Unstable periodic orbits and prediction

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A hierarchical approximation of a generic chaotic attractor can be formulated in terms of unstable periodic orbits. We demonstrate the possibility of extracting the most dominant unstable periodic orbits from a measurement of a time-continuous flow. Since unstable periodic orbits not only represent the static properties of the system but also dominate the dynamics, they can be used to fit models of the flow and to predict the orbit. We show that predictions for the Roessler system using unstable periodic orbits extracted from a time series of moderate length are significantly better than those from other approaches.

I. INTRODUCTION

The past decade has seen substantial progress in the characterization of chaotic dynamical systems. Lyapunov exponents, dimensions, and entropies can now be extracted from experimental data and provide a useful quantitative description.[1] Systems on the borderline to chaos can now be understood by application of powerful techniques such as renormalization. However, for systems far beyond the borderline to chaos, tools for qualitative understanding still are lacking. Recently the unstable periodic orbits (UPO's) of a chaotic dynamical system have attained increased attention.[2-4] As is well known, the closure of the set of unstable periodic orbits defines a strange attractor.[5] Generic strange attractors can be approximated hierarchically by unstable periodic orbits of a given length. The number of unstable periodic orbits grows exponentially with this length, the exponent being the topological entropy K_0.

In Refs. 3 and 4 methods have been developed to extract unstable periodic orbits and to use them to determine the topological entropy and dimensions. However for limited data from a time-continuous system this method cannot be applied directly. For time-continuous flows there are periods t for which no unstable orbits exist. For example, the Roessler[6] system has essentially only return times that are multiples of $\tau_0 \cong 2\pi$.[7] In this article it will be shown that the most probable return times can be obtained by considering spheres around all points on the strange attractor, that have, e.g., been constructed from a scalar time series and computing the radius $\epsilon_p(t)$ of spheres into which there are p returns within a time t. We will demonstrate that this method yields the leading periodic orbits for the Roessler system, for the Mackey-Glass equation, and for an experimental system, using some 10^4 points for each example. Finally we will present a new method for predicting chaotic time series and show that using unstable periodic orbits for prediction is superior to using the time sequence of points on the attractor directly.

II. EXTRACTING UNSTABLE PERIODIC ORBITS FROM TIME-CONTINUOUS SYSTEMS

The invariant density ρ. which represents the long-time behavior of a chaotic system f, usually is approximated directly by the successive points on the attractor $\mathbf{x}_{i+1}=f(\mathbf{x}_i)$, $i=1,\ldots,N$:

$$\rho_N(\mathbf{x})=\frac{1}{N}\sum_i \delta(\mathbf{x}-\mathbf{x}_i) \ . \tag{1}$$

Consider the unstable periodic orbit $\mathbf{x}^*$ of length n, i.e., points $\mathbf{x}_i^*$, $i=1,\ldots,n$, such that $f^n(\mathbf{x}_i^*)=\mathbf{x}_i^*$ and $f(\mathbf{x}_i^*)=\mathbf{x}_{i+1}^*$, f being the flow. Let $K(\mathbf{x}^*)$ denote the instability of the orbit $\mathbf{x}^*$ [i.e., the sum of the positive Lyapunov exponents of $f^n(\mathbf{x}^*)$]. Then $\exp[-K(\mathbf{x}^*)]$ is proportional to the probability of the orbit points $\mathbf{x}_i^*$ and

$$\rho_n(\mathbf{x})=\frac{\sum_{\mathbf{x}_i^*\in \text{fix}(f^n)} \delta(\mathbf{x}-\mathbf{x}_i^*)e^{-K(\mathbf{x}^*)}}{\sum_{\mathbf{x}_i^*\in \text{fix}(f^n)} e^{-K(\mathbf{x}^*)}} \tag{2}$$

is the approximation of the invariant density by the unstable periodic orbits of length n.[2] In Eq. (2) no difference is made between primitive and nonprimitive periodic orbits of length n. In contradistinction to (1), Eq. (2) is invariant under the dynamics and averages using (2) are expected to converge rapidly with n.[3]

In order to extract UPO's from a time series of a time-continuous system, first one has to look at the most probable return times. Therefore it is desirable to have a measure for the probability $P_\epsilon(t)$ for returns in a time t in phase space into a small radius ϵ, especially because in general the usual autocorrelation

$$C(\tau)=\langle \xi(t)\xi(t+\tau)\rangle=(1/N)\sum_i \xi_i\xi_{i+\tau}$$

fails to show sharp peaks (Fig. 1). Given a scalar time series $\{\xi_i\}$ reconstructed optimally by the method of de-

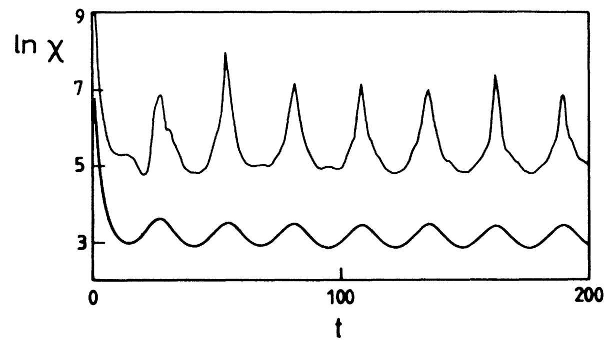

FIG. 1. $\chi=\epsilon_p^{-2}(t)/2m$ and $\chi=C(0)-C(t)$ (from above) from an experimental time series of $N=2\times10^4$ points of a driven damped pendulum with $m=5$, $\tau=5$, and $p=256$ (arbitrary scale).

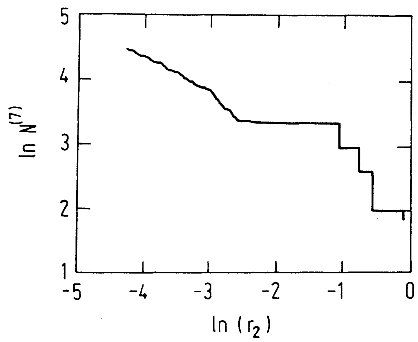

FIG. 2. The number N of periodic points of order 7 obtained with resolution r_2 for the paradigmatic Henon map (Ref. 13) from a time series of 10^4 points.

lays[8,9] which leads to $\{\mathbf{x}_i\}$, $\mathbf{x}_i\in\mathbb{R}^m$, one might define $P_\epsilon(t)$ most directly[10] by simply counting the number of sequences of length t in the time series that returns within a given distance ϵ, that is

$$P_\epsilon(t)=(1/N)\ \ (\text{number of } i:\ |\mathbf{x}_i-\mathbf{x}_{i+1}|<\epsilon)\ . \tag{3}$$

Since $P_\epsilon(t)$ provides only discrete values, we can equivalently look at the inverse function $\epsilon_p(t)$, which is the distance ϵ for which there are p returns in the time series. This function is numerically easy to obtain, since for every t one simply needs to sort the distances $\epsilon_i=|\mathbf{x}_i-\mathbf{x}_{i+t}|$. $\epsilon_p(t)$ then is the pth smallest distance under the ϵ_i. Small $\epsilon_p(t)$ are considered[11] to reveal a high-return probability, i.e., the existence of UPO's of length t. On the other hand, $\epsilon_p(t)$ is directly connected to the autocorrelation function. To show this we consider the distances ϵ_p in the Euclidean reconstruction phase space of embedding dimension m. For $N\to\infty$ it then holds[12] that

$$\frac{1}{N}\sum_p\epsilon_p^2(t)=\frac{m}{N}\sum_i(\xi_i-\xi_{i+t})^2=2m[C(0)-C(t)]\ . \tag{4}$$

Figure 1 demonstrates that $\epsilon_p^{-2}(t)/2m$, unlike the autocorrelation function, shows sharp peaks at certain times t^* which are candidate times for finding the less unstable periodic orbits present in the time series. This function converges at the correct embedding dimension when the delay time for the reconstruction of the attractor is chosen optimally.[9]

For the construction of the UPO's from a given time series, the function $\epsilon_p(t)$ now provides two important pieces of information: the times τ^* for which we have a high probability of finding UPO's and the scale $r_1\geq\epsilon_p(\tau^*)$ at which we must look to find at least p returns within the time series. We then follow the method presented in Ref. 3: A sequence $s_i(\mathbf{x}_i,\dots,\mathbf{x}_{i+\tau^*})$ is considered to be close to an τ-periodic orbit if $|\mathbf{x}_i-\mathbf{x}_{i+\tau^*}|\leq r_1$, where r_1 is an appropriate distance. Two sequences s_i,s_j are considered to belong to the same orbit if they are less than a second distance $r_2>r_1$ apart. Since periodic oprbits are defined only up to a cyclic per-

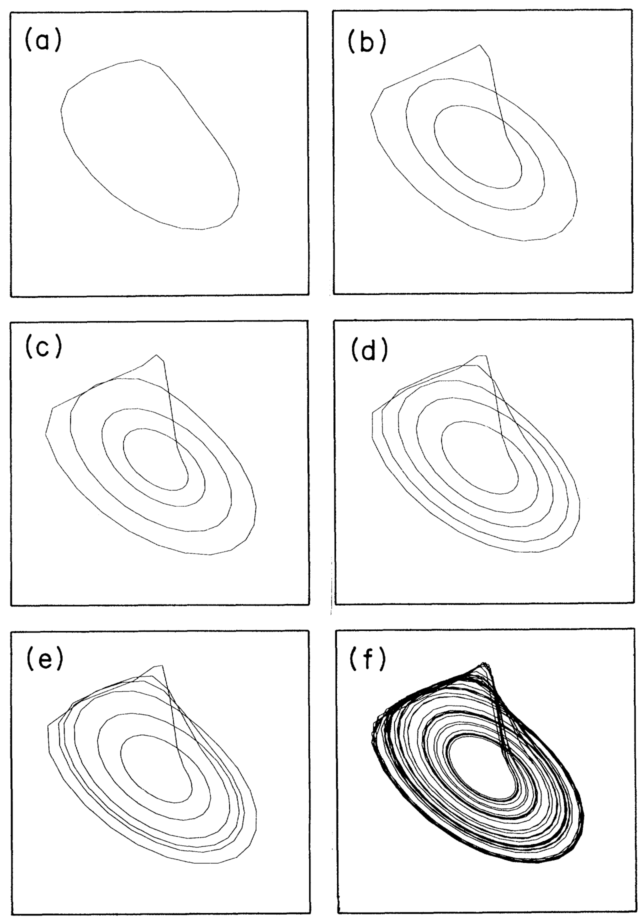

FIG. 3. (a)–(e) Some unstable periodic orbits extracted from a time series of $N=10^4$ points of the Roessler system. (f) Projections of all UPOs obtained from $N=10^4$ points of the Roessler system ($m=4$, $\tau=4\Delta\tau$, $\Delta t=\pi/12$).

mutation Π, we first permute one of the sequences such that $|s_i - \Pi s_j| = d_s$ becomes minimal. If $d_s \leq r_2$, the two sequences are grouped together.

A heuristic criterion for the appropriate choice of r_2 is the requirement that the correct grouping should not alter when r_2 is slightly changed. This comes about as one knows that different unstable periodic orbits tend to separate on the attractor.[5] To illustrate this, Fig. 2 shows the number of periodic points of periodic orbits of length 7 obtained from a time series of $n = 10.000$ points of the paradigmatic Henon map[13] ($a = 1.4$ and $b = 0.3$) versus r_2. There is a broad range for r_2 where the correct grouping does not change.

The orbit points then are obtained as averages of the points of the (possibly permuted) sequences in the group. Figures 3–5 illustrate the result. Plotted are unstable periodic orbits obtained from the time series of 10^4 points from the Roessler system[6] (Fig. 3), the Makey-Glass delay equation[14] (Fig. 4), and experimental data from a driven pendulum (Fig. 5). We found all primitive UPO's up to a length $\tau/\tau_0 \cong 5$ when τ_0 was the first return time in the system corresponding to the first peak in $\epsilon^{-1}(\tau)$. Especially for the Roessler system we found 1,0,2,2,4 primitive periodic orbits for $\tau/\tau_0 \cong 1,2,3,4,5$, respectively. Plotting projections of the orbits onto each other gives an approximate picture of the attractor [Figs. 3–5(f)].

Finally to obtain the instabilities $K(\mathbf{x}^*)$ of an UPO $\mathbf{x}^*$ of length $n(x^*)$, Jacobians J_i are fitted at the orbit points $\mathbf{x}_i^*$ from the corresponding sequence points and their successors. We then multiply the Jacobians around the orbit, and diagonalize the resulting matrix $J = J_n \cdot J_{n-1} \cdots J_1$. The Lyapunov exponents λ of the unstyable periodic orbit $\mathbf{x}^*$ are estimated from their eigenvalues (for details see, e.g., Ref. 15). The instability K of the unstable periodic orbit $\mathbf{x}^*$ is given by the sum of the positive Lyapunov exponents λ^+:

$$K(\mathbf{x}^*) = \sum \lambda^+(\mathbf{x}^*) \ . \tag{5}$$

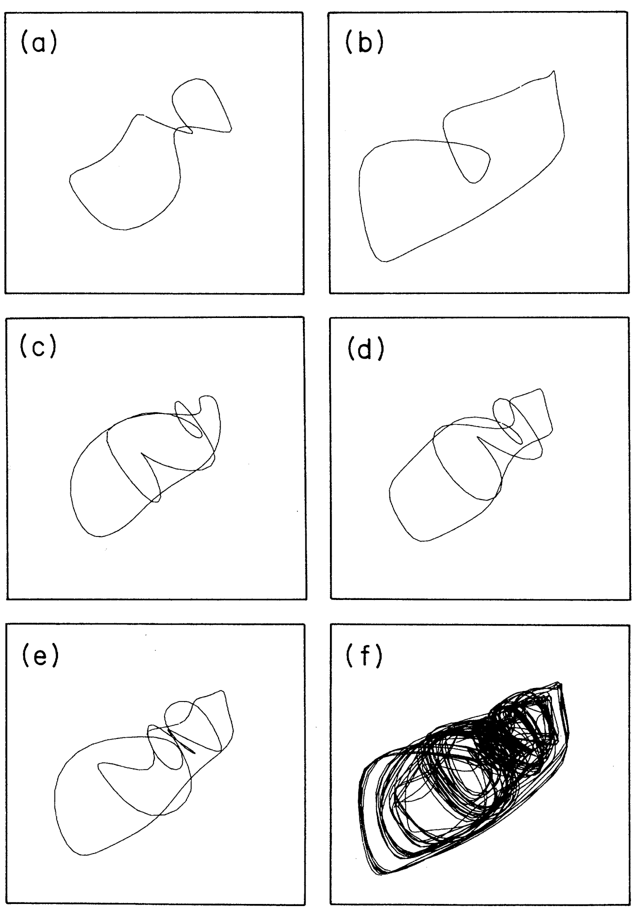

FIG. 4. Same as Fig. 3, but for the Mackey-Glass equation ($\Gamma = 30$, $m = 4$, $\tau = 6\Delta\tau$, $\Delta t = \Gamma/20$).

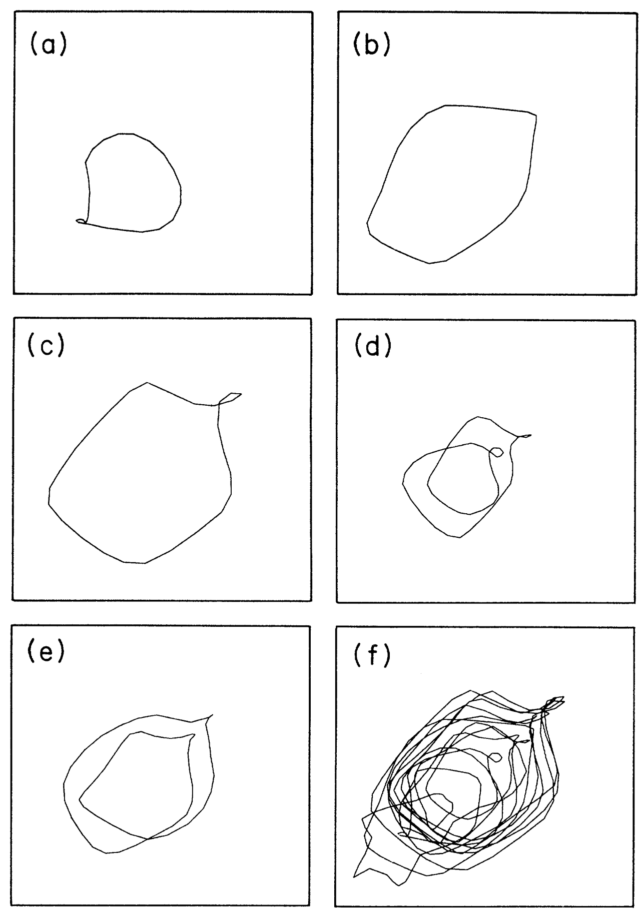

FIG. 5. Same as Fig. 4, but for data from an experimental pendulum.

III. PREDICTION USING UNSTABLE PERIODIC ORBITS

While prediction for unstable systems may at first sight seem a self-contradictory concept, such a prediction is in fact possible for short times, with the minimal error being dominated by the average expansion rate K_1.[16] At least modeling dynamical systems from a time series is possible.[17] By model we mean an approximation $\tilde{f}$ of the flow f. While in the original coordinates the flow f is a map $f:\mathbb{R}^d \Longrightarrow \mathbb{R}^d$, in the case of delay coordinates the nontrivial part of the flow is of the form $f:\mathbb{R}^m \Longrightarrow \mathbb{R}$. This follows directly from the delay method.[8]

Let $\xi(t)$ be the measured time series. Then $\mathbf{x}(t)=[\xi(t),\xi(t+\tau),\ldots,\xi(t+(m-1)\tau)]$ is the reconstructed vector and the flow reduces to

$$f^{\tau}(\mathbf{x}(t))=\mathbf{x}(t+\tau)$$
$$=(\xi(t+\tau),\ldots,\xi(t+(m-1)\tau),\xi(t+m\tau)) \quad (6)$$

containing a nontrivial new component only $\xi(t+m\tau)$. For an infinitely long-time series the delay time τ should be almost arbitrary, but in practical cases the number of data points N is finite and τ has to be chosen very carefully. When choosing τ optimally $m<2d+1$ may be sufficient,[9] a fact which is of great importance in the context of systems modelling and prediction, where there is a need to restrict the number of degrees of freedom to the greatest possible extent. The approximation $\tilde{f}$ of the flow f then usually is a least-squares fit of the model's parameters:

$$dE=d\langle(\tilde{f}-f)^2\rangle=0 \quad (7)$$

which is equivalent to minimizing the (one-step) prediction error of the model $\tilde{f}$. $\langle\ \rangle$ denotes the average with the invariant density.

Instead of modelling the flow f globally using UPO's we here attempt to demonstrate that UPO's should be used directly to predict chaotic time series. The idea is to make use of the fact that an unstable periodic orbit α not only contributes to the static information on the attractor but furthermore represents the flow f, first, by providing to a given UPO point $\mathbf{x}^*_{i,\alpha}$ its successor $f(\mathbf{x}^*_{i,\alpha})=\mathbf{x}^*_{i+1,\alpha}$ and second, by determining the dynamics in the vicinity of the UPO point through the linearized flow given by the Jacobian $J(\mathbf{x}^*_{i,\alpha})=f'(\mathbf{x}^*_{i,\alpha})$.

The successor of a given point $\mathbf{x}_t$ therefore is determined by a nearby UPO point $\mathbf{x}^*_{i,\alpha}$ and the Jacobian, and we have as a first approximation

$$\mathbf{x}_{t+1}\simeq\tilde{f}(\mathbf{x}_t)=f'(\mathbf{x}^*_{i,\alpha})(\mathbf{x}^*_{i,\alpha}-\mathbf{x}_t)+\mathbf{x}^*_{i+1,\alpha} . \quad (8)$$

In the case that within a given distance ϵ of $\mathbf{x}_t$ there are several UPO points $\mathbf{x}^*_{i,\alpha}$ their contributions have to be weighted according to their instabilities and then averaged, and we obtain

$$\tilde{f}(\mathbf{x}_t)=\frac{\sum'_{i,\alpha}[f(\mathbf{x}^*_{i,\alpha})-f'(\mathbf{x}^*_{i,\alpha})(\mathbf{x}^*_{i,\alpha}-\mathbf{x}_t)]\exp[-K(\mathbf{x}^*_{\alpha})]}{\sum'_{i,\alpha}\exp[-K(\mathbf{x}^*_{\alpha})]} . \quad (9)$$

$\sum'$ denotes the sum restricted to UPO points $\mathbf{x}_{i,\alpha}$ for which $|\mathbf{x}_{i-\alpha}-\mathbf{x}_t|\leq\epsilon$. This makes formula (9) a local linear procedure for predicting an orbit in such a way that the unstable periodic orbits in the vicinity "guide" the trajectory.

In order to derive this result we make an ansatz

$$\tilde{f}(\mathbf{x})=f'(\mathbf{x})(\mathbf{x}-\mathbf{x}_t)+b \quad (10)$$

and minimize the one-step prediction error

$$E=\langle[f(\mathbf{x})-\tilde{f}(\mathbf{x})]^2\rangle_{\rho_\epsilon} . \quad (11)$$

Considering now only UPO points within a distance ϵ from $\mathbf{x}_t$ in the average $\langle\cdots\rangle_{\rho_\epsilon}$, that is

$$\langle\cdots\rangle_{\rho_\epsilon}=\int_{|\mathbf{x}-\mathbf{x}_t|<\epsilon}\rho_n(\mathbf{x})\cdots d\mathbf{x}\Big/\int_{|\mathbf{x}-\mathbf{x}_t|<\epsilon}\rho_n(\mathbf{x})d\mathbf{x} , \quad (12)$$

we finally have Eq. (9).

Figure 6 demonstrates the usefulness of UPO's from time series for predictions. A time series of $N=10^4$ scalar points of the Roessler attractor has been used to extract all UPO's of length $\tau/\tau_0\cong 5$ [Fig. 3(f)]. Then a continuation of this time series (10^4 points) has been used to calculate the prediction error $E^2=\langle(\tilde{f}^{\,t}-f^{\,t})^2\rangle$, which has been normalized to standard deviation σ: $F=E/\sigma$. In Fig. 6 we compare this result to a prediction technique[16] where in contrast to the method presented here *every* point within a certain neighborhood contributes to the forecast with equal probability. We chose the neighborhood to consist of $2m$ nearest points according to Ref. 18. While the one-step prediction error of this local linear prediction is small, this method may fail when iterated: In that case some predicted orbits fall off the attractor which leads to huge errors ($E\gg\sigma$) and restricts the forecast time strongly. In Ref. 18 this effect is discussed. There the suspicion is that such bad predictions depend essentially on the proper selectioon of neighbor-

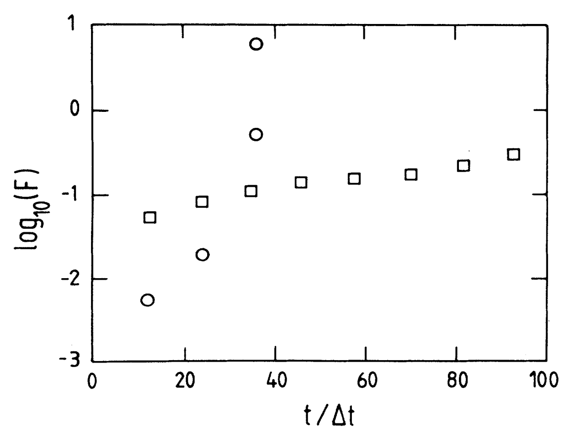

FIG. 6. Normalized prediction error F for iterated local linear prediction (see Ref. 16) (○) and UPO prediction according to Eq. (9) (□) from a time series of 10^4 points of the Roessler system with $M=4$, $\tau=4\Delta t$, $\Delta t=\pi/12$.

hoods, which clearly becomes better when the number of available points is increased. While the problem of choosing optimal neighborhoods has not been treated satisfactorily up to now, Fig. 6 demonstrates that this problem is solved for the Roessler system by our method: The predictions using UPO's get the local dynamics right everywhere. The fits of the Jacobians J by the orbit sequences seem to capture the stable and unstable manifolds in the sense that the predicted orbits are bound to the attractor. This fact makes prediction using unstable periodic orbits an attractive method.

IV. CONCLUSION

We have presented an appropriate procedure for extracting the most dominant unstable periodic orbits, together with their Jacobians, from a measured time series of chaotic time-continuous systems by determining the return times and the relevant scales from a moderate number of data points. As unstable periodic orbits contain the dynamics, they are appropriate for constructing models from time series and can be used directly for prediction. We have presented here a method which uses unstable periodic orbits for predictions and have demonstrated by application to the Roessler system that such predictions are superior to ones using the time series directly. We believe that extraction of UPO's from experimental data will become an important tool for determining the static and dynamic properties of chaotic experimental systems.

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[10]One sees immediately that $P_\epsilon(t)$ averaged over t is the well-known correlation integral $C(\epsilon)$, defined as

$$C(\epsilon)=\lim_{N\to\infty}\frac{1}{N}\sum_t P_\epsilon(t) .$$

[11]Since the correlation integral scales for small ϵ as $C(\epsilon)\propto\epsilon^{D_2}$, we expect $P_\epsilon(\tau^*)$ not to depend essentially on ϵ at possible return times τ^*; that is $P_\epsilon(\tau^*)\propto\epsilon^{\beta(\tau^*)}$ for $\epsilon\to 0$. Keeping P fixed $\epsilon^{-\beta}$ then is proportional to the probability of falling within a small radius ϵ. While such a scaling is observed numerically, the connection of the exponents β to the well-known scaling indices α and their spectra $f(\alpha)$ needs further investigation.

[12]This may shed light on the structure of $C(t)$. While the small ϵ_p represent possible returns (UPO's), the large ϵ_p contribute to the autocorrelation.

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