Kinetic equation of a plasma in a strong magnetic field

Bo-jun Yang
Department of Physics, Beijing University of Post and Telecommunication, Beijing 100088, China

Shu-guang Yao
Department of Radioelectronics, Peking University, Beijing 100871, China

(Received 27 July 1990)

The kinetic equation of a quantum plasma is derived by means of the Wigner distribution function and the Bogoliubov approach in a strong magnetic field. The equation includes correctional binary collisions, which account for the influence of a magnetic field on the collisions of particles and the effect of many-body effects.

I. INTRODUCTION

In previous works[1] the kinetic equation of a quantum plasma was derived for the spatially quasihomogeneous unmagnetized case. Because the influence of the magnetic field on the collisions of particles is neglected, the equation can be used only for weak external fields.

However, many plasmas occur in the presence of magnetic fields. With a stronger magnetic field and a lower particle density, the mean gyroradius R_c can be less than the Debye length D. For example, when $B=1$ T and $n=10^{18}$ m^{-3}, then $D/r_c=3.11$ for electron collisions. In this case the influence of a magnetic field on the collisions of particles cannot be neglected.

The Landau equation has been applied in general to study the nonequilibrium properties for a plasma in a strong magnetic field. The equation is expressed in natural guiding-center variables.[2] In addition, the Bagoliubov-Born-Green-Kirkwood-Yvon hierarchy has been applied. The influence of a strong magnetic field has been accounted for in the collision terms, but quantum effects are neglected, and spatial homogeneity is assumed. Thus this equation can be applied only to study the typical particle's problem.[3]

The principal purpose of the present work is an extension of the improved Boltzmann-Uehling-Uhlenbeck equation to the case for a strong magnetic field.

II. FUNDAMENTAL EQUATION

For convenience we first consider a one-component plasma. In fact, it is assumed that there is an extended neutralizing background of a positive charge. The exchange effect of quantum mechanics is disregarded. The quantum one-body Wigner distribution function $f(x)$ satisfies[1]

$$\frac{\partial f(x_1)}{\partial t}+\frac{\mathbf{p}_1}{m}\cdot\frac{\partial f(x_1)}{\partial \mathbf{q}_1}-\frac{\partial U(q_1)}{\partial \mathbf{q}_1}\cdot\frac{\partial f(x_1)}{\partial \mathbf{p}_1}+\frac{i}{\hbar}(e^{(i\hbar/2)\eta_1}-e^{(i\hbar/2)\eta_1})f(x_1)=-\frac{i}{\hbar}\int dx_2(e^{(i\hbar/2)\theta_{12}}-e^{-(i\hbar/2)\theta_{12}})g(x_1,x_2)\,, \quad (1)$$

where

$$\theta_{12}=\frac{\partial V_{12}(q_1-q_2)}{\partial \mathbf{q}_1}\cdot\left[\frac{\partial}{\partial \mathbf{p}_1}-\frac{\partial}{\partial \mathbf{p}_2}\right],\quad V_{12}(q_1-q_2)=\frac{e^2}{|\mathbf{q}_1-\mathbf{q}_2|}$$

$$\eta_1=\frac{\partial V_1(q_1)}{\partial \mathbf{q}_1}\cdot\frac{\partial}{\partial \mathbf{p}_1},\quad V_1(q_1)=\int V_{12}(q_1-q_2)f(x_2)dx_2\,.$$

$U(q_1)$ is the external potential, $g(x_1,x_2)$ is the two-body correlation function, which satisfies

$$\mathcal{D}_0 g(x_1,x_2)+\sum_{j=1}^{2}\left[\frac{\mathbf{p}_j}{m}\cdot\frac{\partial}{\partial \mathbf{q}_j}g(x_1,x_2)-\frac{\partial U(q_j)}{\partial \mathbf{q}_j}\cdot\frac{\partial g(x_1,x_2)}{\partial \mathbf{p}_j}\right]+\frac{i}{\hbar}\sum_{j=1}^{2}(e^{(i\hbar/2)\eta_j}-e^{-(i\hbar/2)\eta_j})g(x_1,x_2)=H(x_1,x_2)\,, \quad (2)$$

$$H(x_1,x_2)=\frac{i}{\hbar}(e^{(i\hbar/2)\theta_{12}}-e^{-(i\hbar/2)\theta_{12}})f(x_1)f(x_2)+\frac{i}{\hbar}\int dx_3(e^{(i\hbar/2)\theta_{1,3}}-e^{-(i\hbar/2)\theta_{1,3}})g(x_2,x_3)f(x_1)-\frac{i}{\hbar}\int dx_3(e^{(i\hbar/2)\theta_{2,3}}-e^{-(i\hbar/2)\theta_{2,3}})f(x_2)g(x_1,x_3)\,. \quad (3)$$

x indicates all of $\mathbf{q}$ and $\mathbf{p}$. The boundary condition of equation is

$$\lim_{\tau\to\infty} \mathcal{S}^{(2)}_{-\tau} g(x_1,x_2)\to 0 \ . \tag{4}$$

$\mathcal{S}^{(2)}_{-\tau}$ is the displacement operator. Its action is that $\mathbf{q}$ and $\mathbf{p}$ displace τ backward in time for $g(x_1,x_2)$.

The motion of the particle is axisymmetric in a strong magnetic field. It is convenient for choosing the cylindrical coordinate. The derivation of the magnetic field is selected for the z axis. The components of $\mathbf{p}$ are $p_\perp$, θ, and p_z. The condition of the spatially quasihomogeneous is

$$g(x_1,x_2)=g(q_1-q_2;p_1,p_2)=g(q;p_1,p_2) \ .$$

The Fourier transformation

$$\tilde{g}(k;p_1,p_2)=\int g(q;p_1,p_2)e^{-i\mathbf{k}\cdot\mathbf{q}}d\mathbf{q} \ .$$

In the spatially quasihomogeneous case, the variance of $f(x)$ is slower. The variance can be disregarded in the D region. When $r_c<D$, the variance of $f(x)$ can also be disregarded in the r_c region, so that $f(x)$ is dependent only on $p_\perp$ and p_z and independent of to θ. Using

$$g(q;p_1,p_2)=\frac{1}{(2\pi)^3}\int e^{i\mathbf{k}\cdot\mathbf{q}}\tilde{g}(k;p_1,p_2)d\mathbf{k} \ ,$$

$$\tilde{V}(k)=\int \frac{e^2}{q}e^{-i\mathbf{k}\cdot\mathbf{q}}d\mathbf{q}=\frac{4\pi e^2}{k^2} \ ,$$

then Eq. (3) can be rewritten for

$$\begin{aligned}H(x_1,x_2)=\frac{i}{\hbar}\int d\mathbf{k}\frac{1}{(2\pi)^3}e^{i\mathbf{k}\cdot(\mathbf{q}_1-\mathbf{q}_2)}\tilde{V}(k)\Bigg[&(e^{(\hbar\mathbf{k}/2)\cdot(\partial/\partial\mathbf{p}_1-\partial/\partial\mathbf{p}_2)}-e^{-(\hbar\mathbf{k}/2)\cdot(\partial/\partial\mathbf{p}_1-\partial/\partial\mathbf{p}_2)})f(x_1)f(x_2)\\ &-\int(e^{(\hbar\mathbf{k}/2)\cdot(\partial/\partial\mathbf{p})}-e^{-(\hbar\mathbf{k}/2)\cdot(\partial/\partial\mathbf{p}_1)})g(-k;p_2,p_3)f(x_1)d\mathbf{p}_3\\ &+\int(e^{(\hbar\mathbf{k}/2)\cdot(\partial/\partial\mathbf{p}_2)}-e^{-(\hbar\mathbf{k}/2)\cdot(\partial/\partial\mathbf{p}_2)})g(k;p_1,p_3)f(x_2)d\mathbf{p}_3\Bigg] \ .\end{aligned} \tag{5}$$

In the cylindrical coordinate

$$\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{p}}=k_2\frac{\partial}{\partial p_z}+k_\perp\frac{\partial}{\partial p_\perp}\cos(\theta-\alpha) \ ,$$

where $k_\perp$, α, and k_z are the three components of $\mathbf{k}$. Since the displacement operator $\mathcal{S}^{(2)}_{-\tau}$ satisfies the equation

$$\frac{\partial\mathcal{S}^{(2)}_{-\tau}}{\partial\tau}=-\sum_{j=1}^{2}\left[\frac{\mathbf{p}_j}{m}\cdot\frac{\partial}{\partial\mathbf{q}_j}-\frac{\partial U(q_j)}{\partial\mathbf{q}_j}\cdot\frac{\partial}{\partial\mathbf{p}_j}+\frac{i}{\hbar}(e^{(i\hbar/2)\eta_j}-e^{-(i\hbar/2)\eta_j})\right]\mathcal{S}^{(2)}_{-\tau} \ ,$$

the formal solution of $g(x_1,x_1)$ is obtained by means of the displacement approach as[4]

$$g(x_1,x_2|f)=-\int_0^\infty \mathcal{S}^{(2)}_{-\tau}H(x_1,x_2|\mathcal{S}^{(1)}_{\tau}f)d\tau \ , \tag{6}$$

where $\mathcal{S}^{(1)}_{\tau}$ is the one-body displacement operator. Following displacement, $\mathbf{q}$ and $\mathbf{p}$ become

$$p_\perp(\tau)=p_\perp,\quad p_z(\tau)=p_z,\quad \theta(\tau)=\theta-\omega\tau \ ,$$

$$\mathbf{q}(\tau)=\mathbf{q}+\mathbf{i}r_c[\sin\theta(\tau)-\sin\theta]+\mathbf{j}r_c[\cos\theta(\tau)-\cos\theta]+\mathbf{k}v_z\tau \ .$$

III. CALCULATION OF THE CORRELATION FUNCTION

The techniques of Fourier transformation in space are used in the right-hand side of Eq. (1) to obtain

$$\left[\frac{\partial f(x_1)}{\partial t}\right]_{\text{coll}}=\frac{i}{\hbar}\int d\mathbf{k}\frac{1}{(2\pi)^3}(e^{(\hbar\mathbf{k}/2)\cdot(\partial/\partial\mathbf{p}_1)}-e^{-(\hbar\mathbf{k}/2)\cdot(\partial/\partial\mathbf{p}_1)})\tilde{V}(k)h(k,p_1) \tag{7}$$

where

$$h(k,p_1)=\int d\mathbf{p}_2\tilde{g}(k,p_1,p_2) \ .$$

We may substitute Eq. (5) into Eq. (6) performing the Fourier transform to obtain

$$\begin{aligned}h(k,p_1)=-\frac{i}{\hbar}\tilde{V}(k)\int d\mathbf{p}_2\int_0^\infty d\tau\exp((ik_\perp r_{2c})\{\cos[\theta_2(\tau)-\alpha]-\cos(\theta_2-\alpha)\})\exp[-ik_z(p_{2z}-p_{1z})\tau]&\\ \times[(e^{(\hbar\mathbf{k}/2)\cdot(\partial/\partial\mathbf{p}_1^\tau-\partial/\partial\mathbf{p}_2^\tau)}-e^{(\hbar\mathbf{k}/2)\cdot(\partial/\partial\mathbf{p}_1^\tau-\partial/\partial\mathbf{p}_2^\tau)})f(x_1)f(x_2)&\\ -(e^{(\hbar\mathbf{k}/2)\cdot(\partial/\partial\mathbf{p}_1^\tau)}-e^{-(\hbar\mathbf{k}/2)\cdot(\partial/\partial\mathbf{p}_1^\tau)})h^*(k,p_2^\tau)f_1(x_1)&\\ +(e^{(\hbar\mathbf{k}/2)\cdot(\partial/\partial\mathbf{p}_2^\tau)}-e^{-(\hbar\mathbf{k}/2)\cdot\partial/(\partial\mathbf{p}_2^\tau)})h(k,p_1^\tau)f(x_2)] \ .&\end{aligned} \tag{8}$$

We apply the Bessel function expansion

$$h(k,p)=\sum_{n,n'=-\infty}^{\infty} h_n(p_\perp p_z)J_n(k_\perp r_c)J'_{n'}(k_\perp r_c)e^{-i(n-n')(\theta-\alpha)}(i)^{n+n'}$$
$$=\sum_{n=-\infty}^{\infty} i^n h_n(p_\perp,p_z)J_n(k_\perp r_c)e^{-in(\theta-\alpha)}e^{ikr_c\cos(\theta-\alpha)} . \quad (9)$$

We substitute Eq. (9) into Eq. (8), performing some manipulations to obtain

$$h_n(p_{1\perp},p_{1z})=-\frac{1}{\hbar}\widetilde{V}(k)\sum_{n'}\int d\mathbf{p}_2\frac{J_{n'}^2(k_\perp r_{2c})}{(\mathbf{k}\cdot\mathbf{p}_2/m)_{n'}-(\mathbf{k}\cdot\mathbf{p}_1/m)_n-i\varepsilon}$$
$$\times[(e^{(\hbar\mathbf{k}/2)\cdot(\partial/\partial\mathbf{p}_1-\partial/\partial\mathbf{p}_2)_n}-e^{-(\hbar\mathbf{k}/2)\cdot(\partial/\partial\mathbf{p}_1-\partial/\partial\mathbf{p}_2)_{n'}})f(x_1)f(x_1)$$
$$-(e^{(\hbar/2)(\mathbf{k}\cdot\partial/\partial\mathbf{p}_1)_n}-e^{-(\hbar/2)(\mathbf{k}\cdot\partial/\partial\mathbf{p}_1)_n})f(x_1)h_{n'}^*(p_{2\perp},p_{2z})$$
$$+(e^{(\hbar/2)(\mathbf{k}\cdot\partial/\partial\mathbf{p}_2)_{n'}}-e^{-(\hbar/2)(\mathbf{k}\cdot\partial/\partial\mathbf{p}_2)_{n'}})f(x_2)h_n(p_{1\perp},p_{1z})] , \quad (10)$$

where

$$\left[\mathbf{k}\cdot\frac{\mathbf{p}}{m}\right]_n=k_z\frac{p_z}{m}+\frac{n}{V_c}\frac{p_\perp}{m} .$$

Equation (10) can be rewritten as

$$h_n(p_{1\perp},p_{1z})=-\frac{1}{\hbar}\widetilde{V}(k)\sum_{n'}\int d\mathbf{p}_2\frac{J_{n'}^2(k_\perp r_{2c})}{\left[\mathbf{k}\cdot\frac{\mathbf{p}_2}{m}\right]_{n'}-\left[\mathbf{k}\cdot\frac{\mathbf{p}_2}{m}\right]_n-i\varepsilon}$$
$$\times\{[f^+(x_1)f^-(x_2)-f^-(x_1)f^+(x_2)]-[f^+(x_1)-f(x_1)]h_{n'}^*(p_{2\perp}p_{2z})$$
$$+[f^+(x_2)-f^-(x_2)]h_n(p_1,p_{1z})\} , \quad (11)$$

where

$$f^\pm(x)=f\left[q,p_z\pm\frac{\hbar}{2}k_z,p_2\pm\frac{\hbar}{2}\frac{n}{r_c}\right] .$$

The imaginary part of h_n can be solved from Eq. (10) to find

$$\mathrm{Im}h_n=\frac{-\frac{\pi}{\hbar k}\widetilde{V}(k)}{\left|1+\frac{1}{\hbar k}\widetilde{V}(k)\Psi\right|^2}[f^+(x_1)F^- - f^-(x_1)F^+] . \quad (12)$$

Here

$$\Psi=\sum_n\int\frac{d\mathbf{p}_2}{\left[\mathbf{k}\cdot\frac{\mathbf{p}_2}{m}\right]_{n'}-\left[\mathbf{k}\cdot\frac{\mathbf{p}_2}{m}\right]_n-i\varepsilon}J_{n'}^2(kr_{2c})$$
$$\times[f^+(x_2)-f^-(x_2)] ,$$

$$F^\pm=\sum_n\int f^\pm(x)J_n^2(k_\perp r_c)\delta\left[\left[\mathbf{k}\cdot\frac{\mathbf{p}'}{m}\right]_{n'}-\left[\mathbf{k}\cdot\frac{\mathbf{p}}{m}\right]_n\right]d\mathbf{p} .$$

IV. QUANTUM KINETIC EQUATION

Substituting Eqs. (9) and (12) into Eq. (7), one finds

$$\left[\frac{\partial f(x_1)}{\partial t}\right]_{\text{coll}}=\frac{i}{\hbar}\int d\mathbf{k}\frac{1}{(2\pi)^3}\widetilde{V}(k)(e^{\hbar\mathbf{k}\cdot(\partial/\partial\mathbf{p}_1)}-e^{-\hbar\mathbf{k}\cdot(\partial/\partial\mathbf{p}_1)})$$
$$\times\sum_{n,n'=-\infty}^{\infty}h_n(p_\perp,p_z)J_n(k_\perp r_c)J_n(k_\perp r_c)e^{-i(n-n')(\theta-\alpha)}i^{(n+n')} .$$

However, $n=n'$ is required so that the α integral is not zero and one obtains

$$\left[\frac{\partial f(x_1)}{\partial t}\right]_{\text{coll}}=\frac{-1}{\hbar}\int d\mathbf{k}\frac{1}{(2\pi)^3}\widetilde{V}(k)\sum_n(e^{[(\hbar\mathbf{k}/2)\cdot(\partial/\partial\mathbf{p}_1)]_h}-e^{-[(\hbar\mathbf{k}/2)\cdot(\partial/\partial\mathbf{p}_1)]_1})J_n^2(k_\perp r_c)\mathrm{Im}[h_n(p_\perp,p_z)] . \quad (13)$$

Hence the kinetic equation of the one-component plasma is

$$\frac{\partial f(x_1)}{\partial t}+\frac{\mathbf{p}_1}{m}\cdot\frac{\partial f(x_1)}{\partial\mathbf{q}_1}+(\mathbf{F}_{\text{in}}+\mathbf{F}_{\text{out}})\cdot\frac{\partial f(x_1)}{\partial\mathbf{p}_1}=\left[\frac{\partial f(x_1)}{\partial t}\right]_{\text{col}} , \quad (14)$$

$$\left[\frac{\partial f(x_1)}{\partial t}\right]_{\text{coll}}=\frac{1}{\hbar}\int d\mathbf{k}\frac{1}{(2\pi)^3}\sum_n\left(e^{(\hbar/2)(\mathbf{k}\cdot\partial/\partial\mathbf{p}_1)_n}-e^{-(\hbar/2)(\mathbf{k}\cdot\partial/\partial\mathbf{p}_1)_n}\right)J_n^2(k_\perp r_{1c})$$
$$\times\frac{\pi\tilde V^2(k)}{\left|1+\frac{1}{\hbar}\tilde V(k)\Psi\right|^2}\int d\mathbf{p}_2\sum_n\delta\left[\left(\mathbf{k}\cdot\frac{\mathbf{p}_1}{m}\right)_n-\left(\mathbf{k}\cdot\frac{\mathbf{p}_2}{m}\right)_{n'}\right]$$
$$\times J_{n'}(k_\perp r_{2c})[f^+(x_1)f^-(x_2)-f^-(x_1)f^+(x_2)]\ ,\qquad(15)$$

where

$$\mathbf{F}_{\text{in}}\cdot\frac{\partial f(x_1)}{\partial\mathbf{p}_1}=\frac{i}{\hbar}(e^{(i\hbar/2)\eta_1}-e^{-(i\hbar/2)\eta_1})f(x_1)\ ,$$

$$F_{\text{out}}=\frac{\partial U(q_1)}{\partial\mathbf{q}_1}\ ,$$

$$\Psi=\sum_{n''}\int\frac{d\mathbf{p}_3}{\left(\mathbf{k}\cdot\frac{\mathbf{p}_1}{m}\right)_n-\left(\mathbf{k}\cdot\frac{\mathbf{p}_3}{m}\right)_{n''}-i\varepsilon}J_{n''}^2(kr_{3c})[f^+(x_3)-f^-(x_3)]\ .$$

It is not difficult to generalize the equation for a plasma consisting of multicomponents, including the exchange effect of quantum mechanics. Let us suppose there are M components in the plasma. The number of particles is N. a, b, and d indicates the appendant sign of a component. The number of the particles is N_b for a component b. The kinetic equation of the component a is

$$\frac{\partial f_a(x_1)}{\partial t}+\frac{\mathbf{p}_1}{m_a}\cdot\frac{\partial f_a(x_1)}{\partial\mathbf{q}_1}+(\mathbf{F}_{\text{in}}+\mathbf{F}_{\text{out}})\cdot\frac{\partial f_a(x_1)}{\partial\mathbf{p}_1}=\left[\frac{\partial f_a(x_1)}{\partial t}\right]_{\text{coll}}\ ,\qquad(16)$$

$$\left[\frac{\partial f_a(x_1)}{\partial t}\right]_{\text{coll}}=\frac{1}{\hbar}\int d\mathbf{k}\frac{1}{(2\pi)^3}\sum_{b=1}^{M}\frac{N_b}{N}\sum_{n=-\infty}^{\infty}\left(e^{(\hbar/2)(\mathbf{k}\cdot\partial/\partial\mathbf{p}_1)_n}-e^{-(\hbar/2)(\mathbf{k}\cdot\partial/\partial\mathbf{p}_1)_n}\right)\frac{\pi\tilde V_{ab}^2(k)J_n^2(k_\perp r_{1c})}{\left|1+\frac{1}{\hbar}\sum_{d\,(\neq a)}\frac{N_d}{N}\tilde V_{bd}(k)\Psi\right|^2}$$
$$\times\int d\mathbf{p}_2\sum_{n'=-\infty}^{\infty}\delta\left[\left(\mathbf{k}\cdot\frac{\mathbf{p}_1}{m_a}\right)_n-\left(\mathbf{k}\cdot\frac{\mathbf{p}_2}{m_b}\right)_{n'}\right]$$
$$\times J_{n'}^2(k_\perp r_{2c})[f_a^+(x_1)f_b^-(x_2)-f_a^-(x_1)f_b^+(x_2)]\ ,\qquad(17)$$

where

$$f_a^{\pm}(x)=f\left[q,p_z\pm\frac{\hbar}{2}k_z,p_\perp\pm\frac{\hbar}{2}\frac{n}{r_c}\right]\left[1-f\left[q,p_z\mp\frac{\hbar}{2}k_z,p_\perp\pm\frac{\hbar}{2}\frac{n}{r_c}\right]\right]\ ,$$

$$\Psi=\sum_{n''=-\infty}^{\infty}\int d\mathbf{p}_3\frac{1}{\left(\mathbf{k}\cdot\frac{\mathbf{p}_3}{m_d}\right)_{n''}-\left(\mathbf{k}\cdot\frac{\mathbf{p}_1}{m_a}\right)_n-i\varepsilon}J_{n''}^2(kr_{3c})[f_d^+(x_3)-f_d^-(x_3)]\ ,$$

$$\mathbf{F}_{\text{in}}\cdot\frac{\partial f_a(x_1)}{\partial\mathbf{p}_1}=\frac{i}{\hbar}(e^{(i\hbar/2)\eta_a}-e^{-(i\hbar/2)\eta_a})f_a(x_1)+\frac{i}{\hbar}\int dx_2(e^{(i\hbar/2)\theta_{ba}}-e^{-(i\hbar/2)\theta_{ba}})f_a(x_1)f_b(x_2)$$

Equations (16) and (17) are the kinetic equations of a quantum plasma in a strong magnetic field. There are two restrictions in the application of the equations, according to some approximate conditions that are applied in the process for the derivation of the equations. (i) The system is spatially quasihomogeneous, the nonhomogeneous case can be neglected in the Debye length D region. (ii) The variance of the function $f(x)$ can be neglected in the interval of the time $D/\bar v$ where $\bar v$ is the mean speed of the particle. The equation considers correctional binary collision with a many-body effect and the quantum effect, so that the collision integrals can be converged spontaneously.

[1]Bo-jun Yang and Shu-guang Yao, Phys. Rev. C **36**, 667 (1987); Phys. Rev. A **41**, 4504 (1990).

[2]R. Balescu, *Transport Process in Plasma* (North-Holland, Amsterdam, 1988).

[3]Y. L. Klimontovich, *Kinetic Theory of Nonideal Gas and Nonideal Plasmas* (Pergamon, Oxford, 1982).

[4]*Studies in Statistical Mechanics Vol. I,* edited by B. Uhlenbeck (North-Holland, Amsterdam, 1962).