phase by both the pumps giving a back-action force. The amount of this force depends on the selectivity of the electrical circuit. Adding two other pumps at the frequencies $\omega_2-3\omega_1$ and $\omega_2+3\omega_1$ is possible to squeeze the noise in the band between $\omega_2-3\omega_1$ and $\omega_2+3\omega_1$. By adding other pumps we obtain an electrical field which approaches the Fourier transform of a stroboscopic measurement as expressed in (52).

IV. GENERALIZED BACK-ACTION EVASION SCHEMES

We have seen that a stroboscopic measurement can be considered as particular multipumping schemes which have, at the same time, the standard two-pump BAE scheme as the zeroth-order approximation. Furthermore the interpretation of the multipump scheme outlined in the preceding section suggests a simple way to calculate the BAE reduction factor. Here we discuss in more detail a general n-pump scheme with symmetrical frequencies and in particular a four-pump scheme. The starting point is the calculation of the BAE reduction factor for an n-pump scheme simulating, in the limit of $n\to\infty,\tau\to 0$, a stroboscopic measurement of X_1 as in (53). In order to show this we introduce the squeezing factor ρ such that

$$\rho^2=\frac{\Delta X_1^2}{\Delta X_2^2} , \tag{54}$$

where ΔX_1^2 and ΔX_2^2 are the variancies of the two conjugate complex amplitudes of the mechanical oscillator X_1 and X_2. This squeezing factor is related to the BAE reduction factor ($\rho\to 0$ means a noise-free measurement of X_1, corresponding to $r\to\infty$ for the T_b associated to a monitoring of X_1). An uncertainty relation for the two classical conjugate observables due to the back action of the amplifier noise can be written as

$$\Delta X_1 \Delta X_2 \simeq \frac{k_B T_N}{2m\omega_1\omega_2} . \tag{55}$$

This relation can be considered as the classical counterpart of the quantum uncertainty relationship

$$\Delta X_1 \Delta X_2 \simeq \frac{\hbar}{2m\omega_1} , \tag{56}$$

replacing $\hbar$ with $k_B T_N/\omega_2$. The minimum burst temperature for X_1 can be written as

$$T_b=\frac{m\omega_1^2\Delta X_1^2}{2}\simeq\frac{1}{4}T_N\frac{\omega_1}{\omega_2}\rho , \tag{57}$$

which shows the dynamical interpretation of the BAE reduction factor in terms of the squeezing factor $\rho=8/r$. The squeezing factor allows us to express in a straightforward way the BAE reduction factor in terms of mechanical and electrical frequencies and bandwidths. In the squeezed phase the electrical noise contribution to the back-action force will be due only to the two extreme noise sources at $\omega_2\pm 2n\omega_1$. The pumps at the frequencies between $\omega_2\pm(2n-1)\omega_1$ allow the cancellation of the other noise sources in the squeezed phase. The noise sources at $\omega_2\pm 2k\omega_1$ ($k=0,1,\ldots,n-1$) will affect only the quadrature phase. Of course the latter phase will be also affected by the extreme noise sources. The simplifying hypothesis of the calculation is that the mechanical bandwidth is negligible compared to the electrical bandwidth, i.e., $\Delta\omega_1 <, \Delta\omega_2$, and that the mechanical oscillator is sensitive to the down-converted electrical noise only in a resonance region around ω_1. The spectral density for the voltage noise can be written as a Lorentzian distribution

$$S_v(\omega)\simeq\frac{k}{(\omega-\omega_2)^2+(\Delta\omega_2/2)^2} . \tag{58}$$

The contributions to the squeezed phase (X_1) and to the unsqueezed phase (X_2) are given respectively by

$$\Delta X_1^2=\xi\left[\frac{1}{2}\int_{\omega_2-2n\omega_1-(\Delta\omega_1/2)}^{\omega_2-2n\omega_1+(\Delta\omega_1/2)} S_v(\omega)d\omega+\frac{1}{2}\int_{\omega_2+2n\omega_1-(\Delta\omega_1/2)}^{\omega_2+2n\omega_1+(\Delta\omega_1/2)} S_v(\omega)d\omega\right] \tag{59}$$

and

$$\Delta X_2^2=\xi\left[\frac{1}{2}\int_{\omega_2-2n\omega_1-(\Delta\omega_1/2)}^{\omega_2-2n\omega_1+(\Delta\omega_1/2)}S_v(\omega)d\omega+\sum_{k=1}^{n-1}\int_{\omega_2-2k\omega_1-(\Delta\omega_1/2)}^{\omega_2-2k\omega_1+(\Delta\omega_1/2)}S_v(\omega)d\omega\right.$$
$$\left.+\int_{\omega_2-(\Delta\omega_1/2)}^{\omega_2+(\Delta\omega_1/2)}S_v(\omega)d\omega+\sum_{k=1}^{n-1}\int_{\omega_2+2k\omega_1-(\Delta\omega_1/2)}^{\omega_2+2k\omega_1+(\Delta\omega_1/2)}Sv(\omega)d\omega+\frac{1}{2}\int_{w_2+2n\omega_1-(\Delta\omega_1/2)}^{\omega_2+2n\omega_1+(\Delta\omega_1/2)}S_v(\omega)d\omega\right] \tag{60}$$

where ξ is a constant which links the electrical noise to the down-converted mechanical noise. The squeezing factor is simply obtained as the ratio between (59) and (60) according to (54). The integrals are easily calculated, the generic one having the form

$$\int_a^b d\omega\frac{1}{(\omega-\omega_2)^2+(\Delta\omega_2/2)^2}=\frac{2}{\Delta\omega_2}\arctan\frac{(2/\Delta\omega_2)(b-a)}{1+(2/\Delta\omega_2)^2(a-\omega_2)(b-\omega_2)} . \tag{61}$$

In this way the squeezing factor p_n is obtained as

$$\rho_n^2=\frac{\arctan\{(2\Delta\omega_1/\Delta\omega_2)/[1+(4\omega1/\Delta\omega_2)^2n^2]\}}{\arctan(2\Delta\omega_1/\Delta\omega_2)+2\sum_{k=1}^{n-1}\arctan\{(2\Delta\omega_1/\Delta\omega_2)/[1+(4\omega_1/\Delta\omega_2)^2k^2]\}+\arctan\{(2\Delta\omega_1/\Delta\omega_2)/[1+(4\omega_1/\Delta\omega_2)^2n^2]\}} \tag{62}$$

If $\omega_1/\Delta\omega_2 \gg 1$ formula (62) simplifies to

$$\rho_n \simeq \frac{1}{8n}\frac{\omega_2}{\omega_1}\frac{1}{Q_2} , \tag{63}$$

which implies a BAE reduction factor $r_n \propto nQ_2$. Thus the use of a $2n$-mode pumping for a circuit but with a narrow electrical bandwidth is equivalent to the use of a standard BAE 2-mode pumping having an effective quality factor $Q_2' \simeq nQ_2$. The opposite condition gives the possibility of another transduction scheme. Indeed if $\omega_1/\Delta\omega_2 \ll 1$ the formula (62) simplifies to

$$\rho_n \simeq \frac{1}{n} . \tag{64}$$

This configuration corresponds to a low electric quality factor Q_2 and the squeezing factor is different from 1 and is independent of Q_2. This is a new feature of the multipump scheme: the burst noise temperature corresponding to (63) is of the order

$$T_b \simeq 2T_a\frac{1}{nQ_2}=2T_a\frac{\omega_1}{\omega_2}\left[\frac{\omega_2}{\omega_1}\frac{1}{nQ_2}\right] , \tag{65}$$

while in (64) there is no dependence on the electrical quality factor and we obtain, for the burst noise temperature, the value

$$T_b \simeq 2T_a\frac{\omega_1}{\omega_2}\frac{1}{n} , \tag{66}$$

which does not give any limit when the ratio ω_1/ω_2 is arbitrarily small. However, another limitation of the proposed scheme is that the electrical quantity factor cannot be arbitrarily low because of the direct effect of the Johnson noise on the squeezing factor.

In a previous paper we have reported some tests carried out to prove the phase-sensitive transduction for this multipump scheme and we have described some of the experimental problems.[19] In particular one of the problems is that which we have defined as dispersive squeezing. The squeezing effect of the BAE scheme is related to the degree of accuracy with which the pumps are tuned to the proper frequencies. For instance when two pumps at ω_- and ω_+ frequencies are present, mechanical signals at $\omega_2-\omega_-$ and $\omega_+-\omega_2$ will be produced in the down conversion of the electrical noise. If $\omega_-=\omega_2-\omega_1=\omega_-^{(0)}$ and $\omega_+=\omega_2+\omega_1=\omega_+^{(0)}$ the two down-converted forces are at the same resonance frequency ω_1 and will be exactly cancelled in the squeezed phase. For an unperfect tuning, i.e., when $\omega_-=\omega_-^{(0)}+\delta\omega_-$ and $\omega_+=\omega_+^{(0)}+\delta\omega_+$, the two forces will differ in both amplitude and phase. The maximum allowed mismatch of the two pumps is fixed by the mechanical bandwidth $\Delta\omega_1$, i.e.,

$$\delta\omega=\delta\omega_-+\delta\omega_+=\omega_++\omega_--2\omega_2 \ll \Delta\omega_1 . \tag{67}$$

We note that it is very hard to satisfy this cancellation for high-Q values of the mechanical oscillator. If the matching condition (67) is not satisfied the squeezing factor is not constant over the electrical bandwidth and decreases around the maximum value.

On the other hand, when the electrical quality factor is low more useful signals having almost the same amplitude may be obtained at $\omega_2 \pm 2n\omega_1$ frequencies ($n=0,1,2,\dots$). All these signals are uncorrelated as far as the electrical noise is concerned, and correlated for the mechanical noise. In this way more signals are produced and an increase in the sensitivity can be obtained by summing them with adequate filtering procedures. The Fourier spectra for a four-pump–three-signal scheme is shown in Fig. 5. The electrical noise is converted to a mechanical force only in the regions around $\omega_2 \pm 2k\omega_1$ ($k=0,1,2$) within the mechanical bandwidth $\Delta\omega_1$ (dark regions). The dashed curves at low and high frequency indicate, respectively, the bandwidths of the mechanical and the electrical oscillators. The superposi-

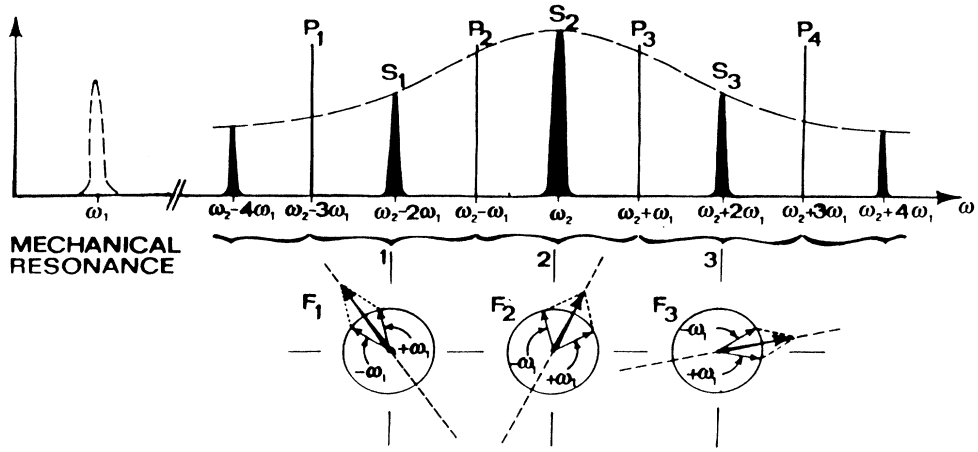

FIG. 5. Fourier spectra for a four-pump–three-signal BAE scheme and corresponding motion of the back-action evasion force vectors in the complex plane.

tion of the back-action forces for each pair of pumps $(P_1-P_2, P_2-P_3, P_3-P_4)$ is shown. The two rotating back-action force vectors due to each parametric down conversion give a squeezed back-action force, and in general the squeezing axes are different for each pair of pumps. Of course, when the pumps are properly phase locked all the back-action force contributions are squeezed in the same phase, with the quadrature phase remaining unaffected by the electrical noise. The four synthesizers send signals to the transducer and the signals at $\omega_2-2\omega_1, \omega_2$ and $\omega_2+2\omega_1$ are processed and summed in order to obtain higher sensitivity. In Fig. 6 a four-pump–three-signal detection scheme is shown.

This approach seems impractical if we want to give a close simulation of stroboscopic measurement. Indeed the complications of a six, eight,. . .,n-mode pumping is not only related to the scaling in the number of pumps and the requirements of a coherent correlation between them, but also to the increase of the total phase noise and the difficulties of a precise setting of all the frequencies around the electrical and the mechanical frequencies. In order to overcome the practical problems related to the use of sinusoidal pump sources it is crucial to investigate the possibility of real stroboscopic measurements on the current generations of resonant gravitational wave antennas.[20] A square-signal modulation of a dc voltage or current may be performed, depending on the use of a charge or a magnetic flux transducer (in the next considerations we will refer to a charge transducer, which seems easier to adapt to the new scheme). This can be achieved by means of a high-speed–low-duty cycle switching network which gives the electric field to the transducer for a short time $\tau(\tau \ll 2\pi/\omega_1)$. In this way no pump noise is introduced, because of the use of three constant voltage levels, 0 and $\pm V_0$. Of course the apparatus will only be sensitive during the small interval τ, and the lock-in reference has to be driven by the clock of the stroboscopic source. There are some problems in view of the real implementation of such a scheme. For instance a switching network that does not introduce relevant noise has to be properly designed. Furthermore, and this seems harder to overcome, the high-speed measurement means a low electromechanical coupling due to the limit of the maximum value of the electric field that it is possible to apply to a capacitive transducer. The effective β for a stroboscopic system can be written as

$$\beta' \simeq \frac{\tau}{T}\beta_0 , \tag{68}$$

where τ is the duration of the electric field, T is the period of the harmonic oscillator, and β_0 is the electromechanical coupling factor defined for a dc operating transducer

$$\beta_0 = \frac{E_0^2}{m\,\omega_1^2 L\,\omega_2^2} . \tag{69}$$

The problem of the small value for β' coupling may be overcome by using a multimode mechanical resona-

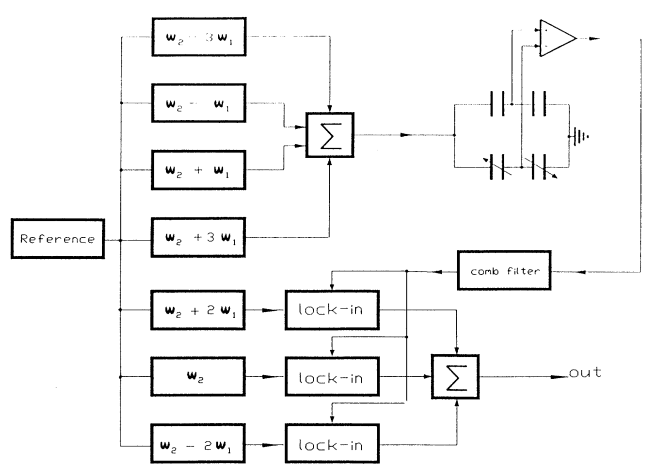

FIG. 6. Detection scheme for a transducer having four pumps and three squeezed signals. The output signal is filtered by a comb amplifier and is reconstructed by means of three lock-in amplifiers.

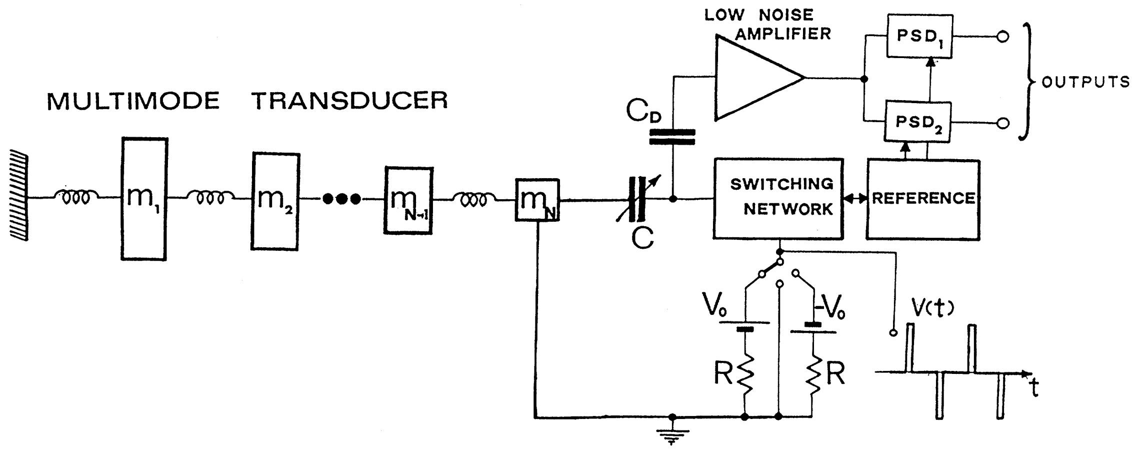

FIG. 7. Detection scheme for a multimode transducer and a quasistroboscopic pumping of the complex amplitude X_1.

tor.[21–23] Multimode transducers have already been proposed in order to increase the electromechanical coupling. The calculation of the bandwidth of a resonant gravitational wave antenna and detailed noise analysis has been done in order to optimize the number of modes.[24,25] QND strategies may be implemented on multimode transducers, although only a two-mode QND configuration has been analyzed in detail until now.[26–28] The use of a multimode transducer allows one to recover the decrease in the sensitivity due to the short time during which the coupling is effective. The increase of the β_0 obtained by lowering the final resonating mass determines an upper limit to the speed at which energy is transported among the modes. Thus the multimode configuration gives limitations on the maximum sampling rate due to the beat period.

V. CONCLUSIONS

We have shown a perturbative approach for the solution of the equation of the motion of two parametrically coupled oscillators. Such an approach suggests the concept of optimal pumping spectrum and therefore a generalized parametric transducer for which amplitudes and frequencies of the pumps are chosen on the basis of extremal problem solving. In this context we have shown that the well-known stroboscopic measurement can be considered as a limit of a multipumping process. Furthermore we propose to use stroboscopic multipumping, in particular four-pumping, as a way to reduce the back action without severe constraint on the electrical quality factor. The increase in the sensitivity due to the use of a four-pump–three-signal transducer has been estimated by using the standard procedure for the calculation of the signal-to-noise ratio in parametric devices. The possibility of performing stroboscopic measurements of the complex amplitude of a harmonic oscillator has been discussed, as well as advantages and drawbacks of such a scheme. The schemes proposed in this paper may be not useful for the current generation of Weber-type antennae. The actual resonant antennae are indeed weakly coupled to a quantum-limited amplifier [dc superconducting quantum interference device (SQUID)]. For this reason the back-action term is irrelevant as compared to the Brownian noise. For the next generation of GW antennae working at 50–100 mK the Brownian term will be drastically reduced due to the lower operating temperature and to the consequent increase of the mechanical Q. We think that the more practical way to implement QND-BAE strategies on the third generation of gravitational wave antennas will be the use of a quasistroboscopic measurements on a multimode transducer (Fig. 7). This raises a number of hardware problems, as well as of software problems linked to the most efficient way to analyze the data in a quantum regime in such a configuration. Furthermore, data-analysis algorithms in a quantum regime of sensitivity are not available at present. It has been suggested that the distribution of the energy associated with the quantum nondemolished complex phase of a harmonic oscillator be monitored to prevent the quantum noise from appearing in the energy distribution of that observable.[29] This opens the possibility of a further test of quantum mechanics and quantum theory of measurement in a novel way by monitoring macroscopic degrees of freedom in a regime of quantum sensitivity.[30–32]

---

*Present address: Università degli Studi di L'Aquila, Dipartimento di Ingegneria Elettrica, 67040 Poggio di Roio L'Aquila.

[1]See, e.g., K. S. Thorne, Rev. Mod. Phys. **52**, 285 (1980).
[2]J. Weber, Phys Rev. **117**, 306 (1960); Phys. Rev. Lett. **22**, 1302 (1969).
[3]V. B. Braginskij and Yu. Vorontsov, Usp. Fiz. Nauk. **114**, 41 (1974) [Sov. Phys. Usp. **17**, 644 (1975)].
[4]V. B. Braginskij, Yu. Vorontsov, and F. Ya. Khalili, Zh. Eksp. Teor. Fiz. **73**, 1340 (1977) [Sov. Phys. JETP **46**, 705 (1977)].
[5]C. M. Caves, K. S. Thorne, R. W. P. Drever, V. D. Sandberg, and M. Zimmermann, Rev. Mod. Phys. **52**, 341 (1980).
[6]M. F. Bocko and W. W. Johnson, Phys. Rev. Lett. **47**, 1184 (1981); **48**, 1371 (1982).

[7]D. Blair, Phys. Lett. A **91**, 192 (1982).
[8]G. W. Spetz, A. G. Mann, W. O. Hamilton, and W. C. Oelfke, Phys. Lett. A **104**, 335 (1984).
[9]M. F. Bocko and W. W. Johnson, Phys. Rev. A **30**, 2135 (1984).
[10]F. Bordoni, S. De Panfilis, F. Fuligni, V. Iafolla, S. Nozzoli, in *Proceedings of the Fourth Marcell Grossmann Meeting on General Relativity,* edited by R. Ruffini (North-Holland, Amsterdam, 1986), p. 523.
[11]E. Barro, R. Onofrio, P. Rapagnani, and F. Ricci in *Proceedings of the First International Symposium on Experimental Gravitational Physics,* edited by P. F. Michelson (World Scientific, Singapore, 1988), p. 363.
[12]J. N. Hollenhorst, Phys. Rev. D **19**, 1669 (1979).
[13]J. C. Decroly, L. Laurent, J. C. Lienard, G. Marechal, and J. Vorsbeitchik, *Parametric Amplifiers* (MacMillan, London, 1973).
[14]F. Fuligni (unpublished).
[15]M. F. Bocko, Ph.D. thesis, University of Rochester, Rochester, NY, 1984.
[16]R. Onofrio, in *Proceedings at the International Workshop on Gravitational Wave Signal Analysis and Processing* (World Scientific, Singapore, 1990).
[17]M. F. Bocko, F. Bordoni, F. Fuligni, and W. W. Johnson in *Physical Systems and 1/f Noise* (Elsevier, Amsterdam, 1986), p. 47.
[18]F. Bordoni, in *Proceedings of the International Workshop on Gravitational Wave Signal Analysis and Processing* (World Scientific, Singapore, 1990).
[19]F. Bordoni and R. Onofrio, Phys. Rev. A **41**, 21 (1990).
[20]R. Onofrio, Phys. Lett. A **148**, 1 (1990).
[21]H. J. Paik, J. Appl. Phys. **47**, 21 (1976).
[22]J. P. Richard, in *Proceedings of the Second Marcell Grossmann Meeting on General Relativity,* Trieste, 1979, edited by R. Ruffini (North-Holland, Amsterdam, 1982), p. 1239.
[23]M. Karim, Phys. Rev. D **30**, 2031 (1984).
[24]M. Basman, Ph.D. thesis, Stanford University, 1985; and Phys. Rev. D **38**, 2327 (1988).
[25]W. M. Folkner, Ph.D. thesis, University of Maryland, 1987.
[26]M. F. Bocko, L. Narici, D. H. Douglass, and W. W. Johnson, Phys. Lett. A **104**, 335 (1984).
[27]R. Onofrio, Phys. Lett. A **120**, 1 (1987).
[28]F. Bordoni, F. Fuligni, and V. Iafolla in *Proceedings of the International Workshop on Gravitational Wave Signal Analysis and Processing* (World Scientific, Singapore, 1990).
[29]R. Onofrio, Europhys. Lett. **11**, 695 (1990).
[30]*Quantum Theory and Measurement,* edited by J. A. Wheeler and W. H. Zurek (Princeton University Press, Princeton, 1983).
[31]*Le Hazard et la Matière/Chance and Matter,* edited by J. Souletie, J. Vannimenus, and R. Stora, Les Houches, Session XLVI (Elsevier, Amsterdam, 1987); see in particular the contributions of A. J. Legett and M. H. Devoret, J. M. Martines, and D. Esteve.
[32]Y. Srivastava and A. Widom, Phys. Rep. **148**, 1 (1987).