Spin effects in a free-electron laser

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An attempt is made to solve the spherical Raman-Nath equation taking electron spin into account. Spin effects are discussed for a free-electron laser in the extreme case $v_b \simeq c$, where c is the speed of light and v_b the velocity of the electron.

Several quantum-statistical properties, such as photon statistics and squeezing, have been investigated for a free-electron laser (FEL) in a number of papers.[1-4] The spherical Raman-Nath equation (SRNE) was obtained[3] from Schrödinger's equation. However, to now, no attempts were made to investigate the effects of electron spin, another quantum feature. The topic of this Brief Report is to solve SRNE taking into account electron spin under the initial condition $C^{n_w,n_r}_{\sigma,s,l(0)}=\delta_{s0}\delta_{l0}$. In particular, more details about spin effects are given in the extreme relativistic limit $v_b \simeq c$, where c is the speed of light and v_b the velocity of the electron in the laboratory frame.

In the following, the Bambini-Renieri frame is used except where otherwise indicated. Using a method similar to that in Ref. 4, one can obtain the Hamiltonian including spin effects and two constants of motion for a helical pumped FEL:

$$H=mc^2+\hbar\omega+\frac{p_z^2}{2m}+\hbar\omega(a_w^\dagger a_w+a_r^\dagger a_r)+\hbar\Omega(a_w^\dagger a_r e^{2iKz}+a_r^\dagger a_w e^{-2iKz})+\hbar F\Omega(a_w^\dagger\sigma_+ e^{iK_1z}+a_w\sigma_- e^{-iK_1z})$$
$$-\hbar G\Omega(a_r^\dagger\sigma_+ e^{-iK_2z}+a_r\sigma_- e^{iK_2z}) , \quad (1)$$

$$a_w^\dagger a_w+a_r^\dagger a_r-\frac{\sigma_3}{2}=\text{const} , \quad (2)$$

$$p_z-\hbar K_1 a_w^\dagger a_w+\hbar K_2 a_r^\dagger a_r=\text{const} , \quad (3)$$

where

$$\Omega=\frac{e^2}{2m\omega\epsilon_0 V},\quad F\Omega=\frac{eK_1}{\sqrt{2}m}\left[\frac{\hbar}{2\omega\epsilon_0 V}\right]^{1/2},\quad K_1G=K_2F,\quad 2K=K_1+K_2,\quad v_bK_1=cK_2,\quad \sigma_3=[\sigma_+,\sigma_-] ,$$

p_z is the axial momentum of the electron, ϵ_0 is the dielectric constant of free space, V is the volume of wiggler field, ω is the frequency of the laser and the wiggler field, K_1 (K_2) is the wave number of the wiggler (laser) field, $a_w^\dagger$ ($a_r^\dagger,\sigma_+$) and a_w (a_r,σ_-) are the creation and destruction operators of the wiggler field (laser field, electron spin), respectively.

We start from a state with electron energy $p_0^2/2m$ and axial spin σ ($=\pm1$), and the laser and wiggler fields are in coherent states with a mean number of photons $|\alpha_{r0}|^2$ and $|\alpha_{w0}|^2$, respectively. The most general FEL state at any time t can be given by the superposition

$$|\Psi_{(t)}\rangle=\sum_{s=0}^{\sigma}\sum_{l=-n_r}^{n_w-s}\sum_{n_r=0}^{\infty}\sum_{n_w=0}^{\infty}C^{n_w,n_r}_{\sigma,s,l(t)}\exp\left[-\frac{it}{\hbar}\left[\frac{p_0^2}{2m}+\hbar\omega n_r+\hbar\omega n_w+\hbar\omega+mc^2\right]\right]$$
$$\times\frac{\exp[-\frac{1}{2}(|\alpha_{r0}|^2+|\alpha_{w0}|^2)]}{\sqrt{n_w!n_r!}}\alpha_{r0}^{n_r}\alpha_{w0}^{n_w}$$
$$\times|p_0-\hbar l(K_1+K_2)-\hbar sK_1;n_r+l;n_w-l-s;\sigma-2s\rangle , \quad (4)$$

where $C^{n_w,n_r}_{\sigma,s,l(t)}$ is the probability amplitude for interchanging l photons and changing the electron spin σ by $2s$. n_w and n_r are the numbers of wiggler and laser photons, respectively. Substituting (1) and (4) into the Schrödinger wave equation,

$$i\hbar\frac{\partial}{\partial t}|\Psi_{(t)}\rangle=H|\Psi_{(t)}\rangle , \quad (5)$$

one obtains the modified SRNE,

$$
\begin{aligned}
i\frac{d}{dt}C^{n_w,n_r}_{\sigma,s,l(t)}=&(\lambda_{(s)}+\mu_{(s)}l+\nu l^2)C^{n_w,n_r}_{\sigma,s,l(t)}\\
&+\Omega[\sqrt{(n_r+l)(n_w-l-s+1)}C^{n_w,n_r}_{\sigma,s,l-1(t)}+\sqrt{(n_r+l+1)(n_w-l-s)}C^{n_w,n_r}_{\sigma,s,l+1(t)}]\\
&+F\Omega[\sqrt{n_w-l-s+1}C^{n_w,n_r}_{\sigma,s-1,l(t)}+\sqrt{n_w-l-s}\,C^{n_w,n_r}_{\sigma,s+1,l(t)}]\\
&-G\Omega[\sqrt{n_r+l}\,C^{n_w,n_r}_{\sigma,s+1,l-1(t)}+\sqrt{n_r+l+1}C^{n_w,n_r}_{\sigma,s-1,l+1(t)}]\ ,
\end{aligned}
\tag{6}
$$

where

$$
\lambda_{(s)}=\frac{\hbar K_1^2}{2m}s^2-\left[\omega+\frac{p_0K_1}{m}\right]s,\quad \mu_{(s)}=\frac{\hbar K_1(K_1+K_2)}{m}s-\frac{p_0(K_1+K_2)}{m},\quad \nu=\frac{\hbar(K_1+K_2)^2}{2m}\ .
$$

It is obvious that in the regime $F\gtrsim\sqrt{n_r+l}$ or $G\gtrsim\sqrt{n_w-l-s}$ the spin effects will play an important role in a FEL. In order to solve (6), one can make the transformation first:

$$
C^{n_w,n_r}_{\sigma,s,l(t)}=(-i)^l e^{i\alpha x}e^{i\beta l x}\exp\left[i\gamma x\left[l+\frac{(n_r-n_w+s)}{2}\right]^2\right]|M^{n_w,n_r}_{\sigma,s,l(x)}\rangle\ ,
\tag{7}
$$

where

$$
x=\Omega t,\quad \alpha=\frac{\nu(n_r-n_w)^2}{4\Omega},\quad \beta=\frac{\nu(n_r-n_w)-\mu_{(0)}}{\Omega},\quad \gamma=-\frac{\nu}{2\Omega}\ ,
$$

then define a series of operators as follows:

$$\hat{L}_+|M^{n_w,n_r}_{\sigma,s,l(x)}\rangle=\sqrt{(n_r+l+1)(n_w-l-s)}|M^{n_w,n_r}_{\sigma,s,l+1(x)}\rangle\ , \tag{8}$$

$$\hat{L}_-|M^{n_w,n_r}_{\sigma,s,l(x)}\rangle=\sqrt{(n_r+l)(n_w-l-s+1)}|M^{n_w,n_r}_{\sigma,s,l-1(x)}\rangle\ , \tag{9}$$

$$\hat{S}_+|M^{n_w,n_r}_{\sigma,s,l(x)}\rangle=\sqrt{n_w-l-s}\,|M^{n_w,n_r}_{\sigma,s+1,l(x)}\rangle\ , \tag{10}$$

$$\hat{S}_-|M^{n_w,n_r}_{\sigma,s,l(x)}\rangle=\sqrt{n_w-l-s+1}|M^{n_w,n_r}_{\sigma,s-1,l(x)}\rangle\ , \tag{11}$$

$$\hat{L}_z=\tfrac{1}{2}[\hat{L}_+,\hat{L}_-]\ , \tag{12}$$

$$\hat{S}_z=\tfrac{1}{2}[\hat{S}_+,\hat{S}_-]\ , \tag{13}$$

$$\hat{I}_1=[\hat{S}_+,\hat{L}_-]\ , \tag{14}$$

$$\hat{I}_2=[\hat{L}_+,\hat{S}_-]\ , \tag{15}$$

$$\hat{I}_3=\hat{S}_-\hat{S}_++\hat{I}_2\hat{I}_1\ . \tag{16}$$

Substituting (7) into (6) and using (8)–(16), one obtains the operational differential equation for $|M^{n_w,n_r}_{\sigma,s,l(x)}\rangle$:

$$
\begin{aligned}
\frac{d}{dx}|M^{n_w,n_r}_{\sigma,s,l(x)}\rangle=&e^{i\gamma x L_z^2}[i\gamma\hat{L}_z^2+iA\hat{I}_3^2+iB\hat{I}_3\hat{L}_z+iC\hat{L}_z+iD\hat{I}_3+iE\hat{S}_z-e^{i\beta x}\hat{L}_++e^{-i\beta x}\hat{L}_--iF(\hat{S}_++\hat{S}_-)\\
&-G(e^{-i\beta x}\hat{I}_1-e^{i\beta x}\hat{I}_2)]e^{-i\gamma x\hat{L}_z^2}|M^{n_w,n_r}_{\sigma,s,l(x)}\rangle\ ,
\end{aligned}
\tag{17}
$$

where

$$
A\Omega=-\frac{\hbar(K_1-K_2)^2}{8m},\quad B\Omega=\frac{\hbar(K_1^2-K_2^2)}{2m},\quad C=-B(n_w+n_r)\ ,
$$

$$
D\Omega=\frac{\hbar(n_w+n_r)(K_1-K_2)^2}{4m}-\frac{\hbar K_1(K_1+K_2)(n_r-n_w)}{2m}-\left[\omega+\frac{p_0K_1}{m}\right]\ ,
$$

and

$$
E\Omega=2(n_w+n_r)\left[\frac{\hbar K_1(K_1+K_2)(n_r-n_w)}{2m}-\frac{\hbar(n_w+n_r)(K_1-K_2)^2}{8m}+\left[\omega+\frac{p_0K_1}{m}\right]\right]\ .
$$

Equation (17) can be solved by using the following transformation:

$$|M^{n_w,n_r}_{\sigma,s,l(x)}\rangle=e^{i\gamma x\hat{L}_z^2}e^{g_{1(x)}\hat{L}_z}e^{g_{2(x)}\hat{L}_+}e^{g_{3(x)}\hat{L}_-}e^{g_{4(x)}\hat{S}_+}e^{g_{5(x)}\hat{S}_-}e^{g_{6(x)}\hat{S}_z}e^{g_{7(x)}\hat{I}_1}e^{g_{8(x)}\hat{I}_2}e^{g_{9(x)}\hat{I}_3}e^{g_{10(x)}\hat{Q}}|M^{n_w,n_r}_{\sigma,s,l(0)}\rangle\ , \tag{18}$$

where $\hat{Q}$ is composed of the operators defined in formulas (8)–(16). Inserting (18) into (17), we can obtain the exact solutions of $\hat{Q}$, $g_{i(x)}$ $(i=1,\ldots,10)$ and $C^{n_w,n_r}_{\sigma,s,l(t)}$ under the initial condition $C^{n_w,n_r}_{\sigma,s,l(0)}=\delta_{s0}\delta_{l0}$. But, in the following, particular emphasis is given to the limiting case $v_b\simeq c$, then $\omega\simeq cK$, $K_1\simeq K_2\simeq K$, $F\simeq G\simeq\sqrt{\nu/4\Omega}$, $A\Omega\simeq 0$, $B\Omega\simeq 0$, $C\Omega\simeq 0$, $D\Omega=-(\omega+p_0K/m)-[\hbar K^2(n_r-n_w)]/m$, $E\Omega\simeq -2D\Omega(n_r+n_w)$, and

$$g_{1(x)}=i\beta x-2\ln\left[\frac{\cos\left[\dfrac{\delta x}{2}-\phi\right]}{\cos\phi}\right]\ , \tag{19}$$

$$g_{2(x)}=-\frac{\delta\sin\left[\dfrac{\delta x}{2}\right]\cos\left[\dfrac{\delta x}{2}-\phi\right]}{2\cos^3\phi}\ , \tag{20}$$

$$g_{3(x)}=\frac{2\cos\phi\sin\left[\dfrac{\delta x}{2}\right]}{\delta\cos\left[\dfrac{\delta x}{2}-\phi\right]}\ , \tag{21}$$

$$g_{9(x)}=iDx\ , \tag{22}$$

$$g_{10(x)}=0\ , \tag{23}$$

where $\delta\simeq\sqrt{4+\beta^2}$, $\tan\phi=i\beta/\delta$. Substituting these results into (18) and (7), one obtains the solution of (6):

$$C^{n_w,n_r}_{\sigma,s,l(t)}=(-i)^l g_{4(x)}^{-s} I^{n_w,n_r}_{\sigma,s,l(x)} P_{(x)}^{l/2}(1-P_{(x)})^{(n_w-n_r-l)/2}$$
$$\times\exp\left\{\tfrac{1}{2}g_{6(x)}+iD(n_w+n_r-s)x+\frac{ix\left[\nu(n_w-n_r)^2+\mu_{(0)}(n_w-n_r)-\dfrac{(n_w-n_r)\nu s}{2}-\dfrac{\nu s^2}{4}\right]}{2\Omega}\right.$$
$$\left.-\frac{ilx\,[\nu(n_w-n_r+s)+2\mu_{(0)}]}{2\Omega}-\frac{il^2\nu x}{2\Omega}+i(n_w-n_r-l)\tan^{-1}\left[\frac{\beta}{\delta}\tan\left[\frac{\delta x}{2}\right]\right]\right\}\ , \tag{24}$$

where $P_{(x)}=4\sin^2(\delta x/2)/\delta^2$ and

$$I^{n_w,n_r}_{\sigma,s,l(x)}=\left[\frac{n_r!}{n_w!(n_r+l)!(n_w-l-s)!}\right]^{1/2}\sum_{j_1,j_2,j_3,j_4=0}^{\infty}\frac{(n_w+j_1)!(n_r+l+j_4)!(n_w-l-s+j_2)!}{(n_r-j_1)!(n_w-l+j_3-j_4)!(l+j_1-j_3+j_4)!}$$
$$\times\frac{(g_2g_3)^{j_1}(g_4g_5)^{j_2}\left[\dfrac{g_7}{g_3g_4}\right]^{j_3}(g_3g_4g_8)^{j_4}}{(j_2-j_3+j_4-s)!j_1!j_2!j_3!j_4!}\ .$$

To continue further calculations, we make two assumptions: (i) the laser initial field is the vacuum state, which gives $n_r=0$, (ii) the axial momentum of electron is positive in the Bambini-Renieri frame after the emission of n_w photons, that is $-2n_w\nu/\mu_{(0)}<1$. Then one can expand $P_{(x)}$ over ν and obtain the following solutions under the limits $|\alpha_{w0}|\to\infty$, $\Omega\to 0$, $|\mu_{(0)}/\Omega|\gg 1$, and $\Omega|\alpha_{w0}|=\text{constant}=\bar{\Omega}$:

$$g_{4(x)}\simeq\frac{\sqrt{\Omega\nu}}{2\omega}(1-e^{i\omega t})\ , \tag{25}$$

$$g_{5(x)}\simeq-\frac{\sqrt{\Omega\nu}}{2\omega}(1-e^{-i\omega t})\ , \tag{26}$$

$$g_{6(x)}\simeq iEx+\frac{i\nu}{\omega}x\ , \tag{27}$$

$$g_{7(x)}\simeq-\frac{i\sqrt{\Omega\nu}}{2\omega}(e^{i\mu_{(0)}t}-e^{i\omega t})\ , \tag{28}$$

$$g_{8(x)}\simeq-\frac{i\sqrt{\Omega\nu}}{2\omega}(e^{-i\mu_{(0)}t}-e^{-i\omega t})\ . \tag{29}$$

The variance is

$$(\Delta P_z)^2=\langle P_z^2\rangle-\langle P_z\rangle^2$$
$$\simeq 4\hbar^2K^2\left[G_0\frac{d}{d\theta}\left[\frac{\sin\theta}{\theta}\right]^2+\bar{\Omega}^2t^2\left[\frac{\sin\theta}{\theta}\right]^2-\eta G_0\left[\frac{\sin\theta}{\theta}\right]^2+3\eta G_0\bar{\Omega}^2t^2\left[\frac{\sin\theta}{\theta}\right]^4\right], \quad (30)$$

and the gain is

$$G=\langle a_r^\dagger a_r\rangle-\langle a_r^\dagger a_r\rangle|_{\nu=0}\simeq G_0\left[\frac{d}{d\theta}\left[\frac{\sin\theta}{\theta}\right]^2-\eta\left[\frac{\sin\theta}{\theta}\right]^2\right], \quad (31)$$

where

$$G_0=\tfrac{1}{2}\nu|\alpha_{w0}|^2\bar{\Omega}^2t^3,\quad \theta=\tfrac{1}{2}\mu_{(0)}t,\quad \eta=\frac{\Omega t}{2}\left[\frac{\sin(\omega t/2)}{\omega t/2}\right]^2,$$

and

$$\langle P_z\rangle=\sum_{s=0}^{\sigma}\sum_{n_w=0}^{\infty}\sum_{l=0}^{n_w-s}\frac{e^{-|\alpha_{w0}|^2}}{n_w!}|\alpha_{w0}|^{2n_w}|C_{\sigma,s,l(x)}^{n_w,n_r}|^2(P_0-2l\hbar K-\hbar sK),$$

$$\langle P_z^2\rangle=\sum_{s=0}^{\sigma}\sum_{n_w=0}^{\infty}\sum_{l=0}^{n_w-s}\frac{e^{-|\alpha_{w0}|^2}}{n_w!}|\alpha_{w0}|^{2n_w}|C_{\sigma,s,l(x)}^{n_w,n_r}|^2(P_0-2l\hbar K-\hbar sK)^2,$$

$$\langle a_r^\dagger a_r\rangle=\sum_{s=0}^{\sigma}\sum_{n_w=0}^{\infty}\sum_{l=0}^{n_w-s}\frac{e^{-|\alpha_{w0}|^2}}{n_w!}|\alpha_{w0}|^{2n_w}|C_{\sigma,s,l(x)}^{n_w,n_r}|^2 l.$$

From (30) and (31), one may conclude the following.

(1) The results of the spin effects have the same order of magnitude for $\sigma=\pm 1$.

(2) Neglecting the spin term in (31) gives the previous results.[2] The spin effects,

$$-\eta G_0\left[\frac{\sin\theta}{\theta}\right]^2=-\frac{\hbar|\alpha_{w0}|K^2\bar{\Omega}^3t^4}{2m}\left[\frac{\sin\theta}{\theta}\right]^2\left[\frac{\sin(\omega t/2)}{\omega t/2}\right]^2,$$

always lower the gain due to $\eta>0$ and disappear in the classical limit $\hbar\to 0$.

(3) The maximum gain, $G_0(0.54-0.55\eta)$, occurs at $\theta_0=-1.3-0.78\eta$ corresponding to $\omega=(mc/p_0t)(1.3+0.78\eta)$. Including the spin effects raises the laser frequency.

(4) For typical experimental parameters,[3] we have $\nu/4\Omega\sim 3\times 10^{10}$, so the regime in which the spin effects are important is $l\gtrsim n_w-s-3\times 10^{10}$ or $l\lesssim 3\times 10^{10}$ with $0\le l\le n_w-s$. This means the spin effects are significant when interchanging the first and the last 3×10^{10} photons.

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