Relative diffusion and formation of clumps in phase space of electrons in localized Langmuir fields

Shao-ping Zhu and X. T. He
Center for Nonlinear Studies, Institute of Applied Physics and Computational Mathematics, P.O. Box 8009, Beijing, China
(Received 12 February 1990; revised manuscript received 4 September 1990)

A different relative diffusion coefficient for electrons interacting with coherent, localized Langmuir wave packets is proposed. It is shown that the coherent, localized wave packets can drive the formation of clumps in phase space, and only when the relative velocity is zero does the lifetime of clumps diverge logarithmically, with the relative positions of the constituents getting closer to one another.

In this paper we report that trajectories of electrons interacting with the localized, coherent Langmuir wave packets exhibit an effect of the strong two-point correlation. These wave packets are important in many aspects of laser fusion, space packets, and turbulent plasma, etc. Thus the clump phenomena in phase space, as proposed by Dupree in 1972,[1] can appear in the case of the spatially inhomogeneous and coherent fields.

Write the localized, coherent fields in the form of a soliton structure:[2]

$$E(x,t)=E_0\operatorname{sech}(gx)\cos(k_0x-\omega_p t)\ , \tag{1}$$

where $g=\beta E_0/\sqrt{8\pi n k_B T}$, β is a parameter, E_0 amplitude, k_0 Langmuir wave number, T electron temperature, n electron number density, and $\omega_p=(4\pi n e^2/m_e)^{1/2}$, where m_e is electron mass.

Extending $\operatorname{sech}(gx)$ to whole space with a periodic length L, and expressing it in terms of wave packets, the equations for electron trajectories in phase space are written in the dimensionless form as follows:

$$\begin{aligned}\frac{dx_i(t)}{dt}&=v_i(t)\ ,\\ \frac{dv_i(t)}{dt}&=-\sum_{l=-l_0}^{l_0}E_l\cos[k_l x_i(t)-t],\quad i=1,2\end{aligned} \tag{2}$$

where $k_l=k_0+(2l\pi/L)$, $l=0,\pm1,\pm2,\ldots,\pm l_0$, $k_0=\pi/L$; E_l is in units of $\sqrt{8\pi n k_B T}$, space is in units of the Debye length $\lambda_e=(T_e/4\pi n e^2)^{1/2}$, time is in units of $1/\omega_p$.

It is shown that if the overlapping criterion is satisfied,[3] the periodically trapping trajectories of the electrons in phase space may lead to stochastic processes that form a stochastic region in phase space, accompanied by the existence of small regular islands in this stochastic region.[2,4] Thus the stochastic motion may lead to strong correlations of neighboring electrons in phase space, because relative diffusion is greatly reduced, and so may tend to enable clumps to form in phase space under localized and coherent fields.

Introducing the relative and barycentric motion coordinates

$$\begin{aligned}r&=x_1-x_2,\quad R=x_1+x_2\ ,\\ u&=v_1-v_2,\quad V=v_1+v_2\ ,\end{aligned} \tag{3}$$

from Eq. (2) we can write the following equations for the above-noted coordinate system in the form

$$\begin{aligned}\frac{dr}{dt}&=u,\quad \frac{du}{dt}=\sum_n 2E_n\sin(\tfrac{1}{2}k_nR-t)\sin(\tfrac{1}{2}k_nr)\ ,\\ \frac{dR}{dt}&=V,\quad \frac{dV}{dt}=-\sum_n 2E_n\cos(\tfrac{1}{2}k_nR-t)\cos(\tfrac{1}{2}k_nr)\ .\end{aligned} \tag{4}$$

According to the definition of the diffusion coefficient, we obtain the relative diffusion coefficient

$$D_-(R,r,V,u)=\frac{1}{2t}\langle[u(t)-u(0)]^2\rangle\ , \tag{5}$$

and the absolute diffusion coefficient

$$D_{ii}(v_i,x_i)=\frac{1}{2t}\langle[v_i(t)-v_i(0)]^2\rangle\ , \tag{6}$$

where the symbol $\langle\ \rangle$ denotes an ensemble average.

Next we express Eqs. (5) and (6) in a specific form. First of all, to integrate Eqs. (2), we obtain the trajectories for electrons as follows:

$$\begin{aligned}x_i(t)&=x_i^0+v_i^0t\\ &\quad-\int_0^t dt'(t-t')\sum_n E_n\cos[k_nx_i(t')-t']\ ,\\ v_i(t)&=v_i^0-\int_0^t dt'\sum_n E_n\cos[k_nx_i(t')-t']\ .\end{aligned} \tag{7}$$

Substituting Eqs. (7) into Eq. (5), and transforming to $t_1=t'$, $t_2=t_1-t''$, Eq. (5) can be rewritten in the form

$$\begin{aligned}D_-=\frac{1}{t}\int_0^t dt_1\int_0^{t_1}dt_2\sum_{n,m}E_nE_m\langle\{&\cos[k_nx_1(t_1)-t_1]\cos[k_mx_1(t_1-t_2)-(t_1-t_2)]\\ &-\cos[k_nx_1(t_1)-t_1]\cos[k_mx_2(t_1-t_2)-(t_1-t_2)]\}+(1\leftrightarrow2)\rangle\ ,\end{aligned} \tag{8}$$

where ($1\leftrightarrow 2$) represents the preceding terms with the indices 1 and 2 interchanged.

We now express the first integrand term inside the braces in the exponent form; then, we obtain

$$\begin{aligned}\langle \cos[k_n x_1(t_1)-t_1]\cos[k_m x_1(t_1-t_2)-(t_1-t_2)]\rangle\\ =\tfrac{1}{2}(\cos\{k_0(R^0+r^0)-2[1-\tfrac{1}{2}k_0(V^0+u^0)]t_1+[1-\tfrac{1}{2}k_{-n}(V^0+u^0)]t_2\}\\ \times\delta_{n,-m}\exp\{-\tfrac{1}{2}\langle[k_n\Delta x_1(t_1)+k_{-n}\Delta x_1(t_1-t_2)]^2\rangle\}\\ +\cos\{[1-\tfrac{1}{2}k_n(V^0+u^0)]t_2\}\delta_{n,m}\ \exp\{-\tfrac{1}{2}\langle[k_n\Delta x_1(t_1)-k_n\Delta x_1(t_1-t_2)]^2\rangle\}),\end{aligned} \tag{9}$$

where the cumulant expansion approximation has been used, and $\Delta x_i(t)\equiv x_i(t)-x_i^0-v_i^0 t$.

Treating the rest of Eq. (8) in a way similar to Eq. (9), then, we obtain

$$\begin{aligned}D_-=\frac{1}{2t}\int_0^t dt_1\int_0^{t_1}dt_2\sum_{i=1}^{2}\sum_n E_n^2(\cos[\xi_1+(-1)^{i-1}\eta_1]\exp\{-\tfrac{1}{2}\langle[k_n\Delta x_i(t_1)+k_{-n}\Delta x_i(t_1-t_2)]^2\rangle\}\\ -\cos[\xi_1+(-1)^{i-1}\eta_2]\exp\{-\tfrac{1}{2}\langle[k_n\Delta x_i(t_1)+k_{-n}\Delta x_i(t_1-t_2)\\ -(-1)^{i-1}k_{-n}\Delta r(t_1-t_2)]^2\rangle\}\\ +\cos[\xi_2+(-1)^{i-1}\eta_3]\exp\{-\tfrac{1}{2}\langle[k_n\Delta x_i(t_1)-k_n\Delta x_i(t_1-t_2)]^2\rangle\}\\ -\cos[\xi_2+(-1)^{i-1}\eta_4]\exp\{-\tfrac{1}{2}\langle[k_n\Delta x_i(t_1)-k_n\Delta x_i(t_1-t_2)\\ +(-1)^{i-1}k_n\Delta r(t_1-t_2)]^2\rangle\}),\end{aligned} \tag{10}$$

where

$$\begin{aligned}&\Delta r(t)=r(t)-r^0-u^0t,\quad \xi_1=k_0R^0-2(1-\tfrac{1}{2}k_0V^0)t_1+(1-\tfrac{1}{2}k_{-n}V^0)t_2\ ,\\ &\eta_1=k_0r^0+k_0u^0t_1-\tfrac{1}{2}k_{-n}u^0t_2,\quad \eta_2=(k_0-k_{-n})r^0+(k_0-k_{-n})u^0t_1+\tfrac{1}{2}k_{-n}u^0t_2,\quad \xi_2=(1-\tfrac{1}{2}k_nV^0)t_1\ ,\\ &\eta_3=-\tfrac{1}{2}k_nu^0t_2,\quad \eta_4=-k_nr^0-k_nu^0t_1+\tfrac{1}{2}k_nu^0t_2\ .\end{aligned} \tag{11}$$

In order to calculate the cumulant expansion in Eq. (10), we assume that the trajectory stochasticity in velocity space can be described by a Wiener process.[5] Then the probability density of finding a particle at time t with velocity increment Δv_i, where $\Delta v_i(t=0)=0$, is given by

$$P(\Delta v_i,t)=\frac{1}{\sqrt{4\pi D_{ii}t}}\exp\left[-\frac{(\Delta v_i)^2}{4D_{ii}t}\right]\quad \text{for } t>0\ . \tag{12}$$

Using the Wiener average instead of an ensemble average, we have

$$\langle[k_n\Delta x_i(t_1)-k_n\Delta x_i(t_1-t_2)]^2\rangle=\tfrac{2}{3}D_{ii}k_n^2t_2^2(3t_1-2t_2)\ , \tag{13}$$

$$\begin{aligned}\langle[k_n\Delta x_i(t_1)+k_{-n}\Delta x_i(t_1-t_2)]^2\rangle=\tfrac{2}{3}D_{ii}\{k_0(4t_1^3-6t_2t_1^3+3t_2t_1)+2k_0(k_n-k_0)[t_1^3-(t_1-t_2)^3]\\ +(k_n-k_0)^2t_2^2(3t_1-2t_2)\}\ ,\end{aligned} \tag{14}$$

$$\begin{aligned}&\langle[k_n\Delta x_i(t_1)+k_{-n}\Delta x_i(t_1-t_2)-(-1)^{i-1}k_{-n}\Delta r(t_1-t_2)]^2\rangle\\ &\quad=\langle[k_n\Delta x_i(t_1)+k_{-n}\Delta x_i(t_1-t_2)]^2\rangle-2(-1)^{i-1}\langle[k_n\Delta x_i(t_1)+k_{-n}\Delta x_i(t_1-t_2)]k_{-n}\Delta r(t_1-t_2)\rangle\\ &\qquad+k_{-n}^2\langle[\Delta r(t_1-t_2)]^2\rangle\ ,\end{aligned} \tag{15}$$

$$\begin{aligned}&\langle[k_n\Delta x_i(t_1)-k_n\Delta x_i(t_1-t_2)+(-1)^{i-1}k_n\Delta r(t_1-t_2)]^2\rangle\\ &\quad=\langle[k_n\Delta x_i(t_1)-k_n\Delta x_i(t_1-t_2)]^2\rangle+2(-1)^{i-1}\langle[k_n\Delta x_i(t_1)-k_n\Delta x_i(t_1-t_2)]k_n\Delta r(t_1-t_2)\rangle+k_n^2\langle[\Delta r(t_1-t_2)]^2\rangle\ .\end{aligned} \tag{16}$$

In Eqs. (15) and (16) there exist the terms $\langle[\Delta r(t)]^2\rangle$ and $\langle\Delta x_i(t_1)\Delta r(t_2\le t_1)\rangle$. It is clear that $\langle|\Delta r(t)|^2\rangle$ is proportional to D_-, so is $\langle\Delta x_i(t)\Delta r(t_2\le t)\rangle$, but $\langle\Delta x_i(t_1)\Delta x_i(t_1-t_2)\rangle$ is proportional to D_{ii}. In our case, D_- is much less than D_{ii} when r^0 and u^0 are smaller, so the contribution of $\langle|\Delta r(t)|^2\rangle$ and $\langle\Delta x_i(t)\Delta r(t_2\le t)\rangle$ to D_- in Eq. (10) is very small. For the convenience of discussions, we calculate them under the unperturbed trajectory approximation, and obtain

$$\langle [\Delta r(t)]^2\rangle = \sum_{i=1}^{2}\sum_n E_n^2\{F(2k_0x_i^0, -(1-k_nv_i^0), -(1-k_{-n}v_i^0), t, t) + F(0, 1-k_nv_i^0, -(1-k_nv_i^0), t, t)$$
$$- F(2k_0x_i^0 - (-1)^{i-1}k_{-n}r^0, -(1-k_nv_i^0), -[1-k_{-n}v_i^0 + (-1)^{i-1}k_{-n}u^0], t, t)$$
$$+ F(-(-1)^{i-1}k_nr^0, 1-k_nv_i^0, -[1-k_nv_i^0 + (-1)^{i-1}k_nu^0], t, t)\}\ , \tag{17}$$

$$\langle \Delta x_i(t_1)\Delta r(t_2 \le t_1)\rangle = \sum_n E_n^2\{F(2k_0x_i^0, -(1-k_nv_i^0), -(1-k_{-n}v_i^0), t_1, t_2) + F(0, 1-k_nv_i^0, -(1-k_nv_i^0), t_1, t_2)$$
$$- F(2k_0x_i^0 - (-1)^{i-1}k_{-n}r^0, -(1-k_nv_i^0), -[1-k_{-n}v_i^0 + (-1)^{i-1}k_{-n}u^0], t_1, t_2)$$
$$+ F(-(-1)^{i-1}k_nr^0, 1-k_nv_i^0, -[1-k_nv_i^0 + (-1)^{i-1}k_nu^0], t_1, t_2)\}\ , \tag{18}$$

where function F is defined as follows:

$$F(a,b,c,t_1,t_2) = \frac{\cos a - \cos(a+bt_1) - \cos(a+ct_2) + \cos(a+bt_1+ct_2)}{2b^2c^2} - \frac{(bt_1+ct_2)\sin a}{2b^2c^2}$$
$$+ \frac{t_1\sin(a+ct_2)}{2bc^2} + \frac{t_2\sin(a+bt_1)}{2b^2c} - \frac{t_1t_2\cos a}{2bc}\ . \tag{19}$$

We find from Eq. (10) that D_- vanishes only for $r^0=0$, $u^0=0$; however, if u^0 is a nonzero constant, D_- is not equal to zero even if r^0 vanishes. Because $\eta_2-\eta_1=-k_{-n}[r^0+u^0(t_1-t_2)]$, $\eta_4-\eta_3=-k_n[r^0+u^0(t_1-t_2)]$ and $t_1 \ge t_2$, we are sure that D_- is of minimum when $r^0>0$, $u^0<0$ or $r^0<0$, $u^0>0$.

By using Eq. (7), the absolute diffusion coefficient in Eq. (6) can be easily rewritten in the form

$$D_{ii} = \frac{1}{2t}\int_0^t dt_1 \int_0^{t_1} dt_2 \sum_n E_n^2(\cos[2k_0x_i^0 - 2(1-k_0v_i^0)t_1$$
$$+ (1-k_{-n}v_i^0)t_2]\exp\{-\tfrac{1}{2}\langle [k_n\Delta x_i(t_1) + k_{-n}\Delta x_i(t_1-t_2)]^2\rangle\}$$
$$+ \cos[(1-k_nv_i^0)t_2]\exp\{-\tfrac{1}{2}\langle [k_n\Delta x_i(t_1) - k_n\Delta x_i(t_1-t_2)]^2\rangle\})\ . \tag{20}$$

With numerical solutions of Eq. (4), according to the definition of Eq. (5), we write the absolute and the relative diffusion coefficients to be computed in the form

$$D_{ii}^p = \frac{1}{2\tau}\langle [v_i(\tau)-v_i(0)]^2\rangle\ ,$$
$$D_-^p = \frac{1}{2\tau}\langle [u(\tau)-u(0)]^2\rangle\ , \tag{21}$$

where

$$\tau = \frac{1}{2}\left[\frac{L}{v_1} + \frac{L}{v_2}\right]$$

is the approximate time for one pass through the periodic structure. The ensemble average is computed by averaging over a set of particles with different start times but with the same initial velocity and positions.

The spectrum of the wave packet is presented in Fig. 1 for parameters $E_0^2/8\pi nk_BT=0.53$, $L=36$, and $\beta=1/\sqrt{3}$, where the overlapping conditions are satisfied except for the modes $n=0,-1$. Figures 2 and 3 show the values of D_- given by Eqs. (10), (4), and (21), where the factors $\langle [\Delta r(t)]^2\rangle$ and $\langle \Delta x_i(t_1)\Delta r(t_2 \le t_1)\rangle$ in Eq. (10) have been neglected to economize computing time. The fact that D_- and D_-^p in Figs. 2 and 3 are almost identical ob-

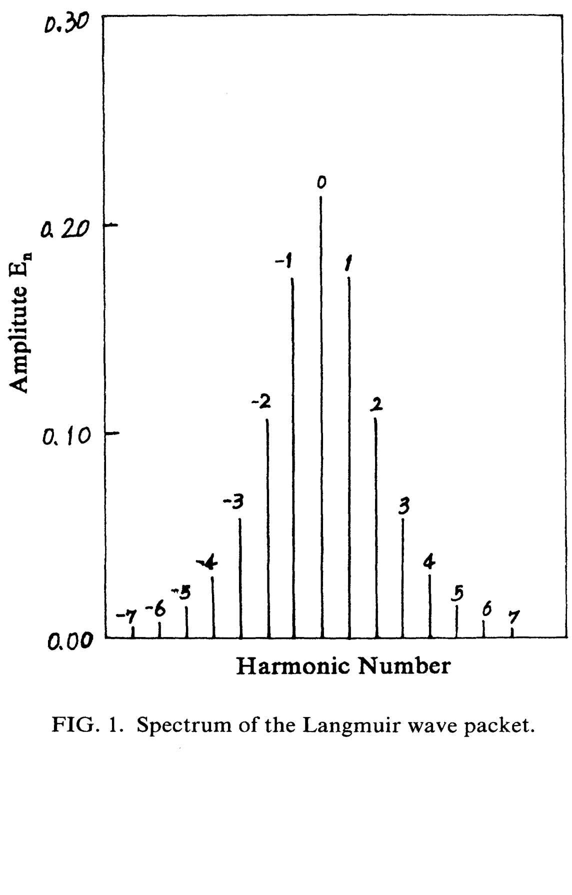

FIG. 1. Spectrum of the Langmuir wave packet.

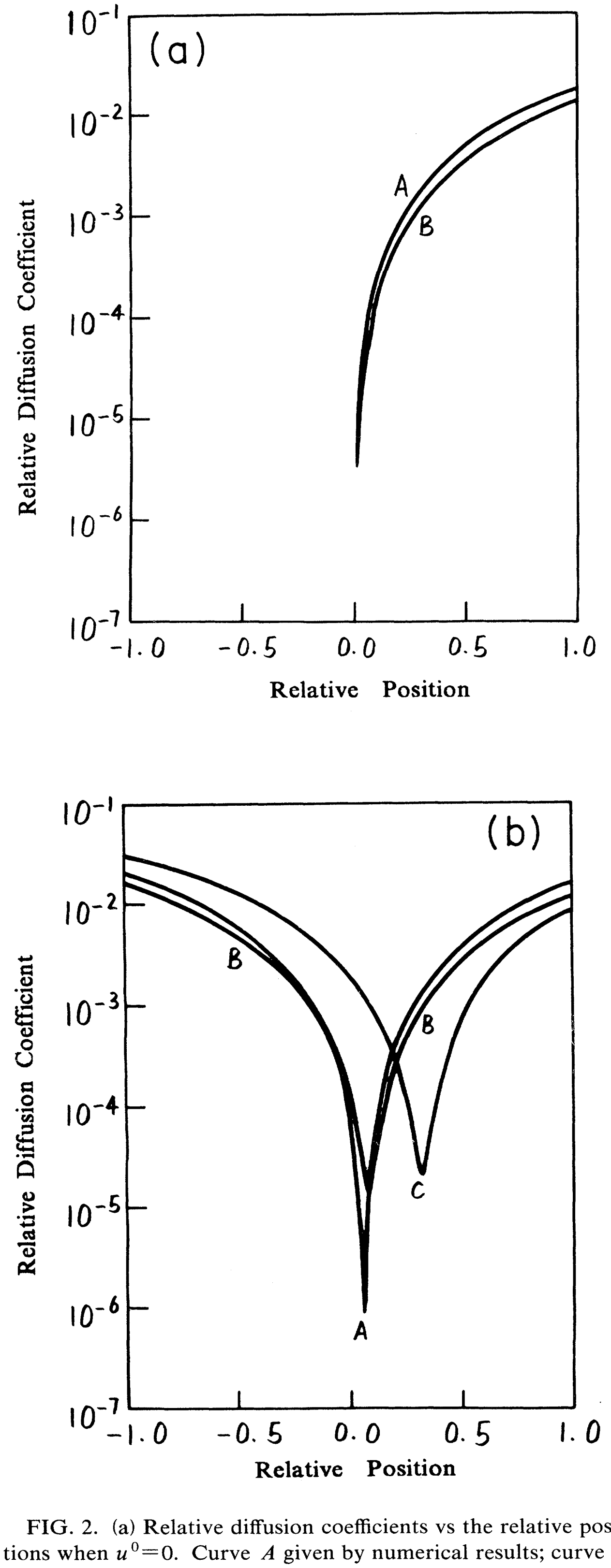

FIG. 2. (a) Relative diffusion coefficients vs the relative positions when $u^0=0$. Curve A given by numerical results; curve B by Eq. (10). (b) Relative diffusion coefficients vs the relative positions. Curves A and C given by numerical results corresponding to $u^0=-0.01,-0.05$, respectively; curve B by Eq. (10) when $u^0=-0.01$.

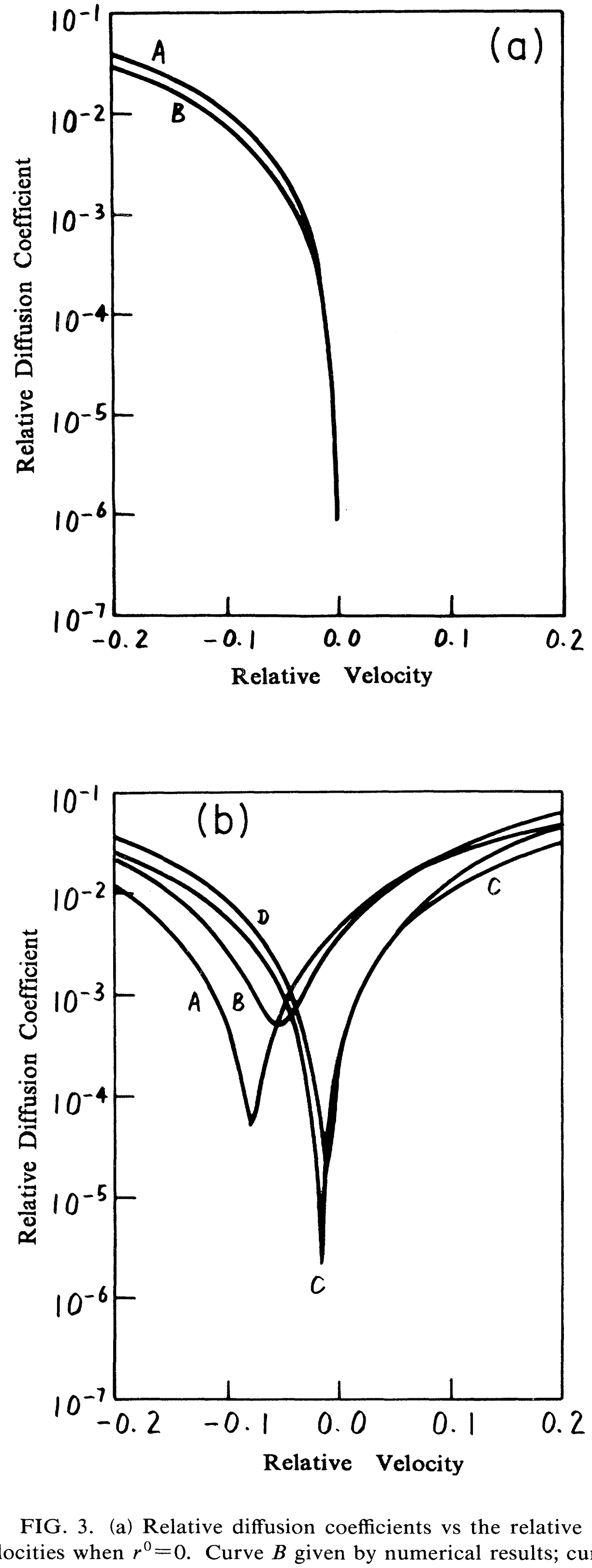

FIG. 3. (a) Relative diffusion coefficients vs the relative velocities when $r^0=0$. Curve B given by numerical results; curve A by Eq. (10). (b) Relative diffusion coefficients vs the relative velocities. Curves A and C given by numerical results corresponding to $r^0=0.1,0.5$, respectively; curves B and D by Eq. (10) corresponding to $r^0=0.1,0.5$, respectively.

viously supports the above approximation to be reasonable.

One can see from Figs. 2 and 3 that D_- of Eq. (10) and D_-^p of Eq. (21) are almost identical, therefore the numerical computation is reliable. Our results show that for $u^0=0$ D_- decreases when r^0 decreases, which is in keeping with that proposed by Dupree,[1] who gave the relative diffusion coefficient as follows:

$$D_-^d=\frac{q^2}{m_e^2}\int_0^\infty dt\sum_{k,\omega}|E_{k,\omega}|^2\exp[i(kv-\omega)t-\tfrac{1}{3}k^2Dt^3]\times(1-\cos kr^0)\ ,\qquad(22)$$

where D is the absolute diffusion coefficient. However, our results show a dependence of D_- on u^0 as well, which is just like r^0, whereas D_-^d of Eq. (22) is not dependent on u^0.

As we expected, D_-^p given by numerical simulation has a nonzero minimum when $r^0+\frac{1}{2}\tau u^0=0$, as shown in Figs. 2 and 3. The fact that D_- has a nonzero minimum means that the lifetime of clumps is finite. It is easy to understand, because D_{11}, D_{22}, and D_- are non-negative, and D_- approaches $D_{11}+D_{22}$ when $|r^0|\to\infty$ or $|u^0|\to\infty$, so D_- varying with r^0 or u^0 must have a minimum. Thus the relative diffusion is greatly reduced close to D_- minima, and the two-point correlation is considerably enhanced.

Analyzing the D_-^p given by Eqs. (4) and (21), we find that the value for $u^0=0$ coincides with the following formula very well:

$$D_-^p=\tfrac{1}{2}k_c^2\langle r^2(\tau)\rangle(D_{11}+D_{22})\ ,\qquad(23)$$

where the characteristic wave number $k_c=0.157$ with the parameters in Figs. 2 and 3. Under the condition of $u^0=0$, and defining the lifetime of clumps τ_{cl}

$$k_c^2\langle r^2(\tau_{cl})\rangle=1\ ,\qquad(24)$$

Fig. 4 shows the dependence of τ_{cl} on the relative positions. We find that the results can be represented very well by

$$\tau_{cl}=18.27-6.1\ln r^0\ ,\qquad(25)$$

τ_{cl} diverges logarithmically as r^0 vanishes.

We also give the coherent time which is defined as

$$\tau_c=\left[\frac{2}{k_c^2(D_{11}^p+D_{22}^p)}\right]^{1/3}\ ,$$

and Fig. 4 shows $\tau_{cl}\gg\tau_c$ for small r^0,u^0.

Clumps can behave like a single discrete particle and are considered to enhance the radiation emission, resistivity, etc.; the above-mentioned discussions assure us that clumps can be formed in this case.

Figure 5 shows the values of D_-^p for different r^0 at different times. Curve A is from particles passing through one periodic structure, and curve B is from particles passing through two periodic structures. The two

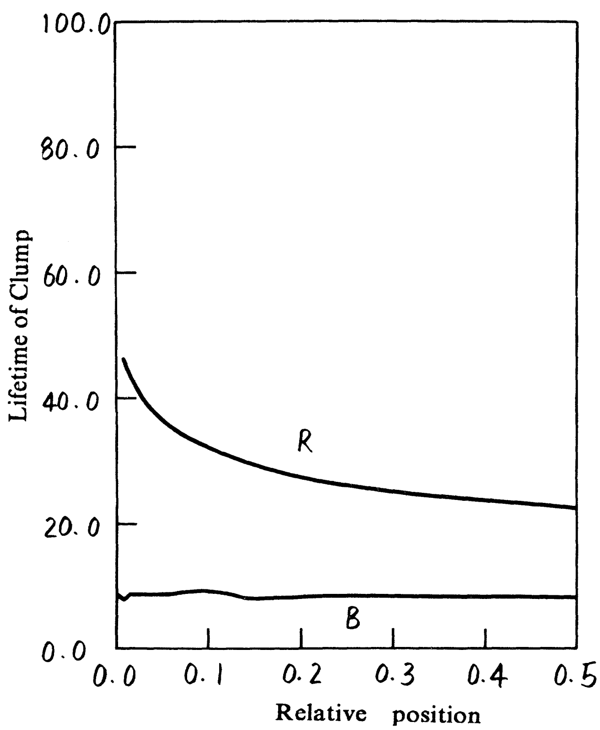

FIG. 4. The lifetime of clumps vs the relative positions when $u^0=0$. Curve R presents the lifetime of clumps, and curve B shows the coherent time.

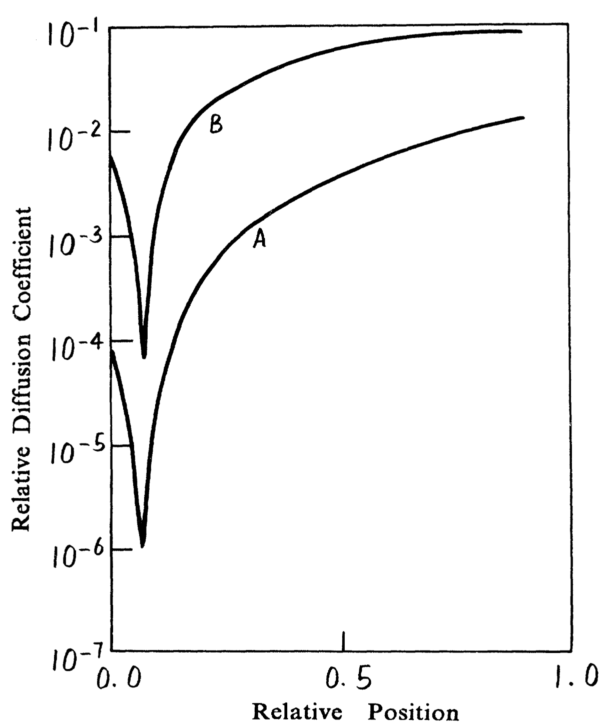

FIG. 5. The relative diffusion coefficients vs the relative positions at different time. Curves A,B corresponding to one pass, two passes, respectively.

curves have the same shape, and are minimized at the same position.

In addition, we find from numerical computation that the values of the relative diffusion coefficient D^p_-, to a certain extent, are the varying function of barycentric position R^0 and velocity V^0, and the profiles of D^p_- for different R^0 and V^0 are similar to each other.

In conclusion we have found a general form of the relative diffusion coefficient for electrons moving in Langmuir field with soliton structure. The behavior of the relative diffusion coefficient substantially differs from the prediction by Dupree. We also show that the coherent, localized wave packets can drive the formation of clumps in phase space. With the numerical solutions of the equations of motion, we verified that only when the relative velocity is zero does the lifetime of clumps indeed diverge logarithmically, with the relative positions of the constituents getting closer to one another.

We thank Professor S. G. Chen and Dr. M. S. Li for helpful discussions. This work was supported by the National Natural Science Foundation of China Grant No. 1880109.

[1]T. H. Dupree, Phys. Fluids. **15**, 334 (1972).
[2]W. Rozmus and J. C. Samson, Phys. Lett. A **126**, 263 (1988).
[3]J. A. Krommes, in *Handbook of Plasma Physics,* edited by A. A. Galeev and R. N. Sudan (North-Holland, Amsterdam, 1984), Vol. 2, p. 183.
[4]X. T. He and Y. Tan, Bull. Am. Phys. Soc. **34**, 2162 (1989).
[5]A. Salat, Phys. Fluids. **31**, 1499 (1988).