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Theory for correlation functions of processes driven by external colored noise

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We study steady-state correlation functions of nonlinear stochastic processes driven by external colored noise. We present a methodology that provides explicit expressions of correlation functions approximating simultaneously short- and long-time regimes. The non-Markov nature is reduced to an effective Markovian formulation, and the nonlinearities are treated systematically by means of double expansions in high and low frequencies. We also derive some exact expressions for the coefficients of these expansions for arbitrary noise by means of a generalization of projection-operator techniques.

I. INTRODUCTION

The analysis of steady-state correlation functions is a useful tool in the characterization of the noise acting on a system. An example of this fact is given by the dye laser. In a series of experimental[1–3] and theoretical papers[4–8] on the static and dynamic properties of a dye laser, it has been concluded on the necessity of modeling this system in terms of stochastic equations driven by external colored noise, instead of the usual internal white-noise assumption of the standard laser theory. In this context, a fundamental role has been played by the steady-state correlation function. A comparison between experimental results[2] and numerical results for the steady-state correlation function associated with white- and colored-noise models[5,8] has determined the colored character of the noise. Furthermore, the existence of a characteristic initial plateau of the correlation function, predicted as a consequence of the colored-noise assumption,[6] has been later corroborated experimentally.[3] Despite its interest, however, there are no analytical techniques available in the literature to deal with steady-state correlation functions for these problems in the complete time regime. Other dynamic properties that may depend strongly on the characteristics of the noise are the relaxation times of steady states and the escape times from unstable or metastable states. References 9 and 10 contain reviews of the role of external noise in different physical systems.

Our aim in this paper is to develop an analytical method to obtain steady-state correlation functions of nonlinear systems driven by colored noise, for both short- and long-time regimes. The calculation of these functions involve two different types of difficulties. The first one is inherent to the non-Markovian character of the process, due to the presence of colored noise. Unlike the Markovian case, the joint probability distribution, from which the two-time correlation function is obtained, obeys a different equation than the single probability distribution. In Refs. 11 and 12 equations for the joint probability distribution and for correlation functions associated with non-Markovian processes have been obtained. The second type of problems involved in the determination of non-Markovian correlation functions are also present for Markovian processes. They are related to the nonlinear character of the stochastic equations, giving rise, for instance, to infinite hierarchies of coupled equations. Apart from a reduced number of exactly solvable problems, approximate methods have been explored, such as the Stratonovich decoupling ansatz[13] and the continued-fraction-expansion method.[14] Both methods give good qualitative results in situations when there are not very different time scales involved, but they only give the short-time behavior in other situations, such as near instability points. Recently, Nadler and Schulten[15] have proposed a unified scheme, based on the introduction of a complementary approach of that of the continued fraction expansion. This methodology contains information on both short- and long-time scales simultaneously, and has been applied in Ref. 16 to the calculation of the steady-state correlation functions in the complete time regime of nonlinear processes driven by multiplicative

Gaussian white noise. It has been obtained an excellent agreement in the comparison with exact results.[16]

In Sec. II we discuss a generalization of the techniques studied in Refs. 15 and 16 to Gaussian colored noise in the framework of a first-order approximation in the correlation time of the noise τ. An application of this method in the context of laser systems will be presented elsewhere.[17] The method contains two main ingredients, one dealing with the non-Markovian nature and the other with the nonlinearity of the process. The first one established the connection between the non-Markovian correlation function, $C(t)$, and effective Markovian correlation functions $C_M(t)$. A similar philosophy was used in Ref. 18, but the result was only valid for the short-time regime. A derivation of an expression of $C(t)$ in the context of an expansion to first order in the intensity of the noise was done in Ref. 19. These expressions were first used for the obtention and discussion of relaxation times in non-Markovian processes in Refs. 18 and 20. However, its practical usefulness for the explicit calculation of correlation functions was quite limited at that time due to the lack of appropriate techniques to deal even with Markovian correlation functions in the complete time regime.

The expression for $C(t)$ contains by construction some interesting generic features associated with the non-Markovian character which make it very convenient as a starting point of any theoretical analysis. For instance, in the short-time regime, it contains the characteristic initial plateau of any non-Markovian correlation function. On the other hand, for time scales much larger than the correlation time of the noise, the result corresponds to the one of a Markovian process with an effective initial distribution. In this way, the result provides an implicit calculation of effective initial distribution as it is done from a different method in Ref. 21.

Once the problem has been reduced to effective Markovian correlation functions, the second step is the application of the method studied in Refs. 15 and 16 for the calculation of $C_M(t)$, that we will call the double-expansion method. It has been obtained in Ref. 16 that this procedure gives excellent quantitative results for these functions for all time regimes, even very near instability points where very different time scales coexist. In this way, we can determine systematically, to any degree of approximation, the complete time behavior of $C_M(t)$, and therefore, that of $C(t)$ as far as the small τ approximation holds.

The method is based on two simultaneous expansions of the Laplace transform of $C_M(t)$ in high and low frequencies. The coefficients of these two separate expansions are related to the derivatives at the origin of times and to the so-called relaxation moments of $C_M(t)$, respectively. The zero-order relaxation moment is the usual relaxation time, and like the other relaxation moments contains information on a global or large time scale, related to the distribution of area under the curve of $C_M(t)$. The goal of the method is to use information of both expansions simultaneously to approximate $C_M(t)$ both in the short- and long-time scale. The interpolation is done by means of Padé approximants, which provide systematic approximations of the $C_M(t)$ as a sum of exponentials. The different orders of approximation of the method are related to the conditions on the derivatives at the origin and the relaxation moments that one imposes to be satisfied by the Markovian correlation function $C_M(t)$. The usefulness of the method relies on the possibility of computing the coefficients of the expansions. This can be done exactly for Markovian processes[15,16] and for the non-Markovian case only the relaxation time has been obtained to first order in τ in Ref. 20. The generalization of this result to the higher-order relaxation moments is presented in Appendix A.

In Sec. III we consider some more formal aspects of the theory of non-Markovian correlation functions. Until the present time most of the standard techniques were restricted to the white-noise case. Here, we present a generalization of the projection-operator technique to deal with completely arbitrary noises. Our starting point is an integrodifferential equation for the joint probability distribution valid for any noise. In Appendix B we give some details of the derivation of this equation. In its application to the Ornstein-Uhlenbeck noise and to first order in τ, it reduces to the equation considered as the starting point in the preceding section.

One remarkable result of the generalization of the projection-operator technique is the formal exact solution of the coefficients of the two frequency expansions for arbitrary noise. However, their combination in the spirit of the double-expansion method for the calculation of correlation functions has to be explored, in principle, for each particular situation.

II. THE METHOD

A. An expression for non-Markovian correlation functions

In this section we restrict ourselves to the study of non-Markovian processes described by stochastic differential equations of the following type:

$$\dot{q}(t)=v(q)+g(q)\xi(t)\ , \tag{2.1}$$

where v and g are general nonlinear functions of q and $\xi(t)$ is the Ornstein-Uhlenbech process with zero mean and correlation function given by

$$\langle \xi(t)\xi(t')\rangle=\frac{D}{\tau}\exp\left[-\frac{|t-t'|}{\tau}\right]. \tag{2.2}$$

D and τ are the intensity and the correlation time of the noise, respectively. We are interested in the calculation of the steady-state correlation function $C(t)$ defined by

$$C(t)=\lim_{t'\to\infty}\frac{\langle \delta q(t+t')\delta q(t')\rangle}{\langle(\delta q)^2\rangle} =\frac{1}{\langle(\delta q)^2\rangle}\int\int dq\,dq'P(q,q';t)\delta q\delta q'\ , \tag{2.3}$$

where $\delta q(t)=q(t)-\langle q\rangle$. $\langle q\rangle$ and $\langle q^2\rangle$ are the steady-state moments. $P(q,q';t)\equiv\lim_{t'\to\infty}P(q,t+t';q',t')$ is the stationary joint probability distribution. The equation obeyed by $P(q,q';t)$, when q is a non-Markovian process, has been studied in Ref. 11. To first order in τ the equation is

$$\frac{\partial}{\partial t}P(q,q';t)=[L_q(\tau)+D\exp(-t/\tau)L_{qq'}(\tau)]P(q,q';t) \tag{2.4}$$

where

$$L_q(\tau)=-\frac{\partial}{\partial q}v(q)+D\frac{\partial}{\partial q}g(q)\frac{\partial}{\partial q}h(q)\ , \tag{2.5}$$

$$L_{qq'}(\tau)=\frac{\partial}{\partial q}g(q)\frac{\partial}{\partial q'}h(q')\ , \tag{2.6}$$

and $h(q)=g(q)-\tau[v(q)g'(q)-v'(q)g(q)]$. The primes on g and v denote derivatives with respect to q. The Fokker-Planck operator $L_q(\tau)$ appears, in the same approximation, in the Fokker-Planck equation for the single probability distribution $P(q,t)$. In the limit $\tau\to 0$, the second term of Eq. (2.4) disappears and the equations of $P(q,q';t)$ and $P(q,t)$ coincide. Integrating Eq. (2.4) we obtain to first order in τ

$$P(q,q';t)=e^{L_q(\tau)}[1+\tau D(1-e^{-t/\tau})L_{qq'}(\tau)]P(q,q';0) \tag{2.7}$$

with $P(q,q';t=0)=P_{\rm st}(q)\delta(q-q')$. $P_{\rm st}(q)$ is the stationary distribution. Introducing the result (2.7) in Eq. (2.3), we get

$$C(t)=C_M(t)+\tau(1-e^{-t/\tau})\gamma_0 C_M^L(t)+O(\tau^2)\ , \tag{2.8}$$

where

$$\gamma_0=-\frac{\langle \delta q L_q^\dagger \delta q\rangle}{\langle(\delta q)^2\rangle}$$

and

$$C_M(t)=\frac{\langle \delta q(t)\delta q\rangle_M}{\langle(\delta q)^2\rangle}\ , \tag{2.9}$$

$$C_M^L(t)=\frac{\langle \delta q(t)L_q^\dagger\delta q\rangle_M}{\langle \delta q L_q^\dagger \delta q\rangle}\ . \tag{2.10}$$

$C_M(t)$ and $C_M^L(t)$ are correlation functions associated with an effective Markovian process with Fokker-Planck operator $L_q(\tau)$. $L_q^\dagger(\tau)$ is the adjoint Fokker-Planck operator. For a Markovian process we can write the average $\langle\ \rangle_M$ as

$$C_M(t)=\frac{1}{\langle(\delta q)^2\rangle}\int dq P_{\rm st}(q)\delta q e^{L_q^\dagger t}\delta q \tag{2.11}$$

and in a similar way for $C_M^L(t)$ [see Eq. (A3)].

The interest of Eq. (2.8) is that the non-Markovian correlation function $C(t)$ is expressed, to first order in τ, in terms of effective Markovian correlation functions $C_M(t)$ and $C_M^L(t)$. A similar philosophy was used in Ref. 18 in the context of a continued-fraction-expansion method for the short-time regime. An expression of $C(t)$ from the perturbative solution of $P(q,q';t)$ to order D was first derived in Ref. 19. To first order in τ, this result reproduces Eq. (2.8), but no analytical techniques were available at that time for a systematic calculation of the Markovian correlation functions in the complete time regime. Our purpose in this section is precisely to apply the double-expansion method to the Markovian correlation functions, $C_M(t)$ and $C_M^L(t)$, in order to get the complete time behavior of $C(t)$. But, before that, some remarks on the general properties of Eq. (2.8) will be useful for the practical application of the method.

First, Eq. (2.8) implies the characteristic initial plateau of any non-Markovian correlation function Ref. 6, thanks to the presence of the term proportional to $e^{-t/\tau}$:

$$\left.\frac{d}{dt}C(t)\right|_{t=0}=\frac{d}{dt}C_M(t)+e^{-t/\tau}\gamma_0 C_M^L(t)+\tau(1-e^{-t/\tau})\gamma_0\left.\frac{d}{dt}C_M^L(t)\right|_{t=0}=0\ . \tag{2.12}$$

Here we have used Eq. (A14) for the derivatives at $t=0$ of Markovian correlation functions.

A more global characterization of $C(t)$ is given by the relaxation time

$$T^0=\int_0^\infty dt C(t)=T_M^0(\tau)+\tau\ . \tag{2.13}$$

A detailed discussion of result (2.13) is given in Ref. 20 where a closed expression for T_M^0, which has a nontrivial dependence on τ, is also derived. As we show in Appendix A, a similar result can be obtained for the so-called relaxation moments

$$T^k=\int_0^\infty dt C(t)t^k=T_M^k(\tau)+\tau k T_M^{k-1}\ . \tag{2.14}$$

T_M^k is the kth order relaxation moment associated with $C_M(t)$. The hierarchy of relaxation moments provides a global characterization in terms of the distribution of area under the curve $C(t)$.

On the other hand, for $t\gg\tau$ the term proportional to $e^{-t/\tau}$ in Eq. (2.8) is negligible and we obtain

$$C(t)=C_M(t)+\tau\gamma_0 C_M^L(t)\ . \tag{2.15}$$

According to (2.11), the result (2.15) can be reinterpreted as the correlation function corresponding to a Markovian process with an effective initial distribution $(1-\tau L_q^\dagger)P_{\rm st}(q)$, instead of $P_{\rm st}(q)$. The calculation of effective initial distributions of non-Markovian processes has been investigated in Ref. 21 by means of a different method. In particular, in the application to Brownian motion, our result (2.15) coincides with the one obtained in Ref. 21. In this way, Eq. (2.15) provides the explicit calculation of effective distributions for the general case.

B. The double-expansion method

As the second step in our procedure we apply the double-expansion method to calculate the Markovian correlation functions defined by Eqs. (2.9) and (2.10). The method has been described in detail in Refs. 15 and 16. The basic idea is to consider two expansions of the Laplace transform of any Markovian correlation function both for high and low frequencies, w:

$$C_M(w)=\frac{1}{w}\sum_{k=0}^{\infty}\mu_M^k\left[\frac{1}{w}\right]^k\ , \tag{2.16}$$

and

$$C_M(w)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}T_M^k w^k , \qquad (2.17)$$

where the coefficients of the expansions are related to the derivatives at $t=0$, μ_M^k, Eq. (A14), and the relaxation moments, T_M^k, Eq. (A4). Equation (2.16) provides information on the short-time behavior of $C_M(t)$. This expansion is the one involved in the usual continued-fraction-expansion method. On the other hand, Eq. (2.17) is related to a more global or long-time characterization of $C_M(t)$. The goal of the method is to obtain a systematic approximation for $C_M(t)$ which combines simultaneously information from both expansions. A convenient way to do so is to interpolate them by means of Padé approximants.

An approximation of order N will then correspond to an ansatz of the form

$$C_M^N(w)=\sum_{n=1}^{N}\frac{a_n}{w+\lambda_n} , \qquad (2.18)$$

where the coefficients a_n and λ_n have to be determined by imposing that the expansions in w and $1/w$ of (2.18) coincide up to the desired order with (2.16) and (2.17). The explicit approximation for $C_M(t)$ is then a superposition of N exponentials of the form

$$C_M^N(t)=\sum_{n=1}^{N}a_n e^{-\lambda_n t} . \qquad (2.19)$$

In practice, the calculation of $C_M(t)$ starts by assuming an expression of the type (2.19) and imposing the normalization condition, $C_M^N(0)=1$. Then, one asks for the conditions on the coefficients such that n derivatives and m relaxation moments are contained exactly, with $n+m+1=2N$. These conditions read[15,16]

$$\sum_{i=1}^{N}a_i(\lambda_i)^k=(-1)^k\mu_M^k;\quad k=0,1,\ldots,n , \qquad (2.20)$$

$$\sum_{i=1}^{N}a_i(\lambda_i)^{-k-1}=\frac{T_M^k}{k!};\quad k=0,1,\ldots,m-1 . \qquad (2.21)$$

The coefficients μ_M^k and T_M^k are known exactly for a generic Markovian process defined by the operator $L_q^{\dagger}(\tau)$. The explicit expressions are given by Eqs. (A4)–(A7) and (A14).

C. Discussion of different orders of approximation

Here we discuss some different approximations of $C(t)$. To do so, we make use of Eqs. (2.8) and (2.12)–(2.15) and the results of Sec. II A.

From Eqs. (2.12)–(2.15) we observe that $C_M(t)$ is important both in the small- and large-time scales. However, $C_M^L(t)$ has no influence in the short-time regime and only contributes essentially for long times. Then, in the calculation of the low-order approximations of the double-expansion method, a good criterion will be to consider conditions both on derivatives at the origin and relaxation moments for $C_M(t)$ and mainly on relaxation moments for $C_M^L(t)$.

At the lowest order, imposing the conditions on the first derivative of $C_M(t)$ and the relaxation time for $C_M^L(t)$, we recognize the Stratonovich decoupling ansatz Ref. 13:

$$C(t)=e^{-\gamma_0 t}+\tau(1-e^{-t/\tau})\gamma_0 e^{-\gamma_0 t}+O(\tau^2) . \qquad (2.22)$$

This approximation contains only one exact derivative $C'(0)=0$, and approximates the relaxation time by $T^0=\gamma_0^{-1}+\tau$ instead of the correct one to order τ, given by Eq. (2.13).

The next simplest approximation, with the right relaxation time of $C(t)$ to order τ (2.13) and the same exact derivative (2.12), will have the form

$$C(t)=(a_1e^{-\lambda_1 t}+a_2e^{-\lambda_2 t})+\tau(1-e^{-t/\tau})\gamma_0 e^{-\gamma_0 t}+O(\tau^2) . \qquad (2.23)$$

In this case, to determine $a_1,a_2,\lambda_1,\lambda_2$ one imposes the conditions on one derivative and two relaxation moments of $C_M(t)$ and the relaxation time of $C_M^L(t)$. In this case the next relaxation moment of $C(t)$ is approximated by $T^1=T_M^1+\tau\gamma_0^{-1}$ instead of the right value (2.14), $T^1=T_M^1+\tau T_M^0$.

III. PROJECTION-OPERATOR TECHNIQUE FOR NON-MARKOVIAN PROCESSES

In this section we derive some exact expressions for correlation functions by means of projection-operator techniques. We present a generalization of the standard technique capable to deal with any noise. As in the usual formulation,[14,15] one obtains high- and low-frequency expansions of the Laplace transform of the correlation function. These two independent expansions give information on the short- and long-time behavior of the correlation function, respectively. The coefficients of the expansions, related to the derivatives at the origin and the relaxation moments, are given in terms of the moments of projection operators. However, it is necessary to remark that, in general, a unified treatment of the two expansions in order to approximate the complete time regime, in the same spirit of that of Nadler and Schulten,[15,16] should be explored for each particular situation. In this aspect and for the Ornstein Uhlenbeck noise with small τ, our strategy has been to search first for an expression of $C(t)$ which contains as much physical information as possible on the behavior for the short- and long-time scales and then apply the unified procedure to the calculation of the remaining Markovian correlation functions, for which the method has been checked and gives excellent results.[15,16] Nevertheless, the exact calculation of the coefficients of the two expansions in the general case is interesting by itself and will be addressed below.

The starting point of this derivation is an integrodifferential equation for the joint probability distribution valid for any noise. This equation has been introduced in Ref. 12 and is adapted to our purposes in Appendix B. In this appendix we show that the equation for the two point probability density can be put in Laplace transform, when expliciting the noise, as

$$wP(q,q';w)-P_{st}(q)\delta(q-q') = \hat{M}(w)P(q,q';w)+S(q,q';w) , \quad (3.1)$$

where $\hat{M}(w)$ is an operator and $S(q,q';w)$ a function (see Appendix B). In the white-noise case $\hat{M}$ is independent of w and $S(q,q';w)$ is zero. This equation is very adequate for using the projection-operator technique in the obtention of expansions for high and low frequencies. In fact, the standard technique[14] starts also with a Laplace transformed equation for the Markovian case. Here, we generalize it to the non-Markovian case.

A. High-frequency expansion

First, we consider a high-frequency expansion. In the Markovian case the continued fraction expansion is obtained by introducing an operator which projects in the subspace associated with the initial state,[14] in our case the stationary state. In the non-Markovian case the method is similar with changes in the projection space and with an operator depending on w.

Let us define a scalar product (F,G) and a projection operator as follows:

$$(F,G)=\int dq\, dq' F(q,q')G(q,q') , \quad (3.2)$$

$$\Pi F=(v,F)\frac{u}{(v,u)} , \quad (3.3)$$

where $u=P_{st}(q)\delta(q-q')+S(q,q';w)$ is the vector defining the projection space and v is the dynamical variable of interest. At this point, we let v be undetermined. Note that in a Markovian case $S(q,q';w)=0$, then u defines the initial stationary condition. The adjoint operator of Π defined as

$$F\Pi^{\dagger}=\frac{v}{(v,u)}(F,u) \quad (3.4)$$

is also a projection operator. Now, as it is usual in projection techniques we apply Π and $(1-\Pi)$ to Eq. (3.1) obtaining two coupled equations. By solving the second equation and substituting into the first one, we obtain

$$w\Pi P=u+\Pi M\Pi P+\Pi M(1-\Pi)[w-(1-\Pi)M]^{-1}(1-\Pi)M\Pi P . \quad (3.5)$$

Taking the scalar product with v in both parts of this equation and rearranging, we get

$$(v,P)=\frac{(v,u)}{w-(v,Mu)/(v,u)-[1/(v,u)](vM^{\dagger}(1-\Pi^{\dagger}),[w-(1-\Pi)M]^{-1}(1-\Pi)Mu)} . \quad (3.6)$$

We can proceed now introducing new functions and projectors as

$$v_1=vM^{\dagger}(1-\Pi^{\dagger}) , \quad (3.7a)$$

$$u_1=(1-\Pi)Mu , \quad (3.7b)$$

$$\Pi_1=(v_1,0)\frac{u_1}{(v_1,u_1)} , \quad (3.7c)$$

$$M_1=(1-\Pi_1)M . \quad (3.7d)$$

Iterating this procedure we get the following result for (v,P):

$$(v,P)=\cfrac{(v,u)}{w-\cfrac{(v,Mu)}{(v,u)}-\cfrac{(v_1,u_1)}{w-\cfrac{(v_1,M_1u_1)}{(v_1,u_1)}\cdots\cfrac{\cfrac{(v_i,u_i)}{(v_{i-1},u_{i-1})}}{w-\cfrac{(v_i,M_iu_i)}{(v_i,u_i)}-\cdots}}} . \quad (3.8)$$

This equation is valid for any dynamical variable $v(q,q')$. To calculate correlation functions we take $v(q,q')=(qq'/\langle q^2\rangle)$, and obtain

$$C(w)=\cfrac{1+S_0(w)}{w-\alpha(w)-\cfrac{K_1(w)}{w-\alpha_1(w)-\cdots\cfrac{K_i(w)}{w-\alpha_i(w)-\cdots}}} \quad (3.9)$$

with

$$\alpha_i(w)=\frac{(v_i,M_i u_i)}{(v_i,u_i)} , \tag{3.10a}$$

$$K_i(w)=\frac{(v_{i-1},u_{i-1})}{(v_i,u_i)} , \tag{3.10b}$$

$$S_0(w)=\frac{\langle qq'S(q,q';w)\rangle}{\langle q^2\rangle} . \tag{3.10c}$$

An equivalent description of (3.9) in terms of the variable t is

$$\frac{dC}{dt}=\alpha^M C(t)+K_1^M\int_0^t dt'C_1(t-t')C(t')+\int_0^t dt'\alpha^w(t-t')C(t')+\int_0^t dt'K_1^w(t-t')\int_0^{t'}dt''C_1(t'-t'')C(t'')+S_0(t) , \tag{3.11}$$

$$\frac{dC_1}{dt}=\alpha_1^M C_1(t)+K_2^M\int_0^t dt'C_2(t-t')C_1(t')+\int_0^t dt'\alpha_1^w(t-t')C_1(t')+\int_0^t dt'K_2^w(t-t')\int_0^{t'}dt''C_2(t'-t'')C_1(t'') \cdots ,$$

where we have split α_i and K_i in contributions which do and do not depend on the frequency:

$$\alpha_i(t)=\mathcal{L}^{-1}\{\alpha_i(w)\}=\alpha_i^M+\alpha_i^w(t) , \tag{3.12}$$
$$K_i(t)=\mathcal{L}^{-1}\{K_i(w)\}=K_i^M+K_i^w(t) .$$

The initial conditions of (3.11) are $C(0)=C_1(0)=\cdots=1$.

From (3.9) and (3.11) it is easy to see that an approximation of order $n\,[C^n(t)=0$ or $K_n=\alpha_n=0]$ leads to an approximate correlation function which reproduces exactly the derivatives in $t=0$ up to order $n+1$.[22] For the first derivatives we obtain

$$\mu^1=\alpha^M+S_0(0) ,$$
$$\mu^2=\alpha^M\mu^1+K_1^M+\alpha^w(0)+S_0^1(0) , \tag{3.13}$$
$$\mu^3=\alpha^M\mu^2+[K_1^M+\alpha^w(0)]\mu^1+K_1^M\alpha_1^M+\alpha^w(0)+K_1^w(0) .$$

In the Markovian case $S_0(0)=\alpha_i^w(0)=K_i^w(0)=0$ and (3.13) gives known results.[14]

B. Low-frequency expansion

Now, we consider the low frequencies. We use the same procedure that in the previous expansion but now using as the starting equation

$$P(q,q';w)+\hat{M}^{-1}(w)S(q,q';w)=w\hat{M}^{-1}P(q,q';w)-\hat{M}^{-1}P_{st}(q)\delta(q-q') \tag{3.14}$$

instead of (3.1). This equation is obtained from the application of $\hat{M}^{-1}$ to (3.1). Now we define $u=-\hat{M}^{-1}P_{st}(q)\delta(q-q')-\hat{M}^{-1}S(q,q';w)$. The scalar product and projector are the same than for the high-frequency expansion. Proceeding as before we obtain for (v,P)

$$(v,P)=\cfrac{(v,u)}{1-w\cfrac{(v,M^{-1}u)}{(v,u)}-\cfrac{w^2\cfrac{(v_1,u_1)}{(v,u)}}{1-w\cfrac{(v_1,M_1^{-1}u_1)}{(v_1,u_1)}-\cdots}} , \tag{3.15}$$

and for the correlations, taking $v=(qq'/\langle q^2\rangle$, we have

$$C(w)=\cfrac{\beta(w)}{1-w\theta(w)-\cfrac{w^2\cfrac{\beta_1(w)}{\beta(w)}}{1-w\theta_1(w)-\cdots}} , \tag{3.16}$$

where

$$\beta_i(w)=(v_i,u_i) , \tag{3.17a}$$

$$\theta_i(w)=\frac{(v_i,M_i^{-1}u_i)}{(v_i,u_i)} . \tag{3.17b}$$

Now it is also easy to see that the kth order approximant reproduces exactly the relation moments T^k. For the first moments $[T^k=(-1)^k(\partial^k/\partial w^k)C(w)|_{w=0}]$ we obtain

$$T^0=\beta(0) , \tag{3.18a}$$
$$T^1=-\beta_1(0)\beta'(0)-\beta(0)\theta(0) , \tag{3.18b}$$
$$T^2=\beta''(0)+2\theta(0)\alpha'(0)+\alpha(0)\theta'(0)+2\beta(0)\theta^2(0)+\beta_1(0)[\alpha(0)+\theta'(0)] . \tag{3.18c}$$

In the white-noise case we have $\hat{M}$ independent of w and $S(q,q';w)=0$. It can be seen that in this case (3.18) reproduces the known results for T^k.

Equations (3.13) and (3.18) give the exact expressions for the first derivatives and relaxation moments valid for any noise. They are essential ingredients in the application of the projection-operator technique.

C. Exponentially correlated noise

Now, we apply the results of this section to an exponentially correlated noise to first order in D and τ. Then, we compare these results with the ones obtained in Sec. II for the Ornstein-Uhlenbeck noise.

For an exponentially correlated noise, the operator $\hat{M}(w)$ and the function $S(w)$ are given to first order in D by (see Appendix B):

$$\hat{M}(w)=-\frac{\partial}{\partial q}v(q)+D/\tau\frac{\partial}{\partial q}g(q)\left[\frac{1}{\tau}+w+\frac{\partial}{\partial q}v(q)\right]^{-1}\frac{\partial}{\partial q}g(q)\,, \tag{3.19}$$

$$S(w)=\frac{\partial}{\partial q}g(q)\left[\frac{1}{\tau}+w+\frac{\partial}{\partial q}v(q)\right]^{-1}\times\delta(q-q')\frac{v(q)}{g(q)}P_{\rm st}(q)\,. \tag{3.20}$$

From Eqs. (3.19) and (3.20), the first derivative of $C(t)$ at $t=0$ can be easily calculated to first order in D. By using (3.10) and (3.19), we get

$$\alpha^M=\frac{1}{\langle q^2\rangle}\left[qq',-\frac{\partial}{\partial q}v(q)\delta(q-q')P_{\rm st}(q)\right]=\frac{\langle qv(q)\rangle}{\langle q^2\rangle}\,. \tag{3.21}$$

By antitransforming Eq. (3.20) we obtain

$$S(t)=\frac{\partial}{\partial q}g(q)\exp\left[-\left[1/\tau+w+\frac{\partial}{\partial q}v(q)\right]t\right]\times\delta(q-q')\frac{v(q)}{g(q)}P_{\rm st}(q)\,. \tag{3.22}$$

Then,

$$S_0(t=0)=\frac{\langle qq'S(q,q',t=0)\rangle}{\langle q^2\rangle}=-\frac{\langle qv(q)\rangle}{\langle q^2\rangle}\,. \tag{3.23}$$

This result implies that $\mu^1=\alpha^\mu+S_0(t=0)=0$, in agreement with Eq. (2.12). In fact, this is an exact result for colored noise valid to any order in D.[12]

Now, we obtain the relaxation time T^0 to first order in D and τ. First, we consider the following expansion:

$$\frac{1/\tau}{1/\tau+w+(\partial/\partial q)v(q)}=1-\tau\left[w+\frac{\partial}{\partial q}v(q)\right]+O(\tau^2) \tag{3.24}$$

in Eqs. (3.19) and (3.20). Now, using the relation given by

$$\hat{M}(w=0)=L_q(\tau)-D\tau\frac{\partial}{\partial q}g\frac{\partial}{\partial q}g\frac{\partial}{\partial q}v \tag{3.25}$$

in Eqs. (3.17) and (3.18), we get T^0 to first in D and τ:

$$T^0=\beta(0)=-\frac{1}{\langle q^2\rangle}(qq',L_q^{-1}P_{\rm st}(q)\delta(q-q'))+\frac{D\tau}{\langle q^2\rangle}\left[qq',L_q^{-2}\frac{\partial}{\partial q}g\frac{\partial}{\partial q}g\frac{\partial}{\partial q}vP_{\rm st}(q)\delta(q-q')\right]-\frac{\tau}{\langle q^2\rangle}\left[qq',\hat{M}(w=0)^{-1}\frac{\partial}{\partial q}v(q)P_{\rm st}(q)\delta(q-q')\right]\simeq T_M^0+\tau \tag{3.26}$$

which reproduces Eq. (2.13).

Finally, we consider the relationship between the stationary joint probability distributions given by Eqs. (3.1) and (2.4). By substituting Eqs. (3.19) and (3.20) in Eq. (3.1), iterating and using (B6), we obtain to first order in D and τ:

$$wP-\delta(q-q')P_{\rm st}(q)=L_q(\tau)P+\hat{S}_\tau\delta(q-q')P_{\rm st}(q)\,, \tag{3.27}$$

where

$$\hat{S}_\tau=D\tau\frac{\partial}{\partial q}g(q)\frac{\partial}{\partial q'}g(q')\,. \tag{3.28}$$

For times $t\gg\tau$ and to first order in τ, we have

$$D\exp(-t/\tau)L_{qq'}(\tau)P(q,q';t)\sim\tau D\delta(t)\frac{\partial}{\partial q}g(q)\frac{\partial}{\partial q'}g(q')\delta(q-q')P_{\rm st}(q)\,. \tag{3.29}$$

Therefore, in this condition Eqs. (2.4) and (3.1) coincide. They correspond to a Markovian process with a Fokker-Planck operator given by $L_q(\tau)$, Eq. (2.15), with an effective initial distribution given by

$$\left[1-D\tau\frac{\partial}{\partial q}g(q)\frac{\partial}{\partial q}g(q)\right]P_{\rm st}(q) \tag{3.30}$$

instead of $P_{\rm st}(q)$. The corresponding steady-state correlation function to first order in τ, associated with this Markovian process is given by Eq.(2.15).

IV. SUMMARY

In this paper we have focused on methodological aspects of the treatment of non-Markovian correlation functions, both from a practical and a formal point of view. First, we have developed a scheme that provides explicit expressions of correlation functions for both short- and long-time regimes. The nonlinearities are treated in a systematic way by means of the double-expansion method, and the non-Markovian nature is reduced to an effective Markovian formulation in the context of the usual small-τ approximation. This approach may be particularly useful in the application to the study of physical systems driven by colored noise, like dye lasers, as will be shown in future work.[17]

From the more formal point of view and in order to formulate the problem of non-Markovian correlation functions for arbitrary values of τ, we have generalized the projection-operator technique for an arbitrary noise. This has allowed us to obtain the exact expressions for the coefficients of the high- and low-frequency expansions of the Laplace transform of the correlation function.

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APPENDIX A

Here we derive the expression for the relaxation moments of the non-Markovian correlation function $C(t)$ to first order in τ. According to (2.8) and (2.14), the kth relaxation moment T^k of $C(t)$ will be given by

$$T^k=T_M^k(\tau)+\gamma_0\tau\int_0^\infty dt\, t^k(1-e^{t/\tau})C_M^L(t)\ . \qquad (A1)$$

The term proportional to $e^{-t/\tau}$ gives rise to contributions of higher-order, so to first order in τ (A1) reduces to

$$T^k=T_M^k(\tau)+\gamma_0\tau T_{ML}^k\ , \qquad (A2)$$

where T_{ML}^k is the relaxation moment of order k corresponding to $C_M^L(t)$.

Now we want to relate T_{ML}^k to the relaxation moments of $C_M(t)$. To do so let us consider a steady-state correlation function of a Markovian process defined by $L_q(\tau)$ of the general form

$$C_{12}(t)=\frac{\langle f_1(q(t))f_2(q)\rangle_M-\langle f_1\rangle\langle f_2\rangle}{\langle f_1f_2\rangle-\langle f_1\rangle\langle f_2\rangle}$$
$$=\frac{1}{\langle f_1f_2\rangle-\langle f_1\rangle\langle f_2\rangle}\int_a^b dq\,[f_2(q)-\langle f_2\rangle]e^{L_q t}[f_1(q)-\langle f_1\rangle]P_{\rm st}(q)\ , \qquad (A3)$$

where a and b are the natural boundaries of the process $q(t)$. The exact expression for the relaxation moments can be derived in a similar way than that of Ref. 16 and reads

$$T_{12}^k=\frac{(-1)^k k!}{\langle f_1f_2\rangle-\langle f_1\rangle\langle f_2\rangle}\int_a^b dq\frac{G_0^{(1)}(q)G_k^{(2)}(q)}{Dg(q)h(q)P_{\rm st}(q)}\ , \qquad (A4)$$

where

$$G_0^{(1)}(q)=-\int_a^q dq'[f_1(q')-\langle f_1\rangle]P_{\rm st}(q') \qquad (A5)$$

and $G_k^{(2)}$ are defined by the recurrence

$$G_k^{(2)}(q)=\int_a^q dq'\left[\int_a^{q'}dq''\frac{G_{k-1}^{(2)}}{Dg\,hP_{\rm st}}-\left\langle\int_a^q dq''\frac{G_{k-1}^{(2)}}{Dg\,hP_{\rm st}}\right\rangle\right]P_{\rm st}(q') \qquad (A6)$$

with

$$G_0^{(2)}(q)=-\int_a^q dq'[f_2(q')-\langle f_2\rangle]P_{\rm st}(q')\ . \qquad (A7)$$

For the particular case of $C_M(t)$ appearing in the text, the f_i are given by (A4)–(A7) with $f_1=f_2=q$. Note that there is a τ dependence contained in $P_{\rm st}(q)$ and $h(q)$. However, in the following we can put $\tau=0$ because we neglect higher-order terms in (A2). Equations (A4)–(A7) then reduce to those of Ref. 16 for white noise $[h(q)=g(q)]$. In our case the functions f_i are

$$f_1(q)=q\ , \qquad (A8a)$$

$$f_2(q)=L_q^\dagger q=Dg(q)g'(q)+v(q)\ . \qquad (A8b)$$

From the form of L_q (2.5) one has

$$G_0^{(2)}(q)=-\int_a^q dq'(Dg\,g'+v)P_{\rm st}=-Dg^2(q)P_{\rm st}(q) \qquad (A9)$$

so we can write

$$G_1^{(2)}(q)=-\int_a^q dq'\left[\int_a^{q'}dq''-\left\langle\int_a^q dq''\right\rangle\right]P_{\rm st}(q')=G_0^{(1)}(q)\ . \qquad (A10)$$

Using now (A10) and proceeding by induction one can prove that for Eqs. (A8)

$$G_k^{(2)}(q)=G_{k-1}^{(1)}(q) \qquad (A11)$$

so that (A4) in this case reduces to

$$T_{12}^{k}=\frac{(-1)^k k!}{\langle \delta q L^{\dagger}\delta q\rangle}\int_a^b dq\frac{G_0^{(1)}G_{k-1}^{(1)}}{Dg^2P_{st}}=\frac{k}{\gamma_0}T_M^{k-1} . \tag{A12}$$

Substituting (A12) into (A2) we get the general result

$$T^k=T_M^k(\tau)+\tau k T_M^{k-1} . \tag{A13}$$

The T_M^k for $k>0$ are given by (A4)–(A7). Notice that in the first term of (A13) one has to keep the nontrivial τ dependence coming from $P_{st}(q)$ and $h(q)$. The particular case of the relaxation time[20] is given by (2.12) and (A4)–(A6).

For completeness let us include here the general expression for the derivatives of (A3) at $t=0$, which reads

$$\frac{d^n}{dt^n}C_{12}(t)=\frac{\langle [L^{\dagger n}f_1(q)](t)f_2(q)\rangle}{\langle f_1f_2\rangle-\langle f_1\rangle\langle f_2\rangle} . \tag{A14}$$

APPENDIX B

In Ref. 12 an integrodifferential equation for the two point probability density is derived. This probability is calculated by averaging the two point δ function $\delta(q-q(t))\delta(q'-q(0))$ over stationary realizations of the noise. A projection-operator method is used for calculating this average. The equation obtained valid for any noise is given by

$$\frac{\partial}{\partial t}P(q,q';t)=-\frac{\partial}{\partial q}v(q)P(q,q';s)+\frac{\partial}{\partial q}g(q)e^{-t(\partial v/\partial q)}\int_0^t ds\langle \xi(t)U(t,s)\xi(s)\rangle e^{s(\partial v/\partial q)}\frac{\partial}{\partial q}g(q)P(q,q';s)$$
$$+\frac{\partial}{\partial q}g(q)e^{-t(\partial v/\partial q)}\int_{-\infty}^0 ds\langle \xi(t)U(t,0)\delta(q-q')U(0,s)\xi(s)\rangle e^{s(\partial v/\partial q)}\frac{\partial}{\partial q}g(q)P_{st}(q) , \tag{B1}$$

where

$$U(t,s)=\overleftarrow{T}\exp\left[\int_s^t du(1-P)A^I(u)\right] , \tag{B2}$$

and

$$A^I(t)=e^{-(\partial v/\partial q)t}\frac{\partial}{\partial q}g(q)\xi(t)e^{(\partial/\partial q)vt} . \tag{B3}$$

P is the projector defined as $PF\equiv\langle F\rangle$, and $\overleftarrow{T}$ is the reversed time ordering operator.

By expanding U in powers of the noise this equation is expressed in terms of the so-called "totally ordered cumulants." The first order in the intensity of the noise is obtained by making $U=1$. If the correlation is exponential with intensity D and correlation time τ, we get to first order in D

$$\frac{\partial}{\partial t}P(q,q';t)=-\frac{\partial}{\partial q}v(q)P(q,q';s)+\frac{\partial}{\partial q}g(q)\exp\left[-\left[\frac{\partial v}{\partial q}+\frac{1}{\tau}\right]t\right]\int_0^t ds\exp\left[\left[\frac{\partial v}{\partial q}+\frac{1}{\tau}\right]s\right]\frac{\partial}{\partial q}g(q)P(q,q';s)$$
$$+D\frac{\partial}{\partial q}g(q)\exp\left[-\left[\frac{\partial v}{\partial q}+\frac{1}{\tau}\right]t\right]\delta(q-q')\int_{-\infty}^0\exp\left[\left[\frac{\partial v}{\partial q}+\frac{1}{\tau}\right]s\right]\frac{\partial}{\partial q}g(q)P_{st}(q)ds . \tag{B4}$$

The last term can be written in compact form taking into account the equation for $P_{st}(q)$:

$$0=-\frac{\partial}{\partial q}v(q)P_{st}(q)+D\frac{\partial}{\partial q}g(q)\int_{-\infty}^0 ds\exp\left[\left[\frac{\partial \sigma}{\partial q}+\frac{1}{\tau}\right]s\right]\frac{\partial}{\partial q}g(q)P_{st}(q)+O(D^2) . \tag{B5}$$

The Laplace transform of Eq. (B4) taking into account (B5) gives Eq. (3.1) with the operator $\hat{M}(w)$ and the function $S(w)$ done by Eqs. (3.19) and (3.20).

Furthermore, Eq. (3.5) is very useful in order to obtain compact expressions. In the Markovian limit a similar equation given by

$$0=-v(q)P_{st}(q)+Dg(q)\frac{\partial}{\partial q}g(q)P_{st}(q) \tag{B6}$$

is used to get Eq. (3.27).

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