Energy loss of heavy ions in dense plasma. I. Linear and nonlinear Vlasov theory for the stopping power

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The plasma physics of heavy-ion stopping in fully ionized matter is developed on the basis of the Vlasov-Poisson equations with particular emphasis on small ion velocities v_p, below the electron thermal velocity $v_{\rm th}$, and on solutions nonlinear in the coupling parameter $\mathcal{Z}=Z_{\rm eff}/(n_0\lambda_D^3)$ between the heavy-ion projectile with effective charge $Z_{\rm eff}$ and the plasma with electron density n_0 and Debye length λ_D. Concerning the stopping power in the low-velocity regime relevant for the Bragg peak at the end of the ion range, results on the friction term $dE/dx \propto v_p$ are presented, and an improved dE/dx formula for plasma is derived in closed form and readily applicable for stopping-power calculations; it is identical to the standard result for $v_p > v_{\rm th}$, but also describes the limit $v_p \to 0$ correctly. For $v_p < v_{\rm th}$, nonlinear results are found to contribute to the stopping power with terms $\propto \mathcal{Z}^{5/2}$ for positive ions and terms $\propto \mathcal{Z}^3$ for negative ions in addition to the basic $\mathcal{Z}^2$ term; they are derived from a low-velocity expansion of a form-factor representation of dE/dx. Concerning high velocities $v_p > v_{\rm th}$, the relevant coupling parameter is $\mathcal{Z}(v_{\rm th}/v_p)^3$, and nonlinear corrections to the stopping power $\propto \mathcal{Z}^3/v_p^5$ are obtained by extending the work of Ashley, Ritchie, and Brandt [Phys. Rev. B **5**, 2393 (1972)] to the plasma case. An interpolation between the low- and the high-velocity results is given; taking, e.g., parameters characteristic for heavy-ion beam inertial fusion the nonlinear corrections further enhance dE/dx up to 10% in the Bragg peak region. An application of the present results to heavy-ion energy loss in an electron-cooling line is also discussed. In the present paper, $Z_{\rm eff}$ is assumed to be constant; the physics determining $Z_{\rm eff}$ is treated in a subsequent article [Peter and Meyer-ter-Vehn, following paper, Phys. Rev. A **43**, 2015 (1991)].

I. INTRODUCTION

This work is motivated by the possibilities of generating dense, hot matter using intense heavy-ion beams and by future applications of this new branch of heavy-ion physics in the fields of inertial confinement fusion (ICF) and heavy-ion-pumped x-ray lasers. Only recently have heavy-ion beams been considered as drivers for ICF (e.g., Refs. 1–4); the most promising features are the high efficiency and high repetition rate of heavy-ion accelerators. The first experimental studies will become possible in the near future. The interaction of the ion beam with a hot dense target plasma is of central importance. The situation may be compared with that of high-power lasers 20 years ago when laser-plasma interaction studies started and opened a new field of plasma physics. Up to now the intensities of heavy-ion beam accelerators have been too small for heating matter to high temperatures. Concerning target heating the accelerator facility Schwerionen-Synchrotron/Elektronen-Speicherring (SIS/ESR) under construction at Gesellschaft für Schwerionenforschung (GSI), Darmstadt will offer several possibilities: one expects that the SIS/ESR beam can heat massive gold targets up to temperatures of 10–30 eV. The present theoretical work is related to future experiments at GSI.[5]

A number of theoretical studies on the ion-beam interaction with plasma are found in the literature.[6–12] Two major results were the increase of ion energy loss in ionized target material, which was also proved experimentally[13] and, secondly, the increase of the effective charge $Z_{\rm eff}$ in the case of heavy ions theoretically, predicted by Nardi and Zinamon,[9] which also increases the stopping power ($dE/dx \propto Z_{\rm eff}^2$).

A self-contained representation of the theory of energy loss of ions penetrating classical dense plasmas may be given by means of the Vlasov-Poisson equations. Based on the linearized form of these equations (so called "dielectric approximation;" e.g., Ref. 14) the stopping power dE/dx is easily derived (Sec. II). For high projectile velocities ($v_{\rm th} \ll v_p \lesssim 2Z_{\rm eff}\alpha c$) the plasma stopping power reduces to the well-known Bohr result[15]

$$-\left[\frac{dE}{dx}\right]_{\rm plasma}=\left[\frac{Z_{\rm eff}e\omega_p}{v_p}\right]^2 \ln\left[\frac{mv_p^3}{Z_{\rm eff}e^2\omega_p}\right], \qquad (1)$$

where $\omega_p^2=4\pi n_0 e^2/m$ is the square of the plasma frequency, and n_0 denotes the (unperturbed) free-electron density in the plasma. This should be compared with the Bethe stopping power in cold gas[16]

$$-\left[\frac{dE}{dx}\right]_{\rm gas}=\left[\frac{Z_{\rm eff}e\omega_p}{v_p}\right]^2 \ln\left[\frac{2mv_p^2}{\bar{I}}\right]. \qquad (2)$$

Here, ω_p is defined as above except that the density of bound electrons is substituted for n_0, and $\bar{I}$ is the average ionization potential of the target atoms. The stopping powers given by Eqs. (1) and (2) are illustrated in Fig. 1 assuming the same fixed effective charge $Z_{\rm eff}$ in both cases. In a fully ionized plasma dE/dx is higher than in cold gas because of the difference in the Coulomb logarithms. The physical reason is that free electrons are

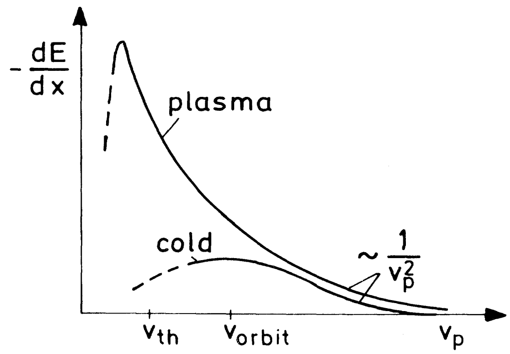

FIG. 1. Schematic drawing of the stopping power dE/dx as a function of projectile velocity for cold gaseous targets and plasma targets. The same constant Z_{eff} is assumed in both cases.

more easily excited by the beam ions (plasma waves) than bound electrons (bound-bound excitations and bound-free ionizations).

The linearization of the Vlasov-Poisson equations is permissible only if the effective charge Z_{eff} is not too large or if the ion is moving very fast. In fact, very high charge states are characteristic for heavy ions stopping in plasma. This will be shown in a subsequent paper by the authors[17] (hereafter referred to as paper II) treating in detail the atomic physics of ionization and recombination processes to determine the projectile charge Z_{eff}. As it turns out, the parameter describing the coupling of projectile and plasma $\mathcal{Z}=Z_{\text{eff}}/(n_0\lambda_D^3)$ may exceed unity in the dense plasma considered here, indicating that nonlinear terms in dE/dx going beyond the Z_{eff}^2 dependence become important, in particular at the end of the ion range. Therefore, in Sec. III a nonlinear description of the stopping power is developed, showing that the Bragg peak in dE/dx of heavy ions will further increase due to nonlinear effects.

It should be noted that the present work investigates the stopping of individual projectile ions. Possible "collective" interaction between beam ions which may take place for very intense ion beams (e.g., beam-plasma instabilities) are not considered. Estimates[18] indicate that even driver beams with intensities 100 TW/cm^2 and more for an ICF reactor (see e.g, HIBALL reactor study, Ref. 19) have beam densities of about 10^{14} cm^{-3}, many orders of magnitude smaller than the target densities of 10^{22} cm^{-3}. We assume therefore that the individual projectile approximation is valid.

II. PLASMA STOPPING POWER IN THE LINEARIZED VLASOV THEORY

Starting from a classical collisionless plasma described by the Vlasov equation we first discuss the linear theory of the plasma stopping power. Parts of this theory go back to the work of Chandrasekhar in the 1940s.

The derivation is based on the solution of the linearized form of the Vlasov-Poisson equations. The linearization of these equations assumes the disturbance of the plasma by the fast ion to be so small that the induced electric field $\mathbf{E}_{\text{ind}}$ is linearly proportional to the outer perturbation $\mathbf{E}_{\text{ext}}$ and, hence, proportional to the ion charge Z_{eff}. This theory is also called dielectric theory because the factor of proportionality between $\mathbf{E}_{\text{tot}}$ and $\mathbf{E}_{\text{ext}}$ is the dielectric tensor of the medium. Since $dE/dx \propto Z_{\text{eff}}e|\mathbf{E}_{\text{ind}}|$, the stopping power is proportional to the square of the effective charge Z_{eff}^2.

In the following first the conditions are stated under which the Vlasov-Poisson equations apply and may be linearized. The limiting forms for small and high velocities are discussed analytically, and a useful analytic approximation is given for arbitrary velocity provided the plasma density is not too high. It is shown that in the special case of highly charged ions at low velocities the conditions for linearization may be violated.

A. Solution of the linearized form of the Vlasov-Poisson equations

The Fermi-Dirac statistics, describing electrons and allowing at most two electrons in the plasma volume h^3, reduces to Maxwell-Boltzmann statistics if $h^3 < 2(\Delta p)^3/n_0 \approx 2(mv_{\text{th}})^3/n_0$, or

$$k_B T > (4\pi^3 n_0 a_0^3)^{2/3}\ \text{Ry} = \left[\frac{n_0}{5.42\times 10^{22}\ \text{cm}^{-3}} \right]^{2/3} \text{Ry} . \tag{3}$$

Here, T and n_0 denote temperature and (unperturbed) density of plasma electrons, respectively, k_B is Boltzmann's constant, a_0 the Bohr radius, and 1 Ry$=$13.6 eV. Under this condition the plasma can be described by the classical Boltzmann equation

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} + \frac{\mathbf{F}}{m}\cdot\frac{\partial f}{\partial \mathbf{v}} = \left[\frac{\partial f}{\partial t} \right]_{\text{coll}} , \tag{4}$$

where $f(\mathbf{r},\mathbf{v},t)$ is the single-particle distribution function, $f(\mathbf{r},\mathbf{v},t)d^3r\,d^3v$ specifies the probability of finding an electron (mass m, charge $-e$) at position $\mathbf{r}$ with velocity $\mathbf{v}$ at time t; $\mathbf{F}(\mathbf{r},\mathbf{v},t)$ is the force acting upon the electron and includes internal forces induced by the plasma particles themselves and the external electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$. The induced fields $\mathbf{E}_{\text{ind}}$ and $\mathbf{B}_{\text{ind}}$ do not contain individual close collisions carried out by the particles. Close collisions are taken into account by the collision term $(\partial f/\partial t)_{\text{coll}}$.

The collision term $(\partial f/\partial t)_{\text{coll}}$ is negligible and the Boltzmann equation reduces to the Vlasov equation if the frequency of collisions with large scattering angle between the electrons is small compared with the plasma frequency ω_p. The cross section for collisions with scattering angles of 90° or more is $\sigma_{90°} = \pi b_{90°}^2 = \pi(e^2/mv_{\text{th}}^2)^2$ and the frequency of such collisions $\nu = n_0\sigma_{90°}v_{\text{th}}$. Thus

$$\frac{\nu}{\omega_p} = \frac{1}{32\sqrt{2}\pi n_0\lambda_D^3} = \frac{1}{24\sqrt{2}\mathcal{N}_D} . \tag{5}$$

Here $\mathcal{N}_D=(4\pi/3)n_0\lambda_D^3$ is the number of electrons in the Debye sphere, $1/\mathcal{N}_D$ the plasma parameter, and $\lambda_D^2=k_BT/4\pi e^2 n_0$. If $\mathcal{N}_D>1$, then $\nu \ll \omega_p$ and the collision term $(\partial f/\partial t)_{\text{coll}}$ may be omitted ("collisionless plasma"), i.e.,

$$k_BT>(288\pi n_0 a_0^3)^{1/3}\ \text{Ry}=\left[\frac{n_0}{7.42\times10^{21}\ \text{cm}^{-3}}\right]^{1/3}\text{Ry}\ . \tag{6}$$

The conditions Eqs. (3) and (6) are shown in the $\{k_BT,n_0\}$ phase-space diagram in Fig. 2 as thick solid lines and specify the regime of validity of the Vlasov equation.

The Vlasov-Poisson equations without external fields read

$$\begin{aligned}&\frac{\partial f}{\partial t}+\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}}+\frac{e}{m}\frac{\partial\Phi}{\partial\mathbf{r}}\cdot\frac{\partial f}{\partial\mathbf{v}}=0\ ,\\ &\nabla^2\Phi=-4\pi Z_{\text{eff}}e\,\delta(\mathbf{r}-\mathbf{v}_pt)\\ &\qquad+4\pi e\int d^3v\, f(\mathbf{r},\mathbf{v},t)-4\pi n_0 e\ ,\end{aligned} \tag{7}$$

where $\mathbf{E}=-\nabla\Phi$ ($\mathbf{B}=0$ assuming nonrelativistic motion). The δ function stands for the projectile ion moving with velocity v_p. The last term in the Poisson equation represents the static plasma ion background.

Introducing dimensionless quantities

$$\begin{aligned}&\mathbf{r}\Longrightarrow k_D\mathbf{r},\quad \mathbf{v}\Longrightarrow\mathbf{v}/\sqrt{k_BT/m}\ ,\quad t\Longrightarrow\omega_pt\ ,\\ &f\Longrightarrow\frac{\sqrt{k_BT/m}^3}{n_0}f,\quad \Phi\Longrightarrow\frac{e}{k_BT}\Phi\ ,\end{aligned} \tag{8}$$

Eqs. (7) read

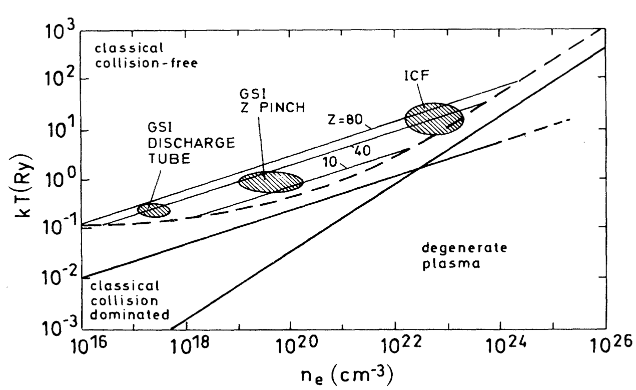

FIG. 2. Phase-space diagram showing the regions of classical collision-free and collision-dominated plasma as well as degenerate plasma. Thin lines correspond to $Z=1$ for effective charge states $Z_{\text{eff}}=10,40,80$. Below these lines for a given projectile charge Z_{eff} nonlinear effects in dE/dx should be taken into consideration. The dashed line corresponds to 99% Saha ionization of a hydrogen plasma. The shaded areas refer to three examples of topical interest discussed in the text where nonlinear interaction of beam ions with plasma becomes important.

$$\begin{aligned}&\frac{\partial f}{\partial t}+\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}}+\frac{\partial\Phi}{\partial\mathbf{r}}\cdot\frac{\partial f}{\partial\mathbf{v}}=0\\ &\nabla^2\Phi=-\frac{Z_{\text{eff}}}{n_0\lambda_D^3}\delta(\mathbf{r}-\mathbf{v}_pt)+\int d^3v\, f(\mathbf{r},\mathbf{v},t)-1\ .\end{aligned} \tag{7'}$$

All equations written in the dimensionless quantities of Eq. (8) will be marked with a prime ('), those in physical quantities are given without a prime. It is evident that the parameter

$$Z=\frac{Z_{\text{eff}}}{n_0\lambda_D^3}=\frac{Z_{\text{eff}}}{N_D}=\frac{4\pi}{3}\frac{Z_{\text{eff}}}{\mathcal{N}} \tag{9}$$

measures the strength of the perturbation by the beam ion. Z is a combination of the ion's effective charge Z_{eff} and the Debye number $\mathcal{N}_D$ of the plasma. Besides the plasma parameter $1/\mathcal{N}_D$, which measures the ratio between average potential and kinetic energies for a plasma in thermal equilibrium, Z is a second coupling parameter of the problem. Only if $Z<1$ the perturbation due to the ion is small, and an expansion of f and Φ in growing orders of Z becomes meaningful.

For large projectile velocities ($v_p \gg v_{\text{th}}$) this condition relaxes because only those few electrons moving with the same speed into the same direction (resonant particles) will interact strongly with the projectile. Ashley, Ritchie, and Brandt[20] showed in their nonlinear theory on the stopping power in cold solids for large v_p that the nonlinear term in lowest order is proportional to $Z/(v_p/v_{\text{th}})^3$.

The solution for the electrostatic potential in first order (e.g., Refs. 21 and 22) is given by

$$\begin{aligned}\Phi_1(\mathbf{r},t)&=\frac{Z_{\text{eff}}e}{2\pi^2}\int d^3k\frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{v}_pt)}}{k^2\epsilon(\mathbf{k},\mathbf{k}\cdot\mathbf{v}_p)}\\ &=\frac{Z_{\text{eff}}e}{2\pi^2}\int d^3k\frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{v}_pt)}}{k^2+k_D^2W\left[\dfrac{\mathbf{k}\cdot\mathbf{v}_p}{k\sqrt{k_BT/m}}\right]}\end{aligned} \tag{10}$$

with the dielectric response function of the field-free plasma

$$\epsilon(\mathbf{k},\omega)=1+\frac{1}{k^2}W\left[\frac{\omega}{k}\right]\ . \tag{11'}$$

The dispersion function $W(\text{Im}\zeta\ge0)$

$$W(\zeta)=\frac{1}{\sqrt{2\pi}}\lim_{\nu\to0+}\int_{-\infty}^{\infty}dx\frac{xe^{-x^2/2}}{x-\zeta-i\nu} \tag{12}$$

has the representation (Fried and Conte[23])

$$\begin{aligned}W(\zeta)=&\,i\left[\frac{\pi}{2}\right]^{1/2}\zeta e^{-\zeta^2/2}+1-\zeta^2\\ &+\frac{\zeta^4}{3}-\cdots+\frac{(-1)^{n+1}\zeta^{2n+2}}{(2n+1)!!}+\cdots\end{aligned} \tag{13}$$

and the asymptotic expansion for large ζ (with $|\text{Im}\zeta/\text{Re}\zeta|\ll1$)

$$W_\infty(\zeta)=i\left[\frac{\pi}{2}\right]^{1/2}\zeta e^{-\zeta^2/2}-\frac{1}{\zeta^2}-\frac{3}{\zeta^4}-\cdots-\frac{(2n-1)!!}{\zeta^{2n}}-\cdots . \tag{14}$$

In passing we note the importance of the zeros of the dielectric function $\epsilon(\mathbf{k},\omega)$ given by the dispersion relation $W(\omega/k)=-k^2$ with $\omega=\mathbf{k}\cdot\mathbf{v}_p$. For $|\omega/k|\gg 1$ Eq. (14) yields

$$\omega^2=1+\frac{3k^2}{\omega^2}+\cdots-i\left[\frac{\pi}{2}\right]^{1/2}\frac{\omega^3}{k^3}e^{-\omega^2/2k^2}, \tag{15'}$$

which can be solved by iteration

$$\omega=\pm(1+3k^2)^{1/2}-i\left[\frac{\pi}{8}\right]^{1/2}\frac{1}{|k|^3}\exp\left[-\frac{1+3k^3}{2k^2}\right], \tag{16'}$$

provided $|k|\ll 1$. This defines plasma waves; the imaginary part $\mathrm{Im}\omega$ is the Landau damping.[24] The beam ion couples to these waves and loses part of its energy to the excitation of the waves.

B. Dielectric theory of the stopping power

The stopping power of an ion is the force $\mathbf{F}$ that the ion experiences from its own induced field:

$$-\left[\frac{dE}{dx}\right]=-\mathbf{F}\cdot\mathbf{e}_x\Big|_{\mathbf{r}=\mathbf{v}_p t}=Z_{\mathrm{eff}}e\frac{\partial\Phi_1}{\partial x}\Big|_{\mathbf{r}=\mathbf{v}_p t}. \tag{17}$$

Taking advantage of the symmetry relation $W(-\zeta)=W^*(\zeta)$ and defining $X(\zeta)=\mathrm{Re}W(\zeta)$, $Y(\zeta)=\mathrm{Im}W(\zeta)$, the substitution of Eq. (10) in Eq. (17) yields

$$-\left[\frac{dE}{dx}\right]=\frac{Z^2N_D}{(2\pi)^2}\int_0^{k_{\max}}dk\,k^3\int_{-1}^{+1}d\mu\frac{\mu Y(\mu v_p)}{[k^2+X(\mu v_p)]^2+Y^2(\mu v_p)} . \tag{18'}$$

In Eq. (18) we introduced a cut-off parameter $k_{\max}$ in order to avoid the logarithmic divergence at large k. This divergence corresponds to the incapability of the linearized Vlasov theory to treat close encounters between the projectile and the plasma electrons properly. The full nonlinear Vlasov equation accurately describes the scattering of individual electrons with the projectile ion in accordance with the Rutherford scattering theory. The exact expression for the energy transfer in a Rutherford two-body collision is

$$\Delta E(b)=\frac{(\Delta\mathbf{p})^2}{2m}=\frac{2Z_{\mathrm{eff}}^2e^4}{mv_r^2}\frac{1}{\left[\dfrac{Z_{\mathrm{eff}}e^2}{mv_r^2}\right]^2+b^2}, \tag{19}$$

where $v_r\simeq(v_p^2+v_{\mathrm{th}}^2)^{1/2}$ is the mean relative velocity between projectile and electron. From the denominator in Eq. (19) it follows that the effective minimum impact parameter is $b_{\min}=Z_{\mathrm{eff}}e^2/mv_r^2$, often called the "distance of closest approach." Thus,

$$k_{\max}=\frac{1}{b_{\min}}=\frac{m(v_p^2+v_{\mathrm{th}}^2)}{|Z_{\mathrm{eff}}|e^2} \tag{20}$$

ensures agreement of Eq. (18') with the Rutherford theory for small impact parameters. When $v_p>2|Z_{\mathrm{eff}}|\alpha c$, then the de Broglie wavelength begins to exceed the classical distance of closest approach. Under these circumstances we choose $k_{\max}=2mv_p/\hbar$.

The k integration in Eq. (18') can be evaluated without further approximation (cf. Sec. 10.3 in Ref. 14):

$$-\left[\frac{dE}{dx}\right]=\frac{Z^2N_D}{2\pi^2}\frac{1}{v_p^2}\int_0^{v_p}d\zeta\,\zeta Y\left[\ln k_{\max}+\tfrac{1}{4}\ln\frac{\left[1+\dfrac{X}{k_{\max}^2}\right]^2+\dfrac{Y^2}{k_{\max}^4}}{X^2+Y^2}-\frac{X}{2Y}\left[\arctan\frac{k_{\max}^2+X}{Y}-\arctan\frac{X}{Y}\right]\right], \tag{21'}$$

where $\zeta=\mu v_p$. In Secs. II C and II D this expression is evaluated for large and small projectile velocities.

C. Stopping power for very large projectile velocities

For large v_p the path of integration along the real μ axis in Eq. (18') comes very close to the pole given by the dispersion relation $\epsilon=0$, which is responsible for the collective oscillations in the plasma. We evaluate this collective contribution to Eq. (18') exploiting the properties of the δ function

$$\lim_{Y\to 0}\frac{Yg(\mu)}{f^2(\mu)+Y^2}=\pi\delta(\mu-\mu_0)\frac{g(\mu_0)}{|f'(\mu_0)|}, \tag{22}$$

where μ_0 is a zero of f, and $f(\mu)=k^2+X(\mu v_p)\simeq k^2-1/\mu^2 v_p^2$. The zero $X=-k^2$ exists only for sufficiently small k, say for $k<1$. The zero is given by $\mu_0=\pm 1/kv_p$; because of the limitation $|\mu|<1$ the path of integration comes close to the zero only if $k>1/v_p$. The collective contribution to the stopping power at large velocities is therefore

$$-\left[\frac{dE}{dx}\right]_{\text{coll}} \simeq \frac{Z^2 N_D}{(2\pi)^2}\int_{1/v_p}^{1} dk\, k^3 2\int_0^1 d\mu\, \pi\delta(\mu-\mu_0)\frac{|\mu_0|}{|(2/\mu_0^3 v_p^2)|}=\frac{Z^2 N_D}{4\pi v_p^2}\ln v_p\ . \tag{23'}$$

The counterpart to the collective stopping power with $k<1$ is the individual particle contribution with $k>1$:

$$-\left[\frac{dE}{dx}\right]_{\text{indiv}} \simeq \frac{Z^2 N_D}{(2\pi)^2}\int_1^{k_{\max}} dk\, k^3 2\int_0^1 d\mu \frac{\mu Y}{k^4} \simeq \frac{Z^2 N_D}{4\pi v_p^2}\ln\left[\frac{4\pi v_p^2}{\mathcal{Z}}\right] . \tag{24'}$$

The combination of Eqs. (25′) and (26′) yields for $v_p \gg 1$

$$-\left[\frac{dE}{dx}\right] = -\left[\frac{dE}{dx}\right]_{\text{coll}} - \left[\frac{dE}{dx}\right]_{\text{indiv}} \simeq \frac{Z^2 N_D}{4\pi v_p^2}\ln\left[\frac{4\pi v_p^3}{\mathcal{Z}}\right] . \tag{25'}$$

This is the well-known Bohr stopping power,[15] identical with Eq. (1).

The ratio $dE_{\text{coll}}/dE_{\text{indiv}}$ indicates which fraction of the ion energy is used up by plasmon excitation. In a plasma with $k_B T=1$ Ry, $n_0=10^{17}$ cm^{-3} ($\lambda_D=8.6$ μm, $\mathcal{N}_D=272$) a fast ion with $E=2.5$ MeV/u and $Z_{\text{eff}}=50$ loses about 33% of its kinetic energy in the excitation of plasma waves. It creates 5.4×10^6 plasmons per cm of path length.

D. Stopping power for small projectile velocities

When an ion moves slowly through a plasma the electrons have much time to experience the ion's attractive potential. They are accelerated towards the ion, but when they reach its trajectory the ion has already moved forward a little bit. Hence, we expect an increased density of electrons at some place in the trail of the ion. This negative charge density pulls back the positive ion and gives rise to the stopping power.

The Taylor expansion of Eq. (21′) for small v_p yields the "friction law"

$$-\left[\frac{dE}{dx}\right] = \mathcal{R}_1 v_p + \mathcal{R}_3 v_p^3 + O(v_p^5) \tag{26'}$$

with the "friction coefficient"

$$\mathcal{R}_1 = \frac{Z^2 N_D}{12\pi\sqrt{2\pi}}\left[\ln(K^2+1)-\frac{K^2}{K^2+1}\right] \tag{27'}$$

and the v_p^3 coefficient

$$\mathcal{R}_3 = \frac{Z^2 N_D}{12\pi\sqrt{2\pi}}\left[-\frac{3}{10}\ln(K^2+1)+\left[\frac{8}{5}-\frac{\pi}{20}\right] -\frac{29}{10}\frac{1}{K^2+1}+\left[\frac{13}{10}+\frac{3\pi}{20}\right] \times\frac{1}{(K^2+1)^2}+\frac{\pi}{10}\frac{1}{(K^2+1)^3}\right] , \tag{28'}$$

where

$$K=8\pi/|\mathcal{Z}|=8\pi n_0 \lambda_D^3/|Z_{\text{eff}}| \ . \tag{29'}$$

Note that in dimensional form $K=\lim_{v_p\to 0} k_{\max} = mv_p^2/(|Z|e^2)$ according to Eq. (20). For plasmas of low and medium densities it is $\mathcal{Z}<1$ and therefore $K\gg 1$. The Coulomb logarithms in Eqs. (27′) and (28′) are then the leading terms. We obtain

$$-\left[\frac{dE}{dx}\right] = \frac{Z^2 N_D}{12\pi\sqrt{2\pi}}\left[(\ln K^2-1)v_p - \left[\frac{3}{10}\ln K^2-\frac{8}{5}+\frac{\pi}{20}\right]v_p^3 + O(v_p^5)\right] . \tag{30'}$$

In the opposite limit, when the density is so high that $K \lesssim 1$ (i.e., $\mathcal{N}_D<|Z_{\text{eff}}|/6$), the stopping power has the limiting form

$$-\left[\frac{dE}{dx}\right] = \frac{Z^2 N_D}{12\pi\sqrt{2\pi}}\left[\frac{K^4}{2}v_p+\frac{\pi}{5}(1-3K^2)v_p^3+O(v_p^5)\right] . \tag{31'}$$

This is a v_p^3 law since in the linearized theory the term linear in v_p vanishes with increasing density. This reduction of the stopping power has its physical origin in the enhancement of the plasma frequency ω_p in the denominator of the Coulomb logarithm and therefore in the higher excitation energy of plasmons. It should be noted, however, that in this case $\mathcal{Z}\gg 1$ and that strong nonlinear contributions will increase the total stopping power again, see Sec. III. In Fig. 3 the full solution of Eq. (21′) is compared with the approximations Eqs. (30′) and (31′) for the linear and cubic velocity dependence.

The most important property of the stopping power at small velocities is $dE/dx \propto v_p$, provided that the density is not too high. This looks like the friction law of a

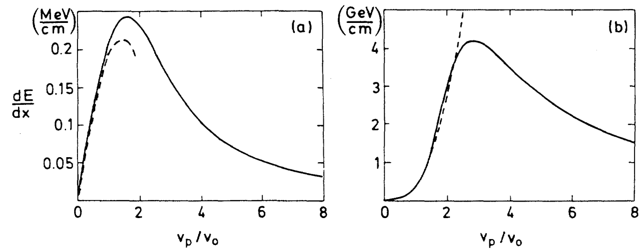

FIG. 3. Stopping power as a function of v_p in units of $v_0=(k_BT/m)^{1/2}$ for ions with charge $Z_{\rm eff}=10$ in a plasma of temperature $k_BT=1$ Ry and density (a) $n_0=10^{17}$ cm^{-3} ($K^2=2.7\times10^4\gg1$) or (b) $n_0=10^{22}$ cm^{-3} ($K^2=0.27\ll1$). Solid lines, numerical evaluation of Eq. (21′); dashed lines, asymptotic forms Eq. (30′) and (31′) for small projectile velocity v_p.

viscous fluid, and accordingly $\mathcal{R}_1$ is called the friction coefficient. However, in the case of an ideal plasma it should be noted that this law does not depend on the plasma viscosity and is not a consequence of electron-electron collisions with small impact parameter. Those collisions are neglected in the Vlasov equation. As described above it is rather the fact that the dressing of the ion takes some time and produces the negative charge behind the ion leading to the drag.

E. Analytic approximation of dE/dx for arbitrary projectile velocities

For $Z<1$ it is $k_{\max}\gg1$, and Eq. (21′) simplifies to become

$$-\left[\frac{dE}{dx}\right]=\frac{Z^2N_D}{2\pi^2}\frac{1}{v_p^2}\int_0^{v_p}d\xi\,\xi Y\times\left[\ln k_{\max}-\tfrac{1}{4}\ln(X^2+Y^2)-\frac{X}{2Y}\times\left[\frac{\pi}{2}-\arctan\frac{X}{Y}\right]\right]. \tag{32′}$$

The first term in the large square brackets is the so-called Chandrasekhar term. One obtains

$$G(v_p)\equiv\frac{2}{\pi}\int_0^{v_p}d\xi\,\xi Y=\mathrm{erf}\left[\frac{v_p}{\sqrt{2}}\right]-\left[\frac{2}{\pi}\right]^{1/2}v_pe^{-v_p^2/2}\,, \tag{33′}$$

with the error function[25] defined as $\mathrm{erf}(x)=2\pi^{-1/2}\int_0^x dt\,\exp(-t^2)$. The Chandrasekhar function G is shown in Fig. 4. The leading term in Eq. (32′) is $G(v_p)\ln(k_{\max})$ if $v_p\ll1$. The integral over the second and third terms in the large square brackets of Eq. (32′) has to be approximated. Utilizing the asymptotic forms of the integrand for $\xi\to0$ and $\xi\to\infty$ we find to a good approximation

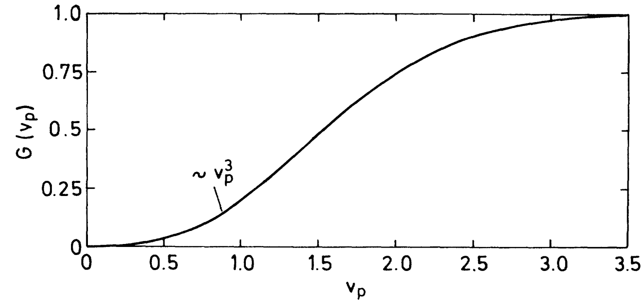

FIG. 4. The Chandrasekhar function $G(v_p)$.

$$H(v_p)\equiv\frac{1}{\pi\ln v_p}\int_0^{v_p}d\xi\left[-\tfrac{1}{2}\xi Y(\xi)\ln[X^2(\xi)+Y^2(\xi)]-\xi X(\xi)\left[\frac{\pi}{2}-\arctan\frac{X(\xi)}{Y(\xi)}\right]\right]\approx-\frac{v_p^3}{3\sqrt{2\pi}\ln v_p}e^{-v_p^2/2}+\frac{v_p^4}{v_p^4+12}\,. \tag{34′}$$

The exact and approximate expressions for $H(v_p)\ln v_p$ are compared in Fig. 5.

With Eq. (33′) for $G(v_p)$ and Eq. (34′) for $H(v_p)$ the stopping power is written in dimensional form

$$-\left[\frac{dE}{dx}\right]=\left[\frac{Z_{\rm eff}e\omega_p}{v_p}\right]^2[G(v_p/\sqrt{k_BT/m})\ln(k_{\max}\lambda_D)+H(v_p/\sqrt{k_BT/m})\times\ln(v_p/\sqrt{k_BT/m})] \tag{35}$$

with $k_{\max}\lambda_D=m(v_p^2+2k_BT/m)[k_BT/(4\pi e^6n_0Z_{\rm eff}^2)]^{1/2}$. We propose to use this approximation for the free-electron stopping power in numerical calculations instead of the formula commonly used in the literature,[26,27] which is obtained from Eq. (35) by substituting G for H.

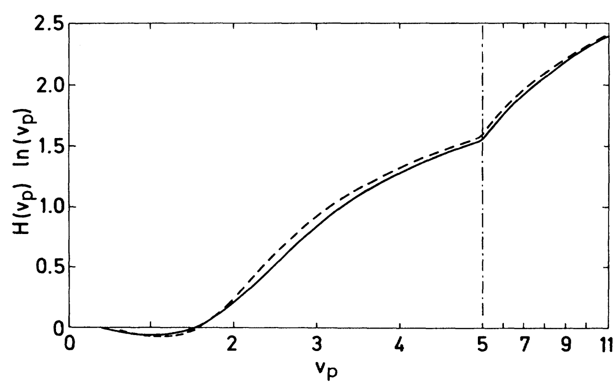

FIG. 5. Comparison of the function $H(v_p)\ln v_p$ (solid line) and the approximate formula given by Eq. (34′) (dashed line). Note the change of scales at $v_p=5$.

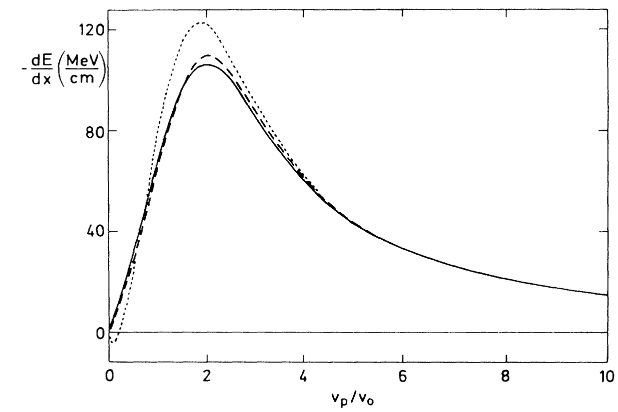

FIG. 6. The stopping power of an ion with $Z_{\rm eff}=10$ in plasma with $k_BT_e=1$ Ry and $n_e=10^{20}$ cm^{-3} ($Z=4.9$) as a function of the projectile velocity v_p in units of $v_0=(k_BT/m)^{1/2}$. The relative error between Eq. (21′) (solid line) and the approximation Eq. (35) (dashed line) is smaller than 3%. The dotted line shows the standard approximation $dE/dx \propto G(v_p/v_0)/v_p^2$ often used in the literature.

Figure 6 reveals that the result Eq. (35) is superior to the $H=G$ approximation which overestimates the maximum of the stopping power up to 20%, and leads to nonphysical negative values for $v_p \ll 1$.

F. Influence of plasma ions on the stopping power in the Vlasov theory

Up to now we treated the plasma ions merely as a static neutralizing background. In a dynamical treatment one has to solve two Vlasov equations, one for electrons and one for ions, and the Poisson equation coupling the two species. For simplification we restrict ourselves to a fully ionized hydrogen plasma. The solution for the electrostatic potential has exactly the same form as Eq. (10), but now the dielectric function reads

$$\epsilon(\mathbf{k},\omega)=1+\frac{1}{k^2}\left[W\left[\frac{\omega}{k}\right]+W\left[\sqrt{M}\frac{\omega}{k}\right]\right], \qquad (36')$$

where $M=1836$ is the proton mass. The only change in all equations is to substitute the dispersion function W, or rather its real and imaginary part X and Y by the sum in Eq. (36′). A certain difficulty is the cutoff parameter $k_{\max}$, which is proportional to the mass of the scattered particle, and therefore one has to decide in each individual case whether to insert the electron mass ($\Longrightarrow k_{\max}$) or the ion mass ($\Longrightarrow Mk_{\max}$).

We calculate the contribution of the ions to dE/dx in three energy regime.

(i) *Large projectile velocities.* For $v_p \gg v_{{\rm th},e}$ the collective contribution given by Eq. (22) has to be evaluated with $f=k^2-X(\mu v_p)-X(\sqrt{M}\mu v_p)\approx k^2-(1/\mu^2v_p^2)(1+1/M)$; thus

$$-\left[\frac{dE}{dx}\right]_{\rm coll} \approx \frac{Z^2N_D}{4\pi v_p^2}\left[1+\frac{1}{M}\right]^3 \ln v_p \approx \frac{Z^2N_D}{4\pi v_p^2}\left[1+\frac{3}{M}\right]\ln v_p . \qquad (37')$$

Analogously the individual particle term gives

$$-\left[\frac{dE}{dx}\right]_{\rm indiv} \approx \frac{Z^2N_D}{4\pi v_p^2}\left[\ln\left[\frac{4\pi v_p^2}{|Z|}\right] + \frac{1}{M}\ln\left[M\frac{4\pi v_p^2}{|Z|}\right]\right] . \qquad (38')$$

Both the collective and the individual particle term change negligibly due to the influence of the plasma ions.

(ii) *Small projectile velocities.* More interesting is the behavior at very small velocities $v_p \ll v_{{\rm th},i}$. With $X(\mu v_p)\approx X(\sqrt{M}\mu v_p)\approx 1$ and $Y(\mu v_p)\approx Y(\sqrt{M}\mu v_p)\approx 0$ the generalization of Eq. (18′) becomes

$$-\left[\frac{dE}{dx}\right]=\frac{Z^2N_D}{(2\pi)^2}\int_{-1}^{+1}d\mu\,\mu\left[Y(\mu v_p)\int_0^{k_{\max}}dk\frac{k^3}{(k^2+2)^2} + Y(\sqrt{M}\mu v_p)\times\int_0^{Mk_{\max}}dk\frac{k^3}{(k^2+2)^2}\right] . \qquad (39')$$

For $k_{\max}\gg 1$ the evaluation yields

$$-\left[\frac{dE}{dx}\right]=\frac{Z^2N_D}{12\pi\sqrt{2\pi}}v_p\sqrt{M}\,\ln\left[\frac{M^2}{2}\frac{64\pi^2}{Z^2}\right] . \qquad (40')$$

Comparison with Eq. (30′) shows that the ions increase the stopping power by a factor $\sqrt{M}\approx 43$.

(iii) *Medium projectile velocities.* When the velocity is between the electronic and the ionic thermal velocity $v_{{\rm th},i}\ll v_p \ll v_{{\rm th},e}$, it is $X(\mu v_p)\approx 1$, $X(\sqrt{M}\mu v_p)\approx Y(\mu v_p)\approx Y(\sqrt{M}\mu v_p)\approx 0$. We find

$$-\left[\frac{dE}{dx}\right]=\frac{Z^2N_D}{12\pi\sqrt{2\pi}}v_p\ln\left[\frac{64\pi^2}{Z^2}\right] + \frac{Z^2N_D}{4\pi}\frac{1}{Mv_p^2}\ln\left[M\frac{8\pi}{|Z|}\right] . \qquad (41')$$

This the sum of the friction law for electrons and the high-velocity form for ions.

We also looked into the question whether a beam ion can couple to the ion-acoustic mode in a plasma and loses energy when pumping ion-acoustic waves. Using the same technique as in the derivation of Eq. (23′) one finds

$$-\left[\frac{dE}{dx}\right]_{\text{acoustic}} \approx \frac{Z^2 N_D}{(2\pi)^2}\int_0^1 dk\, k^3 \int_0^1 d\mu\, \pi \frac{\delta(\mu-\mu_0)|\mu_0|}{\left|\frac{1+k^2/2}{\sqrt{M}}v_p - \frac{\sqrt{M}}{(1+k^2/2)^3}v_p\right|} \approx \frac{Z^2 N_D}{4\pi v_p^2}\frac{1}{4M} \,. \tag{42'}$$

This is negligible compared with $(dE/dx)_{\text{coll}}$ and $(dE/dx)_{\text{indiv}}$. It is straightforward to show that the ion-acoustic contribution increases by a factor $(T_e/T_i)^2$, provided that $T_e \gg T_i$. Only then the beam ions will lose appreciable amounts of energy to the ion-acoustic mode.

From these considerations it follows that the combined stopping power of ions and electrons may be calculated without large error by first treating only the electrons dynamically, then only the ions, and finally adding the two contributions. Figure 7 shows the stopping power by electrons and ions in plasma with (a) $n_0 = 10^{17}$ cm^{-3} and (b) $n_0 = 10^{22}$ cm^{-3}. The peaklike contribution of the ions adds appreciably to the electronic stopping power only at very small projectile velocities. It should be noted that the displayed calculation is done for constant ion charge $Z_{\text{eff}} = 10$; however the charge drops at the end of the range due to electron recombination, and this further reduces the ionic contribution.

G. Influence of a collision term on the stopping power

Finally we consider the question of electron-electron collisions approximated by Krook's term $(\partial f/\partial t)_{\text{coll}} = \nu(f_0 - f) \simeq -\nu f_1$, where f_0 and f_1 are the unperturbed and first-order particle distribution functions, respectively, and the collision frequency ν is given by Eq. (5'). Instead of Eq. (18') one obtains

$$-\left[\frac{dE}{dx}\right] = -\frac{Z^2 N_D}{(2\pi)^2}\int_0^{k_{\max}} dk\, k \int_{-1}^{+1} d\mu\, \mu \,\text{Im}\frac{1}{\epsilon(\mathbf{k}, k\mu v_p + i\nu)} \,. \tag{43'}$$

First we calculate the friction coefficient $\mathcal{R}_1$ for vanishing projectile velocities. The Taylor expansion of the dispersion function $W(\mu v_p + i\nu/k)$ around $i\nu/k$, using $W'(\zeta) = (1/\zeta - \zeta)W(\zeta) - 1/\zeta$ and $K = 8\pi/|Z| \gg 1$, yields in first order of ν

$$\mathcal{R}_1 = \frac{Z^2 N_D}{12\pi\sqrt{2\pi}}\left[\ln K^2 - 1 - \sqrt{2\pi}\left[\frac{1}{2} - \frac{\pi}{16}\right]\nu\right] \,. \tag{44}$$

The contributions by the collisions in this regime never exceed 0.1%. Much higher contributions are to be expected at very high densities when the linearized theory breaks down anyway.

For large velocities we repeat the investigation of the collective and individual particle terms in Sec. II C. The collective term is characterized by the relation $k^2 = -\text{Re}W(\mu v_p + i\nu/k) \approx \text{Re}(\mu^2 v_p^2 + 2i\mu v_p \nu/k - \nu^2/k^2)^{-1}$. The evaluation of the real part shows that ν appears only quadratically; hence, in first order, collisions make no contribution to the collective stopping power. For the individual particle contribution the integral $\int_0^1 d\mu\, \mu \,\text{Im}W(\mu v_p + i\nu/k)$ has to be solved. Using the Taylor expansion $\text{Im}W \simeq Y(\mu v_p) + (\nu/k)X'(\mu v_p)$ one finds

$$-\left[\frac{dE}{dx}\right]_{\text{indiv}} \simeq \frac{Z^2 N_D}{4\pi v_p^2}\left[\ln k_{\max} - \nu\frac{2}{\pi}\left[1 - \frac{1}{k_{\max}}\right]\int_0^{v_p} d\xi\, X(\xi)\right] \,. \tag{45'}$$

The remaining integral can be recast to give Dawson's integral.[25] It is easy to show that

$$-\left[\frac{dE}{dx}\right] \approx \frac{Z^2 N_D}{4\pi v_p^2}\left[\ln\left[\frac{4\pi v_p^3}{Z}\right] - \nu\frac{\sqrt{2}}{\pi}\frac{1}{v_p}\right] \,. \tag{46'}$$

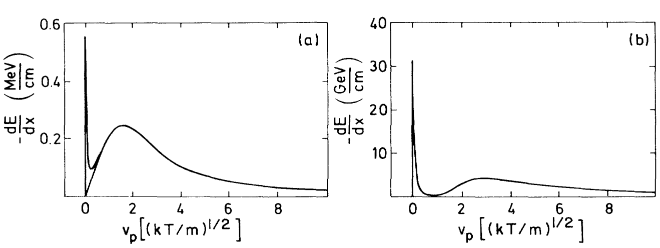

FIG. 7. The electronic and ionic contribution to the stopping power of an ion with constant charge $Z_{\text{eff}} = 10$ in plasma with $k_B T = 1$ Ry and (a) $n_0 = 10^{17}$ cm^{-3} or (b) $n_0 = 10^{22}$ cm^{-3}.

This is a lowering compared with the collision-free theory of at most 1% if one assumes $\mathcal{N}_D \gtrsim 1$ and uses Eq. (5) for ν.

III. PLASMA STOPPING POWER IN THE NONLINEAR VLASOV THEORY

We present results of a nonlinear description of the stopping power for heavy ions in plasma with emphasis on small velocities, i.e., at the end of their range. This region is of special interest because deviations from the linear behavior are expected to be largest. It will be shown that the nonlinear effects in lowest order yield an increase in the stopping power and therefore a further range shortening. In a recent paper[28] an exact nonlinear result could be derived for a one-dimensional plasma and low ion velocity, showing that the friction force is twice as large as predicted by linearized theory. Here, the full three-dimensional case is treated.

It is important for heavy-ion stopping that high nonequilibrium charge states are expected to play an important role at the end of the range of the projectile ions. This will be a major result of paper II. When slowing down, the ions maintain a high charge Z_{eff} and only after they have come to rest they capture the missing electrons. Due to this high Z_{eff}, the condition $\mathcal{Z}=(4\pi/3)Z_{\text{eff}}/\mathcal{N}_D<1$ required for the linear stopping theory to be valid may be violated even though the plasma parameter is $1/\mathcal{N}_D<1$. As one may check in the phase-space diagram in Fig. 2 the condition $\mathcal{Z}<1$ is violated for a number of experimentally relevant examples, and nonlinear effects should be expected.

For small projectile velocities the nonlinear friction coefficient is derived in Sec. III B. The result may be expressed in terms of a form factor F. The deviation of F from the linearized result in lowest nonlinear order yields an additional term in the stopping power, which is proportional to $\mathcal{Z}^3$ for negative ions (Sec. III C) and proportional to $\mathcal{Z}^{5/2}$ for positive ions. In both cases dE/dx will grow due to the nonlinearity.

Also for high projectile velocities v_p nonlinear effects in the stopping power play a role, even though they are smaller. The coupling parameter determining the nonlinearity of the interaction between projectile ion and plasma in this case is $(Z_{\text{eff}}/N_D)(v_{\text{th}}/v_p)^3$. The Z_{eff}^3 term of the stopping power was calculated for fast projectiles in cold matter in a paper by Ashley, Ritchie, and Brandt[20] (ARB). In the following we start the discussion in Sec. III A by applying the high-velocity treatment by ARB to the plasma case and then turn to the even more interesting regime of low velocities in Secs. III B and III C.

A. Nonlinear stopping power at high projectile velocities v_p

Experimental evidence for nonlinear behavior in fast particle stopping was first observed by Barkas, Dyer, and Heckman.[29] They observed a difference in the energy loss of positive and negative (but otherwise identical) pions. The energy loss of the positive particles is larger by some percent than that of the negative particles. The linearized theory with a square dependence on Z_{eff} cannot account for this difference. The ARB theory gives a valid description of the "Barkas effect" for fast projectiles stopping in matter. Jackson and McCarthy[30] generalized the theory to relativistic velocities, Hill and Merzbacher[31] investigated the agreement with quantum theory, and Maynard and Deutsch[32] studied its application to ICF problems.

ARB calculated the energy transfer $\Delta E(b)$ in a distant collision with the impact parameter b between an electron and the projectile ion. The electron is assumed to be bound harmonically with frequency ω. For a plasma electron we take $\omega=\omega_p$. During the collision the electron will be displaced by ξ due to the force by the projectile ion: $\ddot{\xi}+\omega_p^2\xi=\mathbf{f}/m$, where $\mathbf{f}(t,b)$ is the force by the ion, which moves into positive x direction. The equation of motion can be rewritten in the form

$$\xi(t)=\frac{1}{m\omega_p}\int_{-\infty}^{t}dt'\mathbf{f}(t',b)\sin[\omega_p(t-t')] , \tag{47}$$

with the force

$$\mathbf{f}(t,b)=Z_{\text{eff}}e^2\frac{(v_pt+\xi_x)\mathbf{e}_x+(b+\xi_y)\mathbf{e}_y}{[(v_pt+\xi_x)^2+(b+\xi_y)^2]^{3/2}} . \tag{48}$$

The energy transfer is

$$\Delta E(b)=\frac{m}{2}(\dot{\xi}^2+\omega_p^2\xi^2)|_{t\to\infty} = \frac{1}{2m}\left|\int_{-\infty}^{\infty}dt\ \mathbf{f}(t,b)e^{i\omega_p t}\right|^2 . \tag{49}$$

These integral equations describe the full three-body problem, which is nonlinear in Z_{eff}. ARB solved Eqs. (47) and (48) iteratively for small displacements $|\xi|^2 \ll v_p^2t^2+b^2$. For $\mathbf{f}$ there is the expansion

$$\mathbf{f}(t,b)=\mathbf{f}_1(t,b)+\mathbf{f}_2(t,b)+\cdots \tag{50}$$

with

$$\mathbf{f}_1(t,b)=Z_{\text{eff}}e^2\frac{v_pt\mathbf{e}_x+b\mathbf{e}_y}{[(v_pt)^2+b^2]^{3/2}} \tag{50a}$$

$$\mathbf{f}_2(t,b)=\frac{Z_{\text{eff}}e^2}{[(v_pt)^2+b^2]^{3/2}}\left[\xi_x\left[\mathbf{e}_x-\frac{3v_pt}{v_p^2t^2+b^2}(v_pt\mathbf{e}_x+b\mathbf{e}_y)\right]+\xi_y\left[\mathbf{e}_y-\frac{3b}{v_p^2t^2+b^2}(v_pt\mathbf{e}_x+b\mathbf{e}_y)\right]\right] . \tag{50b}$$

In lowest order it is $\xi=0$ and the insertion of Eq. (50a) into Eq. (49) yields[33]

$$\Delta E_1(b)=\frac{2Z_{\text{eff}}^2e^4\omega_p^2}{mv_p^4}\left[K_1^2\left[\frac{\omega_pb}{v_p}\right]+K_0^2\left[\frac{\omega_pb}{v_p}\right]\right] \tag{51}$$

with modified Bessel functions[25] K_0 and K_1. The stopping power reads[33]

$$-\left[\frac{dE}{dx}\right]_1=2\pi n_0\int_{b_{\min}}^{\infty}db\ b\Delta E_1(b)=\left[\frac{Z_{\text{eff}}e\omega_p}{v_p}\right]^2\ln\left[\frac{2e^{-\gamma}v_p}{\omega_pb_{\min}}\right] \tag{52}$$

with $\gamma=0.5772$ (Euler's constant), which is in accordance with the result of the dielectric theory if we identify $b_{\min}=1/k_{\max}$ as the distance of closest approach defined by Eq. (20). The simple description of the electron as a harmonic oscillator hence describes correctly the collective effects of the plasma.

In second order if one inserts the force Eq. (50a) into Eq. (47) and calculates the displacement $\xi(t,b)$, then one knows the force Eq. (50b) and the second-order correction of ΔE. Because of Eq. (49) this correction reads

$$\Delta E_2(b)=\frac{1}{m}\int_{-\infty}^{\infty}dt\ \mathbf{f}_1(t,b)\cos(\omega_p t)\int_{-\infty}^{\infty}dt\ \mathbf{f}_2(t,b)\cos(\omega_p t)+\frac{1}{m}\int_{-\infty}^{\infty}dt\ \mathbf{f}_1(t,b)\sin(\omega_p t)\int_{-\infty}^{\infty}dt\ \mathbf{f}_2(t,b)\sin(\omega_p t)\ . \tag{53}$$

Since in second order $\xi \propto Z_{\rm eff}$ and $f_2 \propto Z_{\rm eff}^2$, then $\Delta E_2(b) \propto Z_{\rm eff}^3$.

ARB calculated the remaining integrals numerically. We investigate only the analytically accessible case of weakly bound electrons ($\omega_p \to 0$); it leads to a new practical result for the energy loss in a plasma not given by ARB. After lengthy calculations one obtains

$$\Delta E_2(b)=-2\frac{Z_{\rm eff}^3 e^6}{m^2 v_p^6 b}\left[\omega_p t_0\int_0^{\infty}dt\frac{\cos^2(\omega_p t)}{(t^2+t_0^2)^{3/2}}\mathrm{si}(\omega_p t)\left[1-\frac{3t_0^2}{t^2+t_0^2}\right]+3t_0\int_0^{\infty}dt\frac{\cos(\omega_p t)}{(t^2+t_0^2)^{5/2}}t\sin(\omega_p t)\mathrm{si}(\omega_p t)\right]\ . \tag{54}$$

This result is linear in ω_p; the ω_p^0 term vanishes, because according to the Rutherford theory for free electrons the relation $dE/dx \propto Z_{\rm eff}^2$ is exact and $Z_{\rm eff}^3$ contributions do not occur. The Barkas-term reads

$$-\left[\frac{dE}{dx}\right]_2=2\pi n_0\int_{b_{\min}}^{\infty}db\ b\Delta E_2(b)$$
$$=\frac{4\pi Z_{\rm eff}^3 e^6 n_0}{m^2 v_p^5}\omega_p\left[\frac{1}{c}\int_0^{\infty}dy\frac{y}{(1+y^2)^{3/2}}\sin(2cy)\mathrm{si}(cy)-\int_0^{\infty}dy\frac{y^2+2}{(1+y^2)^{3/2}}\cos^2(cy)\mathrm{si}(cy)\right]\ , \tag{55}$$

where $c=\omega_p b_{\min}/v_p$. Using l'Hospital's rule for $c\to 0$ it is easy to show that the term in large parentheses in Eq. (55) is $-(3\pi/2)\ln c + A$. ARB obtained $A\approx -2.4$. In the dimensionless quantities of Eq. (8) the final result for the case $\omega_p \to 0$ reads

$$-\left[\frac{dE}{dx}\right]=-\left[\frac{dE}{dx}\right]_1-\left[\frac{dE}{dx}\right]_2$$
$$=\frac{Z^2 N_D}{4\pi v_p^2}\left[\ln\frac{4\pi v_p(v_p^2+2)}{|Z|}+\frac{3}{8}\frac{Z}{v_p^3}\left[\ln\frac{4\pi v_p(v_p^2+2)}{|Z|}-2.4\right]\right]\ . \tag{56'}$$

This is the final result for large v_p and small ω_p, which later will be compared with the stopping power for small v_p. The effective parameter for linearizability in Eq. (56′) is $Z/v_p^3=Z_{\rm eff}/(N_D v_p^3)$.

B. Nonlinear Vlasov-Poisson theory for small projectile velocities v_p

The theory of the Barkas effect described above does not apply to small projectile velocities and cannot describe the stopping power at the end of the range, where the effects of nonlinearity are expected to be largest. In this section we develop a different approach which is valid for all velocities and is particularly useful for the derivation of nonlinear corrections in the regime $v_p \ll v_{\rm th}$. An important point is that the modifications of the linearized theory of dE/dx can be expressed in terms of a form factor $\mathcal{F}(\mathbf{k})$ under the k integral of Eq. (18′). It can be directly evaluated in the case of negative projectile ions (Sec. III C). For positive projectile ions (Sec. III E) there exists the additional problem of how to treat the "trapped" electrons with negative energy comoving with the projectile in its potential trough. In both cases the friction coefficient may be computed numerically without further perturbative assumptions. In the following, however, we restrict ourselves to an analytic calculation of the lowest nonlinear order in the energy loss.

Starting from Eq. (17) we express the stopping power in terms of the induced potential in the rest frame of the projectile ion

$$-\left[\frac{dE}{dx}\right]=-Z_{\rm eff}e^2\frac{\partial}{\partial x}\int d^3r'\frac{n(\mathbf{r}')-n_0}{|\mathbf{r}'-\mathbf{r}|}\Bigg|_{\mathbf{r}=0}$$
$$=-Z_{\rm eff}e^2\frac{1}{v_p}\int d^3r'\frac{\mathbf{r}'\cdot\mathbf{v}_p}{(r')^3}[n(\mathbf{r}')-n_0]\ , \tag{57}$$

where $\Phi_{\rm ind}$ is written as solution of the Poisson equation. We introduce the linearized density in the integrand

$$-\left[\frac{dE}{dx}\right]=-Z_{\rm eff}e^2\frac{1}{v_p}\int d^3r'\frac{\mathbf{r}'\cdot\mathbf{v}_p}{(r')^3}\times[n_1(\mathbf{r}')-n_0]\frac{n(\mathbf{r}')-n_0}{n_1(\mathbf{r}')-n_0}\ ,$$
$$[n_1(\mathbf{r}')-n_0]=\frac{Z_{\rm eff}}{(2\pi)^3}\int d^3k\ e^{i\mathbf{k}\cdot\mathbf{r}'}\left[1-\frac{1}{\epsilon(\mathbf{k},\mathbf{k}\cdot\mathbf{v}_p)}\right]\ . \tag{58}$$

Interchanging integrals this reads in the dimensionless variables of Eq. (8)

$$-\left[\frac{dE}{dx}\right]=\frac{Z^2N_D}{(2\pi)^3}\int\frac{d^3k}{k^2}\frac{i\mathbf{k}\cdot\mathbf{v}_p}{v_p}\left[\frac{1}{\epsilon(\mathbf{k},\mathbf{k}\cdot\mathbf{v}_p)}-1\right]\mathcal{F}(\mathbf{k})\ ,$$
$$\mathcal{F}(\mathbf{k})=\frac{k^2}{4\pi i\mathbf{k}\cdot\mathbf{v}_p}\int d^3r\frac{\mathbf{r}\cdot\mathbf{v}_p}{r^3}e^{i\mathbf{k}\cdot\mathbf{r}}\frac{n(\mathbf{r})-1}{n_1(\mathbf{r})-1}\ . \tag{59'}$$

This representation is a new important element in treating the nonlinear terms. The form factor $\mathcal{F}(\mathbf{k})$ is a measure for the nonlinearity of the problem. In the case $\mathcal{F}(\mathbf{k})\equiv 1$ the linear result Eq. (18′) is recovered.

The form factor $\mathcal{F}(\mathbf{k})$ contains the nonlinear electron density $n(\mathbf{r})$ that we do not know. However, for small projectile velocities $v_p \ll v_{\mathrm{th}}$ the angular integration in Eq. (62′) may be performed assuming the potential of the beam ion to be approximately spherically symmetric[34]

$$-\left[\frac{dE}{dx}\right]=\frac{Z^2N_D}{(2\pi)^2}\int_0^{k_{\max}}dk\,k^3\int_{-1}^{+1}d\mu\frac{\mu Y(\mu v_p)}{[k^2+X(\mu v_p)]^2+Y^2(\mu v_p)}\mathcal{F}(k)\,, \tag{60′}$$

$$\mathcal{F}(k)=k\int_0^\infty dr\frac{n(r)-1}{n_1(r)-1}j_1(kr)$$

with the spherical Bessel function $j_1(x)=(\sin x - x\cos x)/x^2$.

In order to calculate the terms of dE/dx linear in v_p, it is sufficient to know $\mathcal{F}(k,v_p=0)$ for the static case, and the so-called friction coefficient $\mathcal{R}_1$ is obtained with $K=8\pi/|Z|$ [cf. Eq. (29′)] from

$$\begin{aligned}-\left[\frac{dE}{dx}\right]&=\mathcal{R}_1 v_p\,,\\ \mathcal{R}_1&=\frac{Z^2N_D}{6\pi\sqrt{2\pi}}\int_0^K dk\frac{k^3}{(k^2+1)^2}\mathcal{F}(k)\,.\end{aligned} \tag{61′}$$

C. Nonlinear friction coefficient for negative projectile ions

The calculation of the nonlinear friction coefficient is simpler for negative ions than for positive ions, because a repulsive potential has no bound or trapped electrons at small velocities. For negative ions ($Z<0$) with $v_p=0$ the solution of the Vlasov-Poisson equations reads

$$f(\mathbf{r},\mathbf{v})=\frac{1}{[(2\pi)^{1/2}]^3}e^{-E}=\frac{1}{[(2\pi)^{1/2}]^3}e^{-[(v^2/2)-\Phi(\mathbf{r})]}\,. \tag{62′}$$

E is the total energy of an electron, which is a constant of motion. The special solution Eq. (62′) reduces to the Maxwell distribution for $|\mathbf{r}|\to\infty$. It is normalized such that the electron density assumes the unperturbed value at infinity. It is

$$n(\mathbf{r})=\int d^3v\,f(\mathbf{r},\mathbf{v})=e^{\Phi(\mathbf{r})}\,. \tag{63′}$$

Hence the differential equation

$$\frac{1}{r}\frac{d^2}{dr^2}[r\Phi(r)]=-Z\delta(\mathbf{r})+e^{\Phi(r)}-1 \tag{64′}$$

has to be solved. This is equivalent to the integral equation

$$\Phi(r)=\frac{Z}{4\pi r}+\int_0^\infty dr'(1-e^{\Phi(r')})\frac{(r')^2}{r_>}\,, \tag{65′}$$

where $r_>=\max(r,r')$. It may be solved numerically. In the following we will restrict ourselves to a perturbative expansion of Eq. (64′), which yields an analytic solution in lowest order of Z. For $r\neq 0$ it is

$$\frac{1}{r}\frac{d^2}{dr^2}(r\Phi)=e^{\Phi}-1=\Phi+\tfrac{1}{2}\Phi^2+\tfrac{1}{6}\Phi^3+\tfrac{1}{24}\Phi^4+\cdots\,. \tag{66′}$$

The expansion $\Phi=\sum_{\nu=1}^{\infty}\Phi_\nu$ with $\Phi_\nu\propto Z^\nu$ yields

$$\begin{aligned}&\frac{1}{r}\frac{d^2}{dr^2}(r\Phi_1)=\Phi_1, \quad \text{first order}\\ &\frac{1}{r}\frac{d^2}{dr^2}(r\Phi_2)=\Phi_2+\tfrac{1}{2}\Phi_1^2, \quad \text{second order}\\ &\frac{1}{r}\frac{d^2}{dr^2}(r\Phi_3)=\Phi_3+\Phi_1\Phi_2+\tfrac{1}{6}\Phi_1^3, \quad \text{third order}\,.\end{aligned} \tag{67′}$$

The first order gives the usual Debye-Hückel potential

$$\Phi_1(\mathbf{r})=\frac{Z}{4\pi}\frac{e^{-r}}{r}\,. \tag{68′}$$

After the substitution $y=(16\pi^2/Z^2)r\Phi_2$, the second order reduces to the linear differential equation with constant coefficients $y''=y+e^{-2r}/(2r)$. The general solution is

$$\begin{aligned}y=\alpha_1e^{-r}+\alpha_2e^{r}+\alpha_3[&-e^{r}E_1(3r_0)+e^{r}E_1(3r)\\ &+e^{-r}E_1(r_0)-e^{-r}E_1(r)]\end{aligned} \tag{69′}$$

with constants α_1, α_2, α_3, and r_0; $E_1(x)=\int_x^\infty dt\,e^{-t}/t$ is the exponential integral.[25] The upper line of Eq. (69′) is the homogeneous solution, the lower line a particular solution of the inhomogeneous differential equation. The physically meaningful solution with correct behavior at the boundaries reads

$$\begin{aligned}\Phi(r)&=\Phi_1(r)+\Phi_2(r)+\cdots\,,\\ \Phi_1(r)&=+\frac{Z}{4\pi r}e^{-r}\,,\\ \Phi_2(r)&=-\frac{Z^2}{4^3\pi^2 r}[e^{r}E_1(3r)-e^{-r}E_1(r)]-\frac{\ln 3}{4^3\pi^2}\frac{Z^2}{r}e^{-r}\,.\end{aligned} \tag{70′}$$

In the repulsive case ($Z<0$) the nonlinear potential Φ is slightly deeper than in the linearized approximation and the electron density in the vicinity of the negative ion decreases further; $\Phi_2<0$ can also be directly deduced from Eq. (65′). For the calculation of $\mathcal{F}(k)$ from Eq. (60′) we need $(n_2-1)/(n_1-1)$, with $n_1=\exp(\Phi_1)$ and $n_2=\exp(\Phi_1+\Phi_2)$. In the lowest order of Z it is

$$\frac{n_2(r)-1}{n_1(r)-1} \simeq 1+\frac{\Phi_2(r)}{\Phi_1(r)}$$
$$=1-\frac{Z}{16\pi}[e^{2r}E_1(3r)-E_1(r)+\ln 3] \ . \qquad (71')$$

This yields[33]

$$\mathcal{F}(k)=1-\frac{Z}{8\pi k}\left[\ln 3 \arctan\frac{k}{2}+\frac{1}{2}\ln(4+k^2)\arctan k-\int_0^k d\xi\frac{\ln(1+\xi^2)}{4+\xi^2}-\frac{1}{2}\int_0^k d\xi\frac{\ln(4+\xi^2)}{1+\xi^2}\right] . \qquad (72')$$

In good approximation (exact for $k\to 0$ and $k\to\infty$) it is

$$\mathcal{F}(k)\simeq 1-\frac{Z}{16\pi k}\left[2\ln 3 \arctan\frac{k}{2}+\ln\left[1+\frac{k^2}{4}\right]\arctan k\right] . \qquad (73')$$

We insert this into Eq. (61′) and find

$$\mathcal{R}_1=\frac{Z^2N_D}{6\pi\sqrt{2\pi}}\int_0^K dk\frac{k^3}{(k^2+1)^2}\left\{1-\frac{Z}{16\pi k}\left[2\ln 3\arctan\frac{k}{2}+\ln\left[1+\frac{k^2}{4}\right]\arctan k\right]\right\} \qquad (74')$$

with $K=8\pi/|Z|$, cf. Eq. (29′). The integrand in Eq. (74′) was evaluated numerically and is plotted in Fig. 8. Analytically the friction coefficient may be approximated as

$$\mathcal{R}_1\simeq\frac{Z^2N_D}{6\pi\sqrt{2\pi}}\left[\ln(1+K^2)^{1/2}-\frac{1}{2}\frac{K^2}{K^2+1}-Z\frac{\ln 3}{16\pi}\arctan\left[\alpha\frac{K}{2}\right]\left[\arctan K-\frac{K}{K^2+1}\right]\right] , \qquad (75')$$

where $\alpha=0.76$ is an adjusted parameter. For the case $|Z|\ll 8\pi$ this simplifies further to become

$$\mathcal{R}_1\simeq\frac{Z^2N_D}{6\pi\sqrt{2\pi}}\left[\ln K+|Z|\frac{\ln 3}{16\pi}\left[\frac{\pi}{2}\right]^2\right] \qquad (76')$$

and is now independent of α.

For negative ions ($Z<0$) with small v_p the stopping power rises due to the nonliner effects. For $Z=-1$ the increase is 2%, for $Z=-10$ about 25%. Figure 8 also shows the evaluation of Eq. (75′).

It should be noted that this increase is opposite to the behavior at high velocity; the stopping power of fast negative ions decreases due to nonlinear effects as we found within ARB theory.

We can check the sign of the Z^3 term for $Z<0$ without having to rely on the approximations made in the derivation of Eq. (76′). Since the function $y=(16\pi^2/Z^2)r\Phi_2$ solves the equation $y''=y+e^{-2r}/(2r)$ both y and Φ_2 have to be negatively definite: y would be a concave function, cf. Fig. 9(a), and could not satisfy the boundary conditions $y(0)=y(\infty)=0$ provided it is positive in a certain interval $[r_1,r_2]$. Hence, $\Phi_2(r)<0$ for all r in the case of negatively charged ions. This yields $\Phi_1+\Phi_2<\Phi_1<0$ as illustrated in Fig. 9(b), and because of $n=\exp(\Phi)$ it is $n_2<n_1<1$, with $n_1=\exp(\Phi_1)$ and $n_2=\exp(\Phi_1+\Phi_2)$. Due to the monotonous behavior of the densities as functions of r it follows

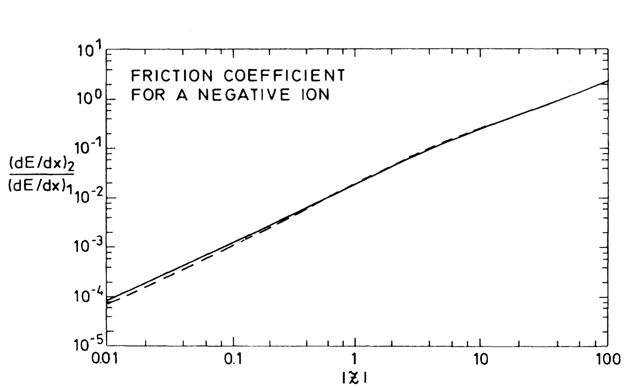

FIG. 8. The ratio of the second-order (proportional to Z^3) and the first-order (proportional to Z^2) terms of the stopping power as a function of Z for negative ions ($Z<0$). Solid line, numerical evaluation of the Z^3 term of Eq. (74′); dashed line, the analytical approximation according to Eq. (75′).

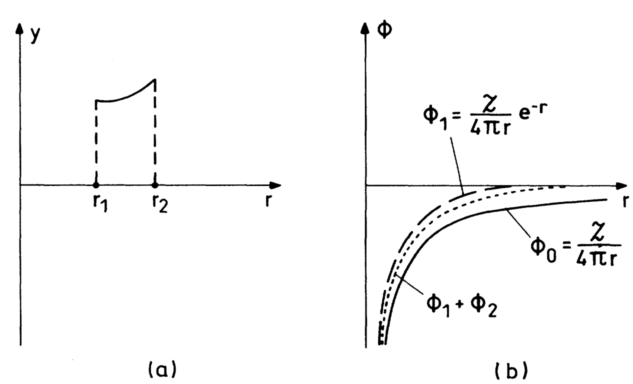

FIG. 9. The illustration concerning the sign of the Z^3 term in the stopping power. (a) The concave section of the function $y(r)$; (b) the electrostatic potential Φ vs radius; different orders of the expansion in Z are shown.

$$Z>0 \Longrightarrow \frac{n_2-1}{n_1-1}>1 \Longrightarrow \left|\frac{dE}{dx}\right|_{\rm nonlin} > \left|\frac{dE}{dx}\right|_{\rm lin} . \quad (77')$$

For $Z>0$ similar considerations would yield the opposite result, namely, a reduction of the stopping power. However, because of the captured electrons in an attractive potential positive ions need a modified treatment outlined in the next section.

D. Nonlinear energy loss of positive ions with small velocities

For a derivation of the friction coefficient for positive ions the question of how to treat trapped electrons must be addressed. In the case of a repulsive potential ($\Phi<0$) the total energy $E=\frac{1}{2}v^2-\Phi(r)$ in the exponent of the distribution function Eq. (62') [$f \propto \exp(-E)$] is always positive. However, in the attractive case all electrons with $v<[2\Phi(r)]^{1/2}$ have a negative total energy, i.e., they are trapped in the (almost) spherically symmetric potential of the slow positive ion. Within the collisionless Vlasov theory there is no exchange between the trapped electrons comoving with the ion and the free plasma electrons—at least not for the stationary problem considered here. It is therefore our standpoint in this paper to exclude the trapped electrons from the plasma calculation and to count them as bound electrons which contribute to the effective ion charge $Z_{\rm eff}$. The calculation of $Z_{\rm eff}$ taking into account the exchange of $E>0$ and $E<0$ electrons due to atomic collision and radiation processes is presented in paper II. This way of handling the trapping problem is not very satisfactory, in particular, for fast ions, which excite a potential wave train trailing the ion so that trapping may occur far away from the ion, but a more detailed treatment is beyond the scope of this paper.

The distribution function of plasma electrons in the case of an attractive potential [instead of Eq. (62')] is therefore given by

$$f(\mathbf{r},\mathbf{v})=\begin{cases} \dfrac{1}{[(2\pi)^{1/2}]^3}e^{-E}=\dfrac{1}{[(2\pi)^{1/2}]^3}e^{-[(v^2/2)-\Phi(\mathbf{r})]} \\ \qquad \text{for } |\mathbf{v}|\ge\sqrt{2\Phi(r)} , \\ 0 \quad \text{for } |\mathbf{v}|<\sqrt{2\Phi(r)} . \end{cases} \quad (78')$$

Only electrons with $E\ge 0$ are treated by the Vlasov theory in this way, whereas electrons with $E<0$ are left for an atomic physics description, i.e., the calculation of $Z_{\rm eff}$. It is

$$n(\mathbf{r})=\int d^3v\, f(\mathbf{r},\mathbf{v})$$
$$=e^{\Phi(r)}-e^{\Phi(r)}\mathrm{erf}[\sqrt{\Phi(r)}]+\frac{2}{\sqrt{\pi}}\sqrt{\Phi(r)} . \quad (79')$$

For slow positive ions ($Z>0$) we have to solve the differential equation

$$\frac{1}{r}\frac{d^2}{dr^2}[r\Phi(r)]=-Z\,\delta(\mathbf{r})+e^{\Phi(r)}-e^{\Phi(r)}\mathrm{erf}[\sqrt{\Phi(r)}] + \frac{2}{\sqrt{\pi}}\sqrt{\Phi(r)}-1 \quad (80')$$

or, equivalently, the integral equation

$$\Phi(r)=\frac{Z}{4\pi r}+\int_0^\infty dr' \left[1-e^{\Phi(r')}+e^{\Phi(r)}\mathrm{erf}[\sqrt{\Phi(r)}] - \frac{2}{\sqrt{\pi}}\sqrt{\Phi(r)}\right]\frac{(r')^2}{r_>} \quad (81')$$

with $r_>=\max(r,r')$ analogously to Eqs. (64') and (65') for the repulsive potential. Solving

$$\frac{1}{r}\frac{d^2}{dr^2}[r\Phi(r)]=\Phi(r)+\frac{4}{3\sqrt{\pi}}\Phi^{3/2}(r)-\tfrac{1}{2}\Phi^2(r)+\cdots \quad (82')$$

by the expansion $\Phi=\sum_{\nu=1}^{\infty}\Phi_\nu$ with $\Phi_\nu \propto Z^\nu$ one obtains

$$\frac{1}{r}\frac{d^2}{dr^2}(r\Phi_1)=\Phi_1, \quad \text{first order}$$
$$\frac{1}{r}\frac{d^2}{dr^2}(r\Phi_2)=\Phi_2-\frac{4}{3\sqrt{\pi}}\Phi_1^{3/2}, \quad \text{second order} \quad (83')$$

The equation in first order is identical with Eq. (70') for negative ions and gives the Debye-Hückel potential $\Phi_1(\mathbf{r})=Z/(4\pi r)\exp(-r)$. With the substitution $w=(6\pi^2/Z^{3/2})r\Phi_2$, the equation of second order becomes $w''=w-e^{-3r/2}/r^{3/2}$ with the general solution

$$w=\alpha_1 e^{-r}+\alpha_2 e^{r}+\alpha_3\left[\sqrt{5}e^{-r}\left\{1-\mathrm{erf}\left[\left[\frac{r}{2}\right]^{1/2}\right]\right\}-e^{r}\left\{1-\mathrm{erf}\left[\left[\frac{5r}{2}\right]^{1/2}\right]\right\}\right] . \quad (84')$$

Adjusting the constants $\alpha_1,\alpha_2,\alpha_3$ to the boundary conditions $w(0)=w(\infty)=0$, one obtains the physical solution

$$\Phi(r)=\Phi_1(r)+\Phi_2(r)+\cdots ,$$
$$\Phi_1(r)=\frac{Z}{4\pi r}e^{-r} , \quad (85')$$
$$\Phi_2(r)=\frac{Z^{3/2}}{r}\frac{1}{6\pi\sqrt{10\pi}}\left[e^{r}\left\{1-\mathrm{erf}\left[\left[\frac{5r}{2}\right]^{1/2}\right]\right\}-\sqrt{5}e^{-r}\left\{1-\mathrm{erf}\left[\left[\frac{r}{2}\right]^{1/2}\right]\right\}+(\sqrt{5}-1)e^{-r}\right] .$$

The nonlinear potential Φ in the attractive case ($Z>0$) is slightly larger than the linearized approximation. Using the same arguments following Eq. (71′), it is $(n_2-1)/(n_1-1)\simeq 1+\Phi_2/\Phi_1$. An integration by parts with $\int dx\, j_1(x)=-j_0(x)$ yields

$$\mathcal{F}(k)=1+\sqrt{Z}\frac{4}{3\sqrt{10\pi}}\frac{1}{k}\int_0^\infty dr\, e^{2r}\left\{1-\operatorname{erf}\left[\left[\frac{5r}{2}\right]^{1/2}\right]\right\}\frac{\sin(kr)}{r}\ . \tag{86′}$$

The exact evaluation of the remaining integral[33] gives

$$\mathcal{F}(k)=1+\sqrt{Z}\frac{4}{3\sqrt{10\pi}}\frac{1}{k}\arctan[f(k)]$$

with

$$f(k)=\frac{\frac{8}{\sqrt{10}}[(1+4k^2)^{1/2}-1]^{1/2}-2k+\frac{2}{5}k(1+4k^2)^{1/2}}{4-\frac{4}{5}(1+4k^2)^{1/2}+\frac{4}{\sqrt{10}}k[(1+4k^2)^{1/2}-1]^{1/2}}\ . \tag{87′}$$

The friction coefficient of Eq. (61′) reads

$$\mathcal{R}_1=\frac{Z^2N_D}{6\pi\sqrt{2\pi}}\left[\ln\sqrt{1+K^2}-\frac{1}{2}\frac{K^2}{K^2+1}+\sqrt{Z}\frac{4}{3\sqrt{10\pi}}\int_0^K dk\frac{k^2}{(k^2+1)^2}\arctan[f(k)]\right]\ , \tag{88′}$$

where $K=8\pi/|Z|$. This result was evaluated numerically and is shown in Fig. 10. An analytic approximation is motivated by the asymptotic form $f\to k/1.62$ for $k\to 0$. The approximation

$$\mathcal{R}_1\simeq\frac{Z^2N_D}{6\pi\sqrt{2\pi}}\left[\ln(1+K^2)^{1/2}-\frac{1}{2}\frac{K^2}{K^2+1}+\sqrt{Z}\frac{2}{3\sqrt{10\pi}}\arctan\left[\alpha\frac{K}{1.62+K}\right]\times\left[\arctan K-\frac{K}{K^2+1}\right]\right] \tag{89′}$$

with the adjusted parameter $\alpha=0.87$ is also plotted in Fig. 10. In the case $Z\ll 8\pi$, Eq. (92′) simplifies to

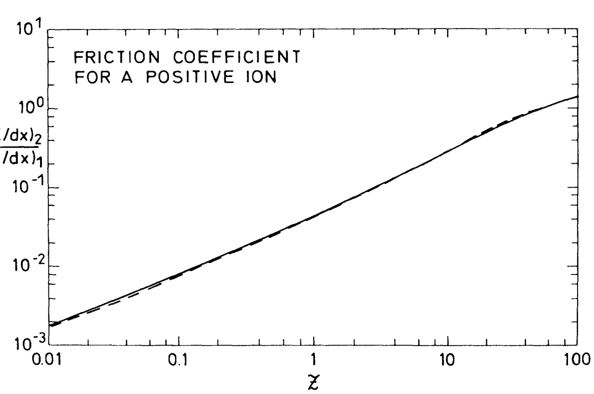

FIG. 10. The ratio of stopping powers $(dE/dx)_2$ and $(dE/dx)_1$ as a function of Z for positive ions ($Z>0$). Solid line, numerical evaluation of the nonlinear $Z^{5/2}$ term [Eq. (88′)]; dashed line, analytical approximation according to Eq. (89′).

$$\mathcal{R}_1\to\frac{Z^2N_D}{6\pi\sqrt{2\pi}}\left[\ln K+\sqrt{Z}\frac{\pi}{3\sqrt{10\pi}}\arctan 0.87\right]\quad\text{for } Z\ll 8\pi\ . \tag{90′}$$

As a consequence of the restriction of the total energy E in Eq. (81′) to positive values the stopping power of slow positive ions is increased by the nonlinear effects. The nonlinear terms in the friction coefficients of positive and negative ions have therefore the same sign, and they are also of the same magnitude. However, the Z scaling is different; the nonlinear term is proportional to Z^3 for negative ions, but proportional to $Z^{5/2}$ for positive ions. The half-integer exponent is typical for analytic results of the nonlinear Vlasov theory, as it was first shown by Bernstein, Greene, and Kruskal[35] in their fundamental work on nonlinear plasma waves (BGK waves). The friction coefficient for positive ions increases in lowest order of the nonlinear effects by almost 4% for $Z=+1$ and by roughly 28% for $Z=+10$ as shown in Fig. 10.

E. Generalization for finite projectile velocities $v_p \lesssim v_{\rm th}$

The relations derived in Secs. III C and III D are only valid to lowest order in v_p. At higher velocities the shielding of the projectile charge is reduced and the approximation of spherical symmetry for the potential will fail. Qualitatively this an be taken into consideration by interpreting $k_D W(\mu v_p/\sqrt{k_BT/m}\,)$ as an effective Debye wave number for a projectile with velocity v_p, see Eq. (10). For not too large velocities it is

$$|W|\simeq|1-\mu^2mv_p^2/k_BT+i(\pi/2)^{1/2}\mu v_p\sqrt{m/k_BT}\,|$$
$$\simeq 1-(1-\pi/4)\mu^2mv_p^2/k_BT\ .$$

Averaging over μ yields in dimensionless quantities

$$\sqrt{\langle|W|\rangle}\simeq\frac{1}{1+\frac{1}{24}(4-\pi)v_p^2}\ . \tag{91′}$$

By virtue of the substitutions $k \Longrightarrow k/\langle |W| \rangle^{1/2}$ and $\mathcal{Z} \Longrightarrow \mathcal{Z}/\langle |W| \rangle^{3/2}$ the relations of the preceding sections may be generalized for the case of finite projectile velocities. Of course this procedure becomes very crude for $v_p \gtrsim v_{\rm th}$ since deviations from spherical symmetry are then essential. Starting out with the k integration in Eq. (60′) and again using the approximations adopted in the derivation of Eqs. (75′) and (89′), we can repeat the calculations for positive (+) and negative (−) ions and find

$$-\left[\frac{dE}{dx}\right]_{\pm} \simeq -\left[\frac{dE}{dx}\right]_1 + S_{\pm}\int_{-1}^{1} d\mu\,\mu Y\left[\frac{1}{4p}\ln\frac{k_{\max}^2-pk_{\max}+q}{k_{\max}^2-pk_{\max}-q}+\frac{1}{2r}\arctan\frac{2k_{\max}-p}{r}+\frac{1}{2r}\arctan\frac{2k_{\max}+p}{r}\right], \quad (92')$$

where

$$S_+ = \frac{4}{3\sqrt{10\pi}}\sqrt{\mathcal{Z}}\langle |W| \rangle^{5/4}\arctan\left[\frac{1.1k_{\max}/\sqrt{\langle |W| \rangle}}{1+\sqrt{5k_{\max}/(\langle |W| \rangle)^{1/2}}}\right] \quad \text{for } \mathcal{Z}>0\,,$$

$$S_- = \frac{2\ln 3}{16\pi}|\mathcal{Z}|\langle |W| \rangle^{2}\arctan\left[0.76\frac{k_{\max}}{2\sqrt{\langle |W| \rangle}}\right] \quad \text{for } \mathcal{Z}<0\,;$$

$(dE/dx)_1$ is the stopping power given in Eq. (21′), and $q^2=X^2+Y^2$, $p^2=2(q-X)$ and $r^2=2(q+X)$. These relations will be used for the following examples.

F. Two examples for nonlinear energy loss of heavy ions in plasma

First we investigate a case that is important for heavy-ion-induced inertial fusion. We show in the subsequent paper II that the effective charge $Z_{\rm eff}$ of fast Bi ions when penetrating a fully ionized Li plasma will be considerably larger than the equilibrium charge $Z_{\rm eq}$ up to the very end of the stopping range. Because of this high nonequilibrium charge and consequently a large coupling parameter $\mathcal{Z}$ [Eq. (9)], the nonlinear stopping effects will be strong. We discuss the case of Bi ions with initial energy 30 MeV/u ($v_p=35\alpha c$) in a fully ionized Li plasma with $k_BT=300$ eV and $\frac{1}{5}$ solid-state density ($n_0=3\times10^{22}$ cm^{-3}). At the end of the range the effective charge of the Bi ions is of order 40. Figure 11 shows the result obtained from Eq. (92′) with $Z_{\rm eff}=40$. In this example the number of electrons in the Debye sphere is $\mathcal{N}_D=51.5$ and the coupling parameter is $\mathcal{Z}=Z_{\rm eff}/n_0\lambda_D^3=3.25$. The nonlinear corrections of the energy loss are most prominent for velocities $v_p \lesssim 2v_{\rm th}$, but do not exceed 10% as seen in Fig. 11.

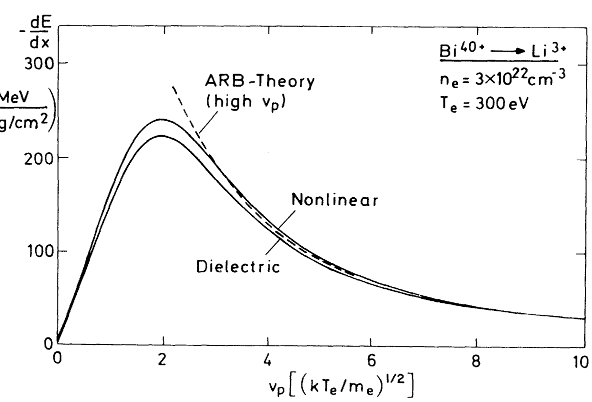

FIG. 11. The nonlinear calculation [in lowest order, Eq. (95′)] for the stopping power of ions with constant charge $Z_{\rm eff}=40$ in a plasma of temperature 300 eV and electron density 3×10^{22} cm^{-3}. Lower solid line: linearized Vlasov theory. Upper solid line: nonlinear stopping power. The enhancement of the energy loss due to nonlinear effects is at the most 10%. Dashed line: the result Eq. (59′) of the theory of Ashley, Ritchie, and Brandt (1972) for high projectile velocities.

At higher projectile velocities the coupling parameter is $\mathcal{Z}/v_p^3$ according to Eq. (56′), and the nonlinear contribution decreases rapidly. The comparison with the theory of Ashley, Ritchie, and Brandt (dashed line) shows that the description of dE/dx presented in the preceding section overestimates the nonlinear effects at velocities $v_p>2v_{\rm th}$. The reason is that the assumption of spherical symmetry of the potential breaks down at high velocities.

As a second example we discuss the question of nonlinear effects when cooling heavy-ion beams in accelerators by means of electrons moving collinearly to the ions with the same velocity as the ion beam (so called electron cooling). For instance, the new accelerator SIS/ESR at GSI, Darmstadt[5] is equipped with such an electron cooling line. In this cooling line the electron beam is initially very cold, i.e., the motion of the electrons relative to each other is small. Energy exchange between the electrons and the accelerator beam ions, e.g., U^{92+}, tends to equilibrate the temperatures and to cool the ion beam. This cooling is governed by the stopping power of the electrons. The electron plasma in the cooling line envisaged at GSI is on the borderline between ideal and nonideal plasmas ($\mathcal{N}_D \lesssim 1$), and the charge of the beam ions is large ($Z_{\rm eff}/N_D \gg 1$). The typical density of the electron beam is $n_0=10^8$ cm^{-3}, and typical temperatures are $k_BT_e \approx 10^{-3}-10^{-4}$ eV so that $\mathcal{N}_D \approx 5.4-0.17$. In the following we investigate the case of $k_BT_e=10^{-3}$ eV in which the plasma then is still ideal. For U^{92+} ions the coupling parameter is $\mathcal{Z}=71$, thus the problem of ion-plasma interaction is highly nonlinear.

Before discussing the result in Fig. 12 let us express

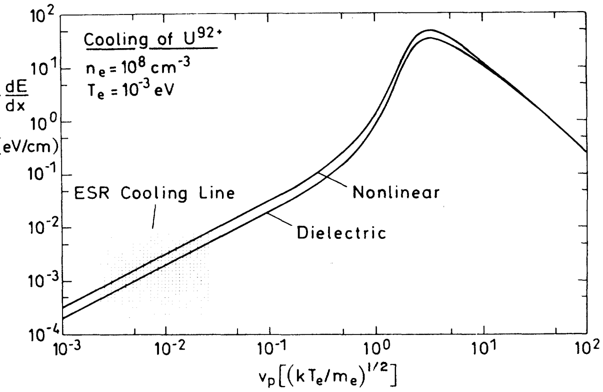

FIG. 12. Nonlinear calculation of the energy loss of U^{92+} ions in the plasma of an electron cooling line with temperature 10^{-3} eV and electron density 10^8 cm^{-3}. Lower line: the linearized Vlasov theory. Upper line: the nonlinear theory in lowest order, Eq. (95′). The friction coefficient is enhanced by roughly 70% due to nonlinear effects (see text concerning validity).

some points of caution. There are important differences between the electron-cooling beam and a classical plasma. The electrons of the cooling line are not held together by a charge-neutralizing background of plasma ions but by a strong magnetic field in the direction of the beam. The magnetic field forces the electrons into the helical motion along the field lines. Also, because of the strong nonlinearity of the problem it is questionable whether the restriction to second order [compare Eq. (85′)] makes sense. Taking the results of the preceding sections nevertheless as a rough estimate for the effect of nonlinear plasma response we obtain the cooling force dE/dx acting on the ion beam as shown in Fig. 12. The temperature of the electron beam is taken as $k_B T = 10^{-3}$ eV and the average velocity v_p of the U^{92+} ions relative to the electron beam is obtained from $m_i v_p^2/2 = k_B T_{i\parallel}$, where $T_{i\parallel}$ is the ion-beam temperature parallel to the magnetic field. The dotted area in Fig. 12 indicates the region of operation of the cooling line at GSI.[36] The solid lines give the cooling force according to the present theory. It is seen that nonlinear effects increase the cooling force and, correspondingly, reduce the cooling time by about 70%. In previous papers[36,37] the cooling force was derived from the linearized dielectric theory.

IV. SUMMARY

The theory of heavy-ion stopping power in plasma is developed on the basis of the Vlasov-Poisson equations. The basic coupling parameter of the problem is found to be $\mathcal{Z} = Z_{\text{eff}}/n_0\lambda_D^3$; the physics determining the effective charge Z_{eff} of the heavy-ion projectile is treated in the subsequent paper, and $Z_{\text{eff}}=\text{const}$ is assumed in the present paper. The stopping power dE/dx of *fast* ions ($v_p \gg v_{\text{th}}$) is well described by the Bethe-Bohr formula Eqs. (1) and (2) and appears naturally as a limiting case in our treatment. The focus of the present paper, however, is on *slow* ions ($v_p \lesssim v_{\text{th}}$) and on nonlinear effects in regions with $\mathcal{Z} > 1$, which actually occur for heavy-ion stopping in dense target plasma close to the end of the range. The results are of fundamental as well as practical interest.

First, the linearized Vlasov theory for dE/dx is reviewed in Sec. II, and the low-velocity terms are derived with emphasis on their density dependence. An approximate expression is given in Eq. (35) which represents the exact linearized result for dE/dx in plasma very accurately for high *and* low projectile velocity. It should be used in future stopping-power calculations instead of the Chandrasekhar expression, which is now commonly used, but gives a poor approximation for $v_p \lesssim v_{\text{th}}$.

The new starting point for treating nonlinear terms is Eq. (59′), which represents the nonlinearities in terms of a form factor $\mathcal{F}(\mathbf{k})$. At low velocities they contribute to dE/dx with terms proportional to $\mathcal{Z}^3$ for negative projectiles and with terms proportional to $\mathcal{Z}^{5/2}$ for positive ones in addition to the $\mathcal{Z}^2$ term of the linearized theory. It is found that, for $v_p \lesssim v_{\text{th}}$, the nonlinear contributions always enhance the stopping power, up to 10% at the maximum of dE/dx near $v_p \approx v_{\text{th}}$ in cases relevant for heavy-ion stopping in plasmas close to solid-state density, as illustrated in Fig. 11. For velocities $v_p \gg v_{\text{th}}$, the nonlinear dE/dx treatment of Ashley, Ritchie, and Brandt is extended to the plasma case, and the explicit result is given by Eq. (56′).

In conclusion, a unified description for the stopping of high- and low-velocity ions in plasma is obtained and the size of nonlinear corrections is given in the various parameter regions.

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[34]Note that this approximation becomes exact in the limit $v_p \to 0$, thus is a rigorous description of the nonlinear friction coefficient. Another remarkable feature of the nonlinear description is that the expression Eq. (59′) for dE/dx does not diverge for $k_{\max} \to \infty$. In this limit

$$\lim_{k\to\infty} \mathcal{F}(\mathbf{k}) \simeq \int_0^\infty d\xi \frac{n(\xi/k)-1}{n_1(\xi/k)-1} j_1(\xi) \simeq \frac{n(0)-1}{n_1(0)-1} ,$$

where $n(0), n_1(0)$ are the full and linearized electron density at the location of the ion, respectively. While the linearized density diverges for $r \to 0$ $[n_1(r)-1 \simeq Z/(4\pi r)e^{-r}]$, the nonlinear density remains finite (for negative ions) or diverges less rapidly (for positive ions). Thus, $\mathcal{F}(\mathbf{k})$ drops quickly to 0 for $|\mathbf{k}| \to \infty$ and a cutoff is not necessary. Notwithstanding this result, the approximate analytic expressions derived in this section are expressed in terms of $k_{\max} \simeq 8\pi/|Z|$. This is a consequence of the perturbative expansion.

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