Higher free-electron laser frequencies with circularly polarized wigglers and axial magnetic fields

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Free-electron laser operation with a circularly polarized wiggler and an appropriately directed axial magnetic field of suitable strength is shown to generate significant coherent emission of radiation into higher frequencies.

Harmonic generation in free-electron lasers (FEL's) has been a subject of great interest. It extends the tunable range of a laser to higher frequencies without using ever-increasing electron-beam energies.[1-5] The single-particle approach followed by Colson[2] clearly shows that these harmonics are generated by a small periodic longitudinal motion acquired by electrons passing through the FEL cavity. Plane-polarized wigglers are, therefore, used in practical devices.[6,7] In this paper we show that a circularly polarized wiggler in the presence of an axial magnetic field of appropriate strength but directed opposite to the electron beam is also capable of generating stable electron trajectories, leading to significant coherent emission of radiation into higher frequencies. The gain mechanism is the usual axial bunching of the electrons as in the case of other FEL devices.

Axial fields are generally applied in free-electron lasers with circularly polarized wigglers for electron-beam confinement and for obtaining gain enhancements (close to magnetoresonance).[8] It is important to note that these fields may also be utilized to generate higher frequencies.

Consider a relativistic electron of energy $\gamma mc^2 = mc^2(1-\beta^2)^{-1/2}$ moving with velocity $\mathbf{v} = c\boldsymbol{\beta}$ through magnetic wiggler and axial guide fields in the presence of a plane electromagnetic wave copropagating along the direction z of particle motion. The configuration of the wiggler field of amplitude B, wavelength $\lambda_0 = 2\pi/k_0$, $(\omega_0 = ck_0)$, phase $k_0 z$, and the constant axial field of strength b is taken as

$$\mathbf{B}_m = \hat{\mathbf{x}} B \cos(k_0 z) - \hat{\mathbf{y}} B \sin(k_0 z) - \hat{\mathbf{z}} b \ . \tag{1a}$$

The radiation of frequency $f\omega_r$ $(f = 1,2,3,\ldots)$, $\omega_r = k_r c$, wavelength $\lambda_r/f = 2\pi/fk_r$, amplitude $E(z,t)$, and phase $\mathcal{E} = fk_r z - f\omega_r t + \phi(z,t)$ in the FEL cavity may be represented by the vector potential

$$\mathbf{A}_r(z,t) = \frac{E(z,t)}{fk_r}(\hat{\mathbf{x}} \sin\mathcal{E} - \hat{\mathbf{y}} \cos\mathcal{E}) \ . \tag{1b}$$

The nonlinear wave equation and pendulum equation for this configuration are derived by following a procedure similar to that described by Colson.[2] For slowly varying amplitude and phase of the radiation field, the Lorentz force equations of electron motion approximate to

$$\frac{d}{dt}(\gamma\beta_x) = -\frac{e}{mc}[b\beta_y - B\beta_z \sin(k_0 z) - E(1-\beta_z)\cos\mathcal{E}] \ , \tag{2a}$$

$$\frac{d}{dt}(\gamma\beta_y) = \frac{e}{mc}[b\beta_x + B\beta_z \cos(k_0 z) + E(1-\beta_z)\sin\mathcal{E}] \ , \tag{2b}$$

$$\frac{d}{dt}(\gamma\beta_z) = -\frac{eB}{mc}[\beta_x \sin(k_0 z) + \beta_y \cos(k_0 z)] + \frac{d\gamma}{dt} \ , \tag{2c}$$

$$\frac{d\gamma}{dt} = \frac{eE}{mc}(\beta_x \cos\mathcal{E} + \beta_y \sin\mathcal{E}) \ . \tag{2d}$$

For β_z close to 1 (or $\gamma \gg 1$), the last terms of Eqs. (2a) and (2b), describing the transverse optical force, may be neglected in comparison with the others due to the transverse force of the static magnetic fields. Thus, to a very good approximation (for a typical value[2] $K \approx 1$) the solutions corresponding to the axial injection of the electron beam ($\beta_z = \beta_0$, $\gamma = \gamma_0 = (1-\beta_0^2)^{-1/2}$ at $t=0$) are given by

$$\beta_x = -\beta_0(K/\gamma)[\cos(k_0 z) - \cos(k_0 z - \Omega t)] \ , \tag{3a}$$

$$\beta_y = \beta_0(K/\gamma)[\sin(k_0 z) - \sin(k_0 z - \Omega t)] \ , \tag{3b}$$

where $K = eB/mc\Omega$, $\Omega = \omega_b + \omega_0\beta_0$ and ω_b $(= eb/mc\gamma_0)$ represents the Larmor frequency of the electron due to the axial field.

Substitution of $\beta_{x,y}$ from Eq. (3) in the relation $\gamma^{-2} = 1 - \boldsymbol{\beta}^2$ gives an expression for β_z which may be expanded for $\gamma \gg 1$ to yield

$$\beta_z = 1 - \tfrac{1}{2}\gamma^{-2}\{1 + 2K^2[1 - \cos(\Omega t)]\} \ . \tag{4}$$

The slow evolution of the dynamical variables about the periodic oscillations on the scale of the magnetic wavelength may now be obtained by taking an average over it. The averaged β_z is thus found to be

$$\bar{\beta}_z = 1 - \tfrac{1}{2}\bar{\gamma}^{-2}(1 + 2K^2) \tag{5a}$$

so that

$$\bar{\gamma}_z \approx \bar{\gamma}(1 + 2K^2)^{-1/2} \approx \gamma_0(1 + 2K^2)^{-1/2} \ , \tag{5b}$$

where the overbars denote the average values.

We now define a slowly evolving dimensionless velocity $v_f(t)$ over the length L of the cavity such that

$$v_f(t)=\frac{L}{c}\{f[(\omega_r+\omega_0)\bar{\beta}_z-(\omega_r-\omega_b)]-\omega_b\} \ . \quad (6)$$

We note that

$$v_1(t)=\frac{L}{c}[\omega_0\bar{\beta}_z-\omega_r(1-\bar{\beta}_z)]$$

is the usual resonance parameter. For $v_1(0)=0$ exactly one wavelength of light passes over an electron as it moves through a period of the wiggler magnet and the coupling between the radiation and electrons is maximized. The dimensionless phase describing the interaction between the electron and electromagnetic (e.m.) radiation then assumes the form

$$\varphi_f(t)=f[(k_r+k_0)\bar{z}-(\omega_r-\omega_b)t]-\omega_b t \ , \quad (7)$$

where

$$\bar{z}(t)=\int_0^t c\bar{\beta}_z(t')dt' \ .$$

The slow motion along the z direction given by $\bar{z}$ together with the oscillating term arising from Eq. (4) give the total axial displacement[9]

$$z(t)\approx\bar{z}(t)+\frac{cK^2}{\gamma^2\Omega}\sin(\Omega t) \ . \quad (8)$$

The second term on the right-hand side of Eq. (8) is responsible for generating higher frequencies.

Using Eqs. (8) and (3) in Eq. (2d) and retaining only the slowly varying terms we can find the rate of exchange of energy between the electrons and e.m. radiation. This is given by

$$\dot{\gamma}=\frac{eEL}{mc^2\gamma_0}\mathcal{K}_f(\kappa)\cos(\varphi_f+\phi) \ , \quad (9a)$$

where

$$\mathcal{K}_f(\kappa)=K[J_f(f\kappa)-J_{(f-1)}(f\kappa)] \ , \quad (9b)$$

$$\kappa=\omega_r K^2/\Omega\gamma_0^2 \ . \quad (9c)$$

J_f is a fth-order Bessel function of the first kind and the overdot represents differentiation with respect to $\tau\equiv tc/L$.

The pendulum equation describing the electron motion is obtained from Eq. (7) by using Eqs. (5) and (9). Thus,

$$\ddot{\varphi}(\tau)=|a_f|\cos(\varphi_f+\phi) \ , \quad (10a)$$

$$|a_f|=\left[f+(f-1)\frac{\omega_b}{\omega_0}\right]\left[\frac{2\pi Ne\mathcal{K}_f(\kappa)LE}{\gamma_0^2 mc^2}\right] \ , \quad (10b)$$

where $N=L/\lambda_0$.

We next use Maxwell's equations to derive the slow evolution of the optical amplitude E and phase ϕ averaged over many optical wavelengths for transverse electron currents (j_x,j_y) in the form

$$\frac{\partial\varepsilon}{\partial z}+\frac{1}{c}\frac{\partial\varepsilon}{\partial t}=-\frac{2\pi}{c}\langle(j_x+ij_y)e^{-i\varepsilon}\rangle \ , \quad (11)$$

where $\varepsilon=Ee^{i\phi}$. j_x and j_y are obtained with the help of Eq. (3) by summing over all single particle currents spread uniformly over the initial positions $\varphi_f(0)$ and velocities $v_f(0)$. Using Eq. (10) and writing $a_f=|a_f|e^{i\phi}$, the wave Eq. (11) assumes the form

$$\dot{a}_f=-\mathcal{R}\langle e^{-i\varphi_f}\rangle \ , \quad (12a)$$

$$\mathcal{R}=\left[f+(f-1)\frac{\omega_b}{\omega_0}\right]\left[\frac{8\pi^2Ne^2\mathcal{K}_f^2L^2\rho_0}{\gamma_0^3mc^2}\right] \ . \quad (12b)$$

The coupled Eqs. (10) and (12) describe the FEL operation for higher frequencies.

These are easily solved in the low-gain limit by expanding the pendulum equation in weak fields for which $|a_f|\ll 1$. The gain

$$G_f=\frac{|a_f(1)|^2}{|a_f(0)|^2}-1$$

and the phase shift $\Delta\phi=\phi(1)-\phi(0)$ for evolution of the system from time $\tau=0$ to $\tau=1$ are then given by

$$G_f=\mathcal{R}\frac{d}{dv_{0f}}\left[\frac{\cos v_{0f}-1}{v_{0f}^2}\right] \ , \quad (13a)$$

$$\Delta\phi=-\frac{1}{2}\mathcal{R}\frac{d}{dv_{0f}}\left[\frac{\sin v_{0f}-v_{0f}}{v_{0f}^2}\right] \ . \quad (13b)$$

The gain is maximum for $v_{0f}=2.6056$ and is given by $G_f=0.135\mathcal{R}$ with the phase shift $\Delta\phi=0.0185\mathcal{R}$. The corresponding frequency of radiation is

$$f\omega_r\sim\frac{2\gamma_0^2}{(1+2K^2)}(f\Omega-\omega_b) \ . \quad (13c)$$

The spontaneous emission per unit solid angle $d\omega$ in the forward direction per unit frequency interval $d(f\omega_r)$ is seen to be

$$\frac{dW_f}{d\omega\, d(f\omega_r)}=2\left[f+(f-1)\frac{\omega_b}{\omega_0}\right]^2\left[\frac{eN\gamma_0}{1+2K^2}\right]^2 \times\frac{M_f^2(\kappa)}{c} \ , \quad (14a)$$

$$M_f^2(\kappa)=K^2[J_f^2(\kappa)+J_{(f-1)}^2(\kappa)] \ . \quad (14b)$$

From Eqs. (12) and (13) the maximum FEL gain is seen to vary as $[f+(f-1)\omega_b/\omega_0]\mathcal{K}_f^2$. Taking $f=5$, $\omega_b=1.5\omega_0$, and $K=1$, the FEL gain is seen to decrease by a factor of 3.7 for an elevenfold increase in frequency from

$$\omega_{r_1}=\frac{2\gamma_0^2\omega_0}{1+2K^2}\left[1-\frac{2.6056}{k_0L}\right]$$

to

$$\omega_{r_{11}}=11\frac{2\gamma_0^2\omega_0}{1+2K^2}\left[1-\frac{2.6056}{11k_0L}\right] \ .$$

The factor multiplying the length L of the cavity in the second term of $\omega_{r_{11}}$ leads to stiff requirements of beam quality.[2] The same frequency enhancement can also be obtained by raising the electron energy from $\gamma_0 mc^2$ to $\sqrt{11}\gamma_0 mc^2$ but the corresponding decrease in gain, as given by Eq. (14), is by the much higher factor of $(11)^{1.5}\sim 36$.

It is interesting to see from Eq. (3) that for $\omega_b = \omega_0\beta_0$, and with $z(t)\sim\bar{z}(t)\sim\beta_0 ct$ [obtained from Eqs. (5) and (8) for $K/\gamma \ll 1$], β_x tends to zero and the electron motion becomes sinusoidal along the Y direction. Equation (13) then shows that the usual odd harmonics of the principal frequency $2\gamma_0^2\omega_0/(1+2K^2)$ are generated in this case.

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[9]It may be noted that in the absence of radiation field (γ=const) Eqs. (3) and (4) can be solved up to second order in K/γ to yield the electron trajectory. This is found to be a superposition of two helices characterized by frequencies ω_0 and ω_b with constant radii $c\beta_0 K/\gamma_0\omega_0$ and $c\beta_0 K/\gamma_0\omega_b$, respectively. For b large enough, so that $\omega_b \gtrsim \omega_0$, stable electron trajectories continuously interacting with e.m. radiation through the length of the cavity are obtained.