PHYSICAL REVIEW A VOLUME 43, NUMBER 3 1 FEBRUARY 1991

Transformations in the angle–angular-momentum phase space

A. Vourdas
Department of Electrical Engineering and Electronics, The University of Liverpool, Brownlow Hill, P.O. Box 147, Liverpool, L69 3BX England
(Received 21 September 1990)

Squeezing transformations in the discrete $Z_{2j+1} \times Z_{2j+1}$ angle–angular-momentum phase space are shown to be associated with the $\mathrm{SL}(2,Z_{2j+1})$ group and important special cases are explicitly constructed. "Spherical" bases are introduced in the direct sum of all the Hilbert spaces H_{2j+1} and the corresponding representations are defined. Transformations of these bases, using area preserving diffeomorphisms on a sphere, are studied and potential applications in quantum-optics models are discussed.

I. INTRODUCTION

In a previous paper[1] we have considered the Hilbert space H_{2j+1} associated with the SU(2) group and introduced the angle (or phase) states $|\theta;jn\rangle$ which are dual to the standard angular-momentum states for which we used the notation $|J;jm\rangle$. We have explained that these angle and angular-momentum states have similarities to the position and momentum states of the harmonic oscillator. For example, the θ states are related to the J states through a Fourier transform, and there exists a Weyl group which creates translations in angle and angular momentum. We have also introduced angle operators $\theta_z,\theta_+,\theta_-$ dual to the standard angular-momentum operators J_z,J_+,J_-.

The purpose of this paper is to continue this work and discuss various transformations on the angle–angular-momentum phase space. Quantum phase-space techniques have led into a deeper understanding of quantum theory and have been used widely in quantum optics. Transformations in phase space have been a valuable tool in both classical and quantum mechanics and for this reason we study them here in the context of the angle–angular-momentum phase space. In Sec. II we consider the analogue of the concept of squeezing. This concept has been studied extensively in the harmonic-oscillator context,[2] with the use of $\mathrm{SL}(2,R)$ (Bogoliubov) transformations, that preserve the Weyl commutator $[x,p]$. In the present context we consider the operators E,F that create translations in the $\theta-J$ phase space and study transformations that preserve the Weyl relation

$$FE=EF\exp\left[i\frac{2\pi}{2j+1}\right] . \tag{1}$$

They form the $\mathrm{SL}(2,Z_{2j+1})$ group. Similar transformations have been considered in Ref. 3 in the context of general finite quantum systems. Here we study a more complex problem because our finite Hilbert space is associated to the angular momentum, and every idea has to be considered in connection with its implications on the angular-momentum structure. We construct explicitly two important special cases of these transformations. The first one is the dilation-contraction transformations in the θ-J phase space (the analogue of the $x'=\lambda x$, $p'=\lambda^{-1}p$ for the harmonic oscillator). It is very interesting to study them in detail because it is not immediately clear what dilation and contraction is, in a discrete (and finite) phase space. The second special case is finite Fourier transforms that map the J states and operators into the θ states and operators. Both of these cases provide valuable physical insight into the nature of these transformations and more generally into the nature of the $\theta-J$ phase space.

In Sec. III we study the Hilbert space H which is the direct sum of all the Hilbert spaces H_{2j+1}. In this Hilbert space we consider the "spherical" bases $|J;\alpha\beta\rangle$ and $|\theta;\alpha\beta\rangle$ such that

$$\begin{aligned}\langle J;\alpha\beta|J;jm\rangle &= Y_{jm}(\alpha,\beta) ,\\ \langle \theta;\alpha\beta|\theta,jm\rangle &= Y_{jm}(\alpha,\beta) .\end{aligned} \tag{2}$$

The first of those is the widely used $|a,\beta\rangle$ basis, denoted here as $|J;\alpha\beta\rangle$. The second one is its θ counterpart $|\theta;\alpha\beta\rangle$. The α,β in $|J;\alpha\beta\rangle$ are angles on a sphere that we call J sphere; and the α,β in $|\theta;\alpha\beta\rangle$ are angles on a sphere that we call θ sphere. We then define four representations with respect to the four bases $|J;jm\rangle$, $|\theta;jm\rangle$, $|J;\alpha\beta\rangle$, $|\theta;\alpha\beta\rangle$; we call them J, θ, spherical-J, and spherical-θ representations correspondingly. Transformations that connect the J with the θ representation and also the spherical-J with the spherical-θ representation are studied in detail.

In Sec. IV we consider a very rich class of transformations, namely area-preserving diffeomorphisms of the J and θ spheres. They are generalizations of the rotation transformations and the corresponding operators J_{jm} (and θ_{jm}) contain the J_+,J_-,J_z (and $\theta_+,\theta_-,\theta_z$) as special cases. Area-preserving diffeomorphisms of simple manifolds have been studied recently in the context of string theory in particle physics.[4,5] Here we apply these transformations on the J and θ spheres and hope that the resulting formalism will be useful in models which are based on the angular-momentum formalism. In particular, they could be useful in problems such as the under-

 ©1991 The American Physical Society

standing of time evolution of systems with complicated Hamiltonians, the solution of nonlinear (soliton) systems, where such transformations could reduce the problem into another one which is soluble, the quantization of systems with nonflat phase space,[6] etc.

In Sec. V we show that the Hilbert space H is isomorphic to the Hilbert space of the two-mode harmonic oscillator. We then apply some of the ideas of this paper on the model

$$h_1=\omega_1 a_1^\dagger a_1+\omega_2 a_2^\dagger a_2+\lambda a_1^\dagger a_2+\lambda^* a_1 a_2^\dagger \ , \tag{3}$$

which is widely used in quantum optics in connection with frequency converters, interferometers, beam splitters,[7] etc. This example shows how the formalism presented can be used for the understanding of the time evolution of various systems.

We conclude in Sec. VI with a discussion of the results.

II. $SL(2,Z_{2j+1})$ TRANSFORMATIONS IN THE $Z_{2j+1}\times Z_{2j+1}$ ANGLE-ANGULAR-MOMENTUM PHASE SPACE

We consider the $(2j+1)$-dimensional Hilbert space H_{2j+1} associated with angular momentum j. We shall use the notation and a lot of the relations proved in Ref. 1 and we shall limit the present study to the case of integer j (Bose sector). In H_{2j+1} we consider the J basis of the angular-momentum states $|J;jm\rangle$, and also the dual θ basis of the phase states $|\theta;jm\rangle$. We also consider the angular-momentum J_z,J_+,J_- and the phase operators $\theta_z,\theta_+,\theta_-$. We have proved that the "Cartesian operators" J_+,J_- (and also the θ_+,θ_-) can be expressed in terms of the "polar operators" J_r,E (and θ_r,F) as

$$\begin{aligned} &J_+=J_rE, \quad \theta_+=\theta_rF \ ,\\ &J_-=E^\dagger J_r, \quad \theta_-=F^\dagger\theta_r \ , \end{aligned} \tag{4}$$

where

$$\begin{aligned} &E=\exp\left[-i\frac{2\pi}{2j+1}\theta_z\right] ,\\ &F=\exp\left[i\frac{2\pi}{2j+1}J_z\right] ,\\ &J_r=(J_+J_-)^{1/2} \ ,\\ &\theta_r=(\theta_+\theta_-)^{1/2} \ . \end{aligned} \tag{5}$$

Let Z_{2j+1} be the set of integers modulo $2j+1$. The set $Z_{2j+1}\times Z_{2j+1}$ is the angle-angular-momentum phase space for this Hilbert space. The first Z_{2j+1} is associated with the angles (which take the values $2\pi n/(2j+1)$ with $-j\le n\le j$); the second Z_{2j+1} is associated with angular momenta (which take the values $-j\le m\le j$). It has been explained in Ref. 1 that the operators E^kF^l play the role of the Weyl group in this phase space

$$\begin{aligned} &F^lE^k=E^kF^l\exp\left[i\frac{2\pi}{2j+1}kl\right] ,\\ &E^{2j+1}=F^{2j+1}=1 \ ,\\ &E^k|J;jm\rangle=|J;jm+k\rangle \ ,\\ &F^l|\theta;jm\rangle=|\theta;jm+l\rangle \ . \end{aligned} \tag{6}$$

It has also been explained that this is a discrete Weyl group (the k,l are integers), and therefore Eqs. (5) and (6) do not lead to the Weyl algebra $[J_z,\theta_z]=i1$. In fact, the $[J_z,\theta_z]$ has been calculated explicitly and found to be different from $i1$. However, it is clear that we do have a Weyl group and that the operators E,F perform translations in angular momentum and angle correspondingly. References 8 have studied the Wigner function in this phase space. References 9 have explained in a different context that operators similar to E^kF^l provide a basis for the $SU(2j+1)\times U(1)$ algebra. This basis is an alternative to the usual Cartan-Weyl basis. It is clear that the E^kF^l generate all the unitary transformations within H_{2j+1}.

We consider transformations similar to the Bogoliubov transformations in the position-momentum phase space of the harmonic oscillator

$$\begin{aligned} &x'=\lambda x+kp \ ,\\ &p'=\mu x+\nu p \ ,\\ &\lambda,k,\mu,\nu\in R \ ,\\ &\lambda\nu-\mu k=1 \ ,\\ &[x',p']=[x,p]=i \ , \end{aligned} \tag{7}$$

which are based on the group $SL(2,R)$ and which have led to the concept of squeezing in quantum optics. Since the Weyl group of Eq. (6) is discrete we shall work with the E,F rather than J_z,θ_z. We consider the transformations

$$\begin{aligned} &E'=E^\lambda F^k \ ,\\ &F'=E^\mu F^\nu \ ,\\ &\lambda,k,\mu,\nu\in Z_{2j+1} \ ,\\ &\lambda\nu-\mu k=1(\mathrm{mod}2j+1) \ . \end{aligned} \tag{8}$$

We can easily show that the E',F' also form a Weyl group

$$\begin{aligned} &F'E'=E'F'\exp\left[i\frac{2\pi}{2j+1}\right] ,\\ &(E')^{2j+1}=(F')^{2j+1}=1 \ . \end{aligned} \tag{9}$$

Transformations (8) form a group. Indeed we can easily show that the product of two such transformations is a transformation of the same type, that each transformation has an inverse, and that there is a unity transformation. This group is the $SL(2,Z_{2j+1})$. We can now proceed and explore how the various operators and states change under these transformations. For simplicity we shall study in detail two important special cases. The first one is the transformations (8) with $k=\mu=0$,

$\lambda\nu=1(\text{mod}2j+1)$. In this case number theory proves that the $\lambda,2j+1$ (and also the $\nu,2j+1$) are coprime, i.e., their greatest common divisor is one. The ν is the "inverse" of λ and will be denoted as $\tilde{\lambda}$,

$$\begin{aligned}&\lambda\tilde{\lambda}=1(\text{mod}2j+1)\ ,\\&\lambda,\tilde{\lambda}\in Z_{2j+1}\ .\end{aligned} \tag{10}$$

We now consider the unitary operators

$$\begin{aligned}U_\lambda&=\sum_{n=-j}^{j}|J;j,\lambda n\rangle\langle J;j,n|\\&=\sum_{n=-j}^{j}|J;j,n\rangle\langle J;j,\tilde{\lambda}n|\\&=\sum_{n=-j}^{j}|\theta;j,n\rangle\langle\theta;j,\lambda n|\\&=\sum_{n=-j}^{j}|\theta;j,\tilde{\lambda}n\rangle\langle\theta;j,n|\ ,\end{aligned} \tag{11}$$

$$U_\lambda U_\lambda^\dagger=U_\lambda^\dagger U_\lambda=1\ . \tag{12}$$

The fact that $\lambda,2j+1$ are coprime implies that as n takes all the values between $-j$ and j the λn also takes all the values between $-j$ and $j(\text{mod}2j+1)$; and this is important in the proof of the unitarity of the operators U_λ. The equalities in Eq. (11) can easily be proved if we use the relations between the J and θ states, given in Ref. 1. Note also that

$$\begin{aligned}&U_\lambda^\dagger=U_{\tilde{\lambda}}\ ,\\&U_{\lambda=1}=1\ .\end{aligned} \tag{13}$$

We can easily prove

$$\begin{aligned}&E'=U_\lambda E U_\lambda^\dagger=E^\lambda\ ,\\&F'=U_\lambda F U_\lambda^\dagger=F^{\tilde{\lambda}}\ .\end{aligned} \tag{14}$$

We now act with the unitary operators U_λ on the angular-momentum operators J_z,J_+,J_- and call $J_{\lambda z},J_{\lambda+},J_{\lambda-}$, the resulting operators correspondingly. They will also obey the SU(2) algebra

$$\begin{aligned}&J_{\lambda z}=U_\lambda J_z U_\lambda^\dagger\ ,\\&J_{\lambda+}=U_\lambda J_+ U_\lambda^\dagger\ ,\\&J_{\lambda-}=U_\lambda J_- U_\lambda^\dagger\ ,\end{aligned} \tag{15}$$

$$[J_{\lambda+},J_{\lambda-}]=2J_{\lambda z},\quad [J_{\lambda z},J_{\lambda\pm}]=\pm J_{\lambda\pm}\ . \tag{16}$$

In order to see how the operators $J_{\lambda z},J_{\lambda+},J_{\lambda-}$ act on a state $|J;j,l\rangle$ we first write this state as $|J;j,l=\lambda n\rangle$ with $-j\le n\le j$; we can always do this in a unique way. Then we easily prove

$$\begin{aligned}&J_{\lambda z}|J;j,\lambda n\rangle=n|J;j,\lambda n\rangle\ ,\\&J_{\lambda+}|J;j,\lambda n\rangle=[j(j+1)-n(n+1)]^{1/2}|J;j,\lambda(n+1)\rangle\ ,\\&J_{\lambda-}|J;j,\lambda n\rangle=[j(j+1)-n(n-1)]^{1/2}|J;j,\lambda(n-1)\rangle\ .\end{aligned} \tag{17}$$

We see that the operator $J_{\lambda+}$ translates the state $|J;j,\lambda n\rangle$ up in the ladder by λ steps. We also see that

$$\begin{aligned}&J_{\lambda+}|J;j,\lambda j\rangle=0\ ,\\&J_{\lambda-}|J;j,-\lambda j\rangle=0\ .\end{aligned} \tag{18}$$

As an example let us take the case $\lambda=2$, $j=2$. Starting from the state $|j;2,-4=1(\text{mod}5)\rangle$ and acting successively with the operator J_{2+}, we get the following ladder of states:

$$\begin{aligned}|J;2,-4=1\rangle\to|J;2,-2\rangle\to|J;2,0\rangle\to|J;2,2\rangle\\\to|J;2,4=-1\rangle\\\to0\end{aligned} \tag{19}$$

with numerical coefficients that have been omitted. If we relabel the states so that

$$|J_\lambda;j,n\rangle\equiv U_\lambda|J;j,n\rangle=|J;j,\lambda n\rangle\ , \tag{20}$$

then with respect to the $|J_\lambda;j,n\rangle$ the $J_{\lambda z},J_{\lambda+},J_{\lambda-}$ play the role of the standard angular-momentum operators (e.g., the $J_{\lambda+}$ translates the $|J_\lambda;j,n\rangle$ up in the ladder by one step, etc.).

We next consider the operators $U_{-\lambda}$ $(\lambda>0)$ and prove easily

$$\begin{aligned}&J_{(-\lambda)z}=-J_{\lambda z}\ ,\\&J_{(-\lambda)+}=J_{\lambda-}\ ,\\&J_{(-\lambda)-}=J_{\lambda+}\ .\end{aligned} \tag{21}$$

The θ counterpart of Eq. (15) is

$$\begin{aligned}&\theta_{\lambda z}=U_\lambda\theta_z U_\lambda^\dagger\ ,\\&\theta_{\lambda+}=U_\lambda\theta_+ U_\lambda^\dagger\ ,\\&\theta_{\lambda-}=U_\lambda\theta_- U_\lambda^\dagger\ ,\end{aligned} \tag{22}$$

$$[\theta_{\lambda+},\theta_{\lambda-}]=2\theta_{\lambda z},\quad [\theta_{\lambda z},\theta_{\lambda\pm}]=\pm\theta_{\lambda\pm}\ . \tag{23}$$

We consider now the state $|\theta;j,l\rangle$ and write it as $|\theta;j,l=\tilde{\lambda}m\rangle$ with $-j\le m\le j$. We can easily see that

$$\begin{aligned}&\theta_{\lambda z}|\theta;j,\tilde{\lambda}m\rangle=m|\theta;j,\tilde{\lambda}m\rangle\ ,\\&\theta_{\lambda+}|\theta;j,\tilde{\lambda}m\rangle=[j(j+1)-m(m+1)]^{1/2}\\&\qquad\times|\theta;j,\tilde{\lambda}(m+1)\rangle\ ,\\&\theta_{\lambda-}|\theta;j,\tilde{\lambda}m\rangle=[j(j+1)-m(m-1)]|\theta;j,\tilde{\lambda}(m-1)\rangle\ .\end{aligned} \tag{24}$$

We see that the $\theta_{\lambda+}$ translates the state $|\theta;j,\tilde{\lambda}m\rangle$ up in the ladder by $\tilde{\lambda}$ steps. We also see that if we relabel the states so that

$$|\theta_\lambda;j,m\rangle\equiv U_\lambda|\theta;j,m\rangle=|\theta;j,\tilde{\lambda}m\rangle\ , \tag{25}$$

then with respect to the $|\theta_\lambda;j,m\rangle$ the $\theta_{\lambda z},\theta_{\lambda+},\theta_{\lambda-}$ play the role of the angle operators $\theta_z,\theta_+,\theta_-$ (e.g., the $\theta_{\lambda+}$ translates the $|\theta_\lambda;j,m\rangle$ up in the ladder by one step, etc.).

The above construction clarifies the nature of the dilation-contraction transformations in our discrete θJ phase space. The second special case that we consider is the transformation (8) with $\lambda=\nu=0$, $k=-\mu=1$. Let us consider the finite Fourier transforms

$$\begin{aligned} U&=\sum_n |\theta;jn\rangle\langle J;jn| \\ &=\sum_n |J;jn\rangle\langle\theta;j-n| \\ &=(2j+1)^{-1/2}\sum_{n,m}\exp\left[i\frac{2\pi}{2j+1}mn\right]|J;jm\rangle\langle J;jn| \\ &=(2j+1)^{-1/2}\sum_{n,m}\exp\left[i\frac{2\pi}{2j+1}mn\right]|\theta;jm\rangle\langle\theta;jn| \ , \end{aligned} \tag{26}$$

$$UU^\dagger=U^\dagger U=1 \ . \tag{27}$$

They have been used in the context of general finite-dimensional systems in Ref. 3 and in the context of finite Fourier transforms for signal-processing applications in Ref. 10. Here we use them in our context and discuss their effect on the angular-momentum operators and states and also on the angle operators and states. The proof of the equalities in Eq. (26) is straightforward. Using Eq. (26) we can prove

$$\begin{aligned} &U^4=1 \ , \\ &U|J;j,n\rangle=|\theta;j,n\rangle \ , \\ &U|\theta;j,n\rangle=|J;j,-n\rangle \ , \\ &UJ_zU^\dagger=\theta_z \ , \\ &U\theta_zU^\dagger=-J_z \ , \\ &UEU^\dagger=F \ , \\ &UFU^\dagger=E^\dagger \ , \\ &UJ_+U^\dagger=\theta_+ \ , \\ &U\theta_+U^\dagger=J_- \ , \\ &UJ_-U^\dagger=\theta_- \ , \\ &U\theta_-U^\dagger=J_+ \ . \end{aligned} \tag{28}$$

Equations (28) show that these transformations map the J states and operators into the θ states and operators. It is also clear that the E is mapped into F and the F into E^{-1}, i.e., this is the special case $\lambda=\nu=0$, $k=-\mu=1$ of the general transformations (8).

III. J AND θ REPRESENTATIONS

We consider the Hilbert space H which is the direct sum of all the Hilbert spaces H_{2j+1} for all j,

$$H=\sum_j H_{2j+1} \ . \tag{29}$$

In this space we consider the J basis of angular-momentum states

$$\begin{aligned} &\{|J;jm\rangle \mid -j\le m\le j;\ j=0,1,2,\dots\}, \\ &\langle J;jm|J;lk\rangle=\delta_{jl}\delta_{mk} \ , \\ &\sum_m |J;jm\rangle\langle J;jm|=\pi_{2j+1} \ , \\ &\sum_j \pi_{2j+1}=1 \ , \\ &\pi_{2j+1}\pi_{2l+1}=\delta_{jl}\pi_{2j+1} \ , \end{aligned} \tag{30}$$

where π_{2j+1} are projection operators in the subspace H_{2j+1}. We also consider the θ basis of phase states

$$\begin{aligned} &\{|\theta;jm\rangle \mid -j\le m\le j;\ j=0,1,2,\dots\} \ , \\ &\langle\theta,jm|\theta;lk\rangle=\delta_{jl}\delta_{mk} \ , \\ &\sum_m|\theta;jm\rangle\langle\theta;jm|=\pi_{2j+1} \ . \end{aligned} \tag{31}$$

An arbitrary state $|f\rangle$ in H can be expressed in these two bases as

$$\begin{aligned} &|f\rangle=\sum_{j,m}S_{jm}|J;jm\rangle=\sum_{j,m}t_{jm}|\theta;jm\rangle \ , \\ &S_{jm}=(2j+1)^{-1/2}\sum_n t_{jm}\exp\left[i\frac{2\pi mn}{2j+1}\right] \ . \end{aligned} \tag{32}$$

The $\{S_{jm}\}$ and $\{t_{jm}\}$ represent the state $|f\rangle$ in the J and θ representations correspondingly.

In Ref. 1 we have defined various operators (e.g., J_z,θ_z, etc.) within the Hilbert space H_{2j+1}; in this sense each of these operators has the index $2j+1$ which for simplicity has been omitted. Summation over j defines these operators in the Hilbert space H. If we call Λ any of these operators, we have

$$\begin{aligned} &\Lambda=\sum_j\Lambda_{2j+1} \ , \\ &\Lambda_{2j+1}=\pi_{2j+1}\Lambda\pi_{2j+1} \ , \\ &\Lambda_{2j_1+1}\Lambda_{2j_2+1}=0 \quad \text{for } j_1\neq j_2 \ . \end{aligned} \tag{33}$$

Another basis in H is the usual $|\alpha,\beta\rangle$ basis where α,β are angles on a sphere. Within our notation we shall denote this basis as $|J;\alpha\beta\rangle$ and we shall introduce later another dual basis with the notation $|\theta;\alpha\beta\rangle$. We shall refer to them as spherical bases. We define

$$\begin{aligned} &|J;\alpha\beta\rangle=\sum_{j,m}Y^*_{jm}(\alpha,\beta)|J;jm\rangle \ , \\ &0\le a\le\pi;\quad 0\le\beta<2\pi \ , \\ &\int|J;\alpha\beta\rangle\langle J;\alpha\beta|d\cos\alpha\, d\beta=1 \ , \\ &\langle J;\alpha_1\beta_1|J;\alpha_2\beta_2\rangle=\delta(\cos\alpha_1-\cos\alpha_2)\delta(\beta_1-\beta_2) \ , \end{aligned} \tag{34}$$

where $Y_{jm}(a,\beta)$ are the usual spherical harmonics. The α,β span a sphere that we shall call J sphere. We also define

$$|\theta;a\beta\rangle=\sum_{j,m} Y^*_{jm}(\alpha,\beta)|\theta;jm\rangle\ ,$$
$$0\le a\le\pi;\quad 0\le\beta<2\pi\ , \tag{35}$$

where here α,β are angles on a sphere that we shall call θ sphere. The $|\theta;a\beta\rangle$ obey relations similar to these given in Eq. (34).

We introduce the dual spherical harmonics $X_{jm}(\alpha,\beta)$ as

$$X_{jm}(\alpha,\beta)=\langle J;\alpha\beta|\theta;jm\rangle$$
$$=\langle\theta;\alpha\beta|J;j-m\rangle$$
$$=(2j+1)^{-1/2}\sum_n Y_{jn}(\alpha,\beta)\exp\left[i\frac{2\pi}{2j+1}nm\right]. \tag{36}$$

Like the $Y_{jm}(\alpha,\beta)$, the $X_{jm}(\alpha,\beta)$ obey the orthogonality and completeness relations

$$\sum_{j,m}X^*_{jm}(\alpha,\beta)X_{jm}(\gamma,\delta)=\delta(\cos\alpha-\cos\gamma)\delta(\beta-\delta)\ ,$$
$$\int d\cos\alpha\, d\beta\, X^*_{jm}(\alpha,\beta)X_{lk}(\alpha,\beta)=\delta_{jl}\delta_{mk}\ . \tag{37}$$

Note that

$$|J;\alpha\beta\rangle=\sum_{j,m}X^*_{jm}(\alpha,\beta)|\theta,jm\rangle\ ,$$
$$|\theta;\alpha\beta\rangle=\sum_{j,m}X^*_{j,-m}(\alpha,\beta)|J;jm\rangle\ . \tag{38}$$

We can now use the states $|J;\alpha\beta\rangle$ and $|\theta;\alpha\beta\rangle$ to introduce new representations. The arbitrary state $|f\rangle$ of Eq. (32) can be represented by the function

$$f_J(\alpha,\beta)=\langle J;\alpha\beta|f\rangle=\sum_{j,m}S_{jm}Y_{jm}(\alpha,\beta)$$
$$=\sum_{j,m}t_{jm}X_{jm}(\alpha,\beta)\ , \tag{39}$$

where a,β are angles on the J sphere. It can also be represented by the function

$$f_\theta(\alpha,\beta)=\langle\theta;\alpha\beta|f\rangle=\sum_{j,m}t_{jm}Y_{jm}(\alpha,\beta)$$
$$=\sum_{j,m}S_{jm}X_{j,-m}(\alpha,\beta)\ , \tag{40}$$

where α,β are angles on the θ sphere. We call them spherical-J and spherical-θ representations correspondingly. The scalar product is in these representations, expressed as

$$\langle g|f\rangle=\int d\cos a\, d\beta\, g^*_J(\alpha,\beta)f_J(\alpha,\beta)$$
$$=\int d\cos\alpha\, d\beta\, g^*_\theta(a,\beta)f_\theta(a,\beta)\ . \tag{41}$$

We next introduce a transformation that connects the spherical-J to the spherical-θ representation. We start with the transformations of Eq. (26) which now we denote as U_{2j+1}, with the subscript indicating the fact that they act on the Hilbert space H_{2j+1}. We define the unitary operators

$$U=\sum_j U_{2j+1}\ ,$$
$$U^4=1\ , \tag{42}$$
$$UU^\dagger=U^\dagger U=1\ .$$

The matrix elements of U

$$U(\alpha,\beta|\gamma,\delta)=\langle J;\alpha\beta|U|J;\gamma\delta\rangle$$
$$=\langle J;\alpha\beta|\theta;\gamma\delta\rangle$$
$$=\sum_{j,m}X_{jm}(\alpha,\beta)Y^*_{jm}(\gamma,\delta)$$
$$=\sum_{j,m}Y_{jm}(\alpha,\beta)X^*_{j,-m}(\gamma,\delta) \tag{43}$$

can be used to connect the J with the θ representation as follows:

$$f_J(\alpha,\beta)=\int U(\alpha,\beta|\gamma,\delta)f_\theta(\gamma,\delta)d\cos\gamma\, d\delta\ ,$$
$$f_\theta(\alpha,\beta)=\int U^*(\alpha,\beta|\gamma,\delta)f_J(\gamma,\delta)d\cos\gamma\, d\delta\ . \tag{44}$$

We refer to Eq. (44) as the spherical $J-\theta$ transform.

An arbitrary operator ψ and its action on a state $|f\rangle$ can be expressed in the spherical J representation as

$$\psi(\alpha,\beta|\gamma,\delta)=\langle J;\alpha\beta|\psi|J;\gamma\delta\rangle\ ,$$
$$\langle J;\alpha\beta|\psi|f\rangle=\int\psi(\alpha,\beta|\gamma,\delta)f_J(\gamma,\delta)d\cos\gamma\, d\delta\ . \tag{45}$$

We next give the expressions for the angular momentum and angle operators in the spherical J representation,

$$\langle J;\alpha\beta|J_z|J;\gamma\delta\rangle=-i\delta(\cos\alpha-\cos\gamma)\delta'(\beta-\delta)\ , \tag{46}$$

$$\langle J;\alpha\beta|J_+|J;\gamma\delta\rangle=e^{i\beta}[-\sin a\,\delta'(\cos\alpha-\cos\gamma)\delta(\beta-\delta)$$
$$+i\cot\alpha\,\delta(\cos\alpha-\cos\gamma)\delta'(\beta-\delta)]\ , \tag{47}$$

$$\langle J;\alpha\beta|J_-|J;\gamma\delta\rangle=e^{i\beta}[\sin\alpha\,\delta'(\cos\alpha-\cos\gamma)\delta(\beta-\delta)$$
$$+i\cot\alpha\,\delta(\cos\alpha-\cos\gamma)\delta'(\beta-\delta)]\ , \tag{48}$$

$$\langle J;\alpha\beta|\theta_z|J;\gamma\delta\rangle=\sum_{j,m}mX_{jm}(\alpha,\beta)X^*_{jm}(\gamma,\delta)\ , \tag{49}$$

$$\langle J;\alpha\beta|\theta_+|J;\gamma\delta\rangle=\sum_{j,m}[j(j+1)-m(m+1)]^{1/2}$$
$$\times X_{j,m+1}(\alpha,\beta)X^*_{j,m}(\gamma,\delta)\ , \tag{50}$$

$$\langle J;\alpha\beta|\theta_-|J;\gamma\delta\rangle=\sum_{j,m}[j(j+1)-m(m+1)]^{1/2}$$
$$\times X_{j,m-1}(\alpha,\beta)X^*_{jm}(\gamma,\delta)\ . \tag{51}$$

The proof is straightforward if we take into account the expressions for these operators given in Ref. 1.

IV. AREA-PRESERVING DIFFEOMORPHISMS ON THE J AND θ SPHERES

Let us consider area-preserving diffeomorphisms on the J sphere, i.e., transformations,

$$\cos\alpha' = A(\cos\alpha,\beta) ,$$
$$\beta' = B(\alpha,\beta) , \qquad (52)$$
$$\frac{\partial(\cos\alpha',\beta')}{\partial(\cos\alpha,\beta)} = 1 ,$$

where A,B are differentiable functions of α,β with Jacobian equal to one. We shall use them to transform the spherical J basis into another one with similar properties. We define the state $|J';\alpha';\beta'\rangle$ as

$$|J';\alpha',\beta'\rangle \equiv |J;\alpha,\beta\rangle ,$$
$$\langle J';\alpha',\beta'|J';\gamma',\delta'\rangle = \delta(\cos\alpha'-\cos\gamma')\delta(\beta'-\delta') , \qquad (53)$$
$$\int |J';\alpha',\beta'\rangle\langle J';\alpha',\beta'| d\cos\alpha' d\beta' = 1 .$$

Notice that the fact that the area is preserved is important for the proof of (53). We can now define the spherical J' representation as

$$f_{J'}(\alpha',\beta') = f_J(\alpha,\beta) = \langle J';\alpha'\beta'|f\rangle \qquad (54)$$

and prove easily that due to the fact that the area is preserved, the scalar product of two states is given by a formula similar to that of Eq. (41).

Originally such transformations have been studied by Arnold[11] in a hydrodynamical context and more recently they have been studied in the context of string theory in particle physics.[4,5] Following these references, we consider the infinitesimal version of transformations (52)

$$\cos\alpha' = \cos\alpha + Q_1(\alpha,\beta)\delta\varepsilon ,$$
$$\beta' = \beta + Q_2(\alpha,\beta)\delta\varepsilon , \qquad (55)$$
$$\frac{\partial Q_1}{\partial\cos\alpha} + \frac{\partial Q_2}{\partial\beta} = 0 ,$$

where the last constraint expresses that the Jacobian is equal to one. For a topologically trivial manifold like a sphere (with Betti number zero) this constraint implies that there exists a "potential" $g(\alpha,\beta)$ such that

$$Q_1 = -\frac{\partial g}{\partial\beta}, \quad Q_2 = \frac{\partial g}{\partial\cos\alpha} . \qquad (56)$$

Using (54), (55), and (56) we get

$$\frac{f_J(\alpha',\beta')-f_J(\alpha,\beta)}{\delta\varepsilon} = \frac{\partial g}{\partial\cos\alpha}\frac{\partial f_J}{\partial\beta} - \frac{\partial f_J}{\partial\cos\alpha}\frac{\partial g}{\partial\beta} . \qquad (57)$$

Motivated from this, we give the following definition. Let $g(\alpha,\beta)$ be a function defined on the J sphere. We define the operator J_g as

$$\langle J;\alpha\beta|J_g|f\rangle = \{g(\alpha,\beta),f_J(\alpha,\beta)\}$$
$$\equiv \frac{\partial(g(\alpha,\beta),f_J(\alpha,\beta))}{\partial(\cos\alpha,\beta)}$$
$$\equiv \frac{\partial g}{\partial\cos\alpha}\frac{\partial f_J}{\partial\beta} - \frac{\partial g}{\partial\beta}\frac{\partial f_J}{\partial\cos\alpha} , \qquad (58)$$

where $|f\rangle$ is an arbitrary state and $f_J(\alpha,\beta) = \langle J;\alpha\beta|f\rangle$ its spherical-J representation. We can easily prove that

$$[J_{g_1},J_{g_2}] = J_{\{g_1,g_2\}} \qquad (59)$$

and also that

$$J_g\beta = \frac{\partial g}{\partial\cos\alpha} ,$$
$$J_g\cos\alpha = -\frac{\partial g}{\partial\beta} , \qquad (60)$$
$$J_g f_J(\alpha,\beta) = f_J(J_g\alpha,J_g\beta) ,$$
$$[\exp(tJ_g)]f_J(\alpha,\beta) = f_J([\exp(tJ_g)]\alpha,[\exp(tJ_g)]\beta) .$$

The last of these equations could be used to describe the evolution of a system with Hamiltonian J_g. Note that we use the same symbol J_g for both the operator and its spherical-J representation [Eq. (45)]. Some other properties of these operators are given in the Appendix.

We next analyze the function g with respect to a basis [e.g., the $Y_{jm}(\alpha,\beta)$] and express the J_g as

$$g(\alpha,\beta) = \sum g_{jm}Y_{jm}(\alpha,\beta) ,$$
$$J_g = \sum_{j,m} g_{jm}J_{jm} , \qquad (61)$$
$$\langle J;\alpha\beta|J_{jm}|f\rangle = \{Y_{jm}(\alpha,\beta),f_J(\alpha,\beta)\}$$
$$= \frac{\partial(Y_{jm}(\alpha,\beta),f_J(\alpha,\beta))}{\partial(\cos\alpha,\beta)} .$$

The J_{jm} create area-preserving diffeomorphisms on the J sphere. For $j=1$ we substitute the Y_{1m} in Eq. (61) and get the expressions for the angular-momentum operators (up to a normalization constant). Therefore the J_{1m} are the standard angular-momentum operators J_+,J_z,J_- (up to a normalization constant). This is not surprising because the solid rotations are indeed area-preserving transformations. Note that

$$\{Y_{j_1m_1},Y_{j_2m_2}\} = \sum_{j_3,m_3}\sigma(Y_{j_1m_1},Y_{j_2m_2},Y_{j_3m_3})Y_{j_3m_3} , \qquad (62)$$

where the σ coefficients have been given in Ref. 5. For our purposes, we also need the Poisson brackets for the X_{jm} which we defined in (36). From (36) and (62) we easily see that

$$\{X_{j_1m_1},Y_{j_2m_2}\} = \sum_{j_3,m_3}\sigma(X_{j_1m_1},Y_{j_2m_2},Y_{j_3m_3})Y_{j_3m_3}$$
$$= \sum_{j_3,m_3}\sigma(X_{j_1m_1},Y_{j_2m_2},X_{j_3m_3})$$
$$\times X_{j_3m_3} , \qquad (63)$$

where the various σ coefficients are related with appropriate finite Fourier transforms, e.g.,

$$\sigma(X_{j_1m_1},Y_{j_2m_2},X_{j_3m_3}) = \sum_{j_3,m_3}\sigma(Y_{j_1n_1},Y_{j_2m_2},Y_{j_3n_3})$$
$$\times\exp\left[i\frac{2\pi}{2j_1+1}m_1n_1\right]$$
$$\times\exp\left[i\frac{2\pi}{2j_3+1}m_3n_3\right] . \qquad (64)$$

Similar expressions can be given for the $\{X_{j_1m_1},X_{j_2m_2}\}$. Using (59) we easily prove that the J_{jm} obey the infinite-dimensional algebra

$$[J_{j_1m_1},J_{j_2m_2}]=\sum_{j_3,m_3}\sigma(Y_{j_1m_1},Y_{j_2m_2},Y_{j_3m_3})J_{j_3m_3} . \tag{65}$$

In analogy with the J operators we also introduce the θ operators associated with area-preserving diffeomorphisms on the θ sphere. Let h be a function defined on the θ sphere. We define the operators θ_h as

$$\langle\theta;\alpha\beta|\theta_h|f\rangle=\frac{\partial(h(\alpha,\beta),f_\theta(\alpha,\beta))}{\partial(\cos\alpha,\beta)} , \tag{66}$$

where $|f\rangle$ is an arbitrary state and $f_\theta(\alpha,\beta)=\langle\theta;\alpha\beta|f\rangle$ its spherical-θ representation. We now analyze the function h in the Y_{jm} basis and express θ_h as

$$\begin{aligned} h(\alpha,\beta)&=\sum_{j,m}h_{jm}Y_{jm} ,\\ \theta_h&=\sum_{j,m}h_{jm}\theta_{jm} ,\\ \langle\theta;\alpha\beta|\theta_{jm}|f\rangle&=\frac{\partial(Y_{jm}(\alpha,\beta),f_\theta(\alpha,\beta))}{\partial(\cos\alpha,\beta)} . \end{aligned}\tag{67}$$

Arguments similar to those given above for the J case show that for $j=1$ the θ_{1m} are the angle operators $\theta_z,\theta_+,\theta_-$ (up to a normalization constant).

We have presented the J_{jm} operators in the spherical-J representation and the θ_{jm} operators in the spherical-θ representation. It is possible to find formulas for the J_{jm} operators in the spherical-θ representation (and also for the θ_{jm} operators in the spherical-J representation), but the expressions are complicated and we do not present them here.

We now consider a general state $|f\rangle$ and calculate the $J_{j_1m_1}|f\rangle$ and $\theta_{j_1m_1}|f\rangle$ in terms of its components $S_{j_2m_2}=\langle J;j_2m_2|f\rangle$ and $t_{j_2m_2}=\langle\theta;j_2m_2|f\rangle$,

$$J_{j_1m_1}|f\rangle=\sum_{j_3,m_3}\sigma(Y_{j_1m_1},Y_{j_2m_2},Y_{j_3m_3})\times S_{j_2m_2}|J;j_3m_3\rangle , \tag{68}$$

$$\theta_{j_1m_1}|f\rangle=\sum_{j_3,m_3}\sigma(Y_{j_1m_1},Y_{j_2m_2},Y_{j_3m_3})\times t_{j_2m_2}|\theta;j_3m_3\rangle . \tag{69}$$

V. APPLICATIONS

We consider the two-mode harmonic oscillator

$$h_0=\omega_1a_1^\dagger a_1+\omega_2a_2^\dagger a_2 . \tag{70}$$

Its Hilbert space is the direct produce $H_A\times H_B$, where H_A is the Hilbert space of the first mode and H_B is the Hilbert space of the second mode. We shall show that the $H_A\times H_B$ is isomorphic to the Hilbert space H of Eq. (29).

We consider the Schwinger representations of SU(2) where[12]

$$\begin{aligned} J_+&=a_1^\dagger a_2 ,\\ J_-&=a_1a_2^\dagger ,\\ J_z&=\tfrac{1}{2}(a_1^\dagger a_1-a_2^\dagger a_2) ,\\ J^2&=[\tfrac{1}{2}(a_1^\dagger a_1+a_2^\dagger a_2)][\tfrac{1}{2}(a_1^\dagger a_1+a_2^\dagger a_2)+1] . \end{aligned}\tag{71}$$

The number eigenstates $|N_1,N_2\rangle$ correspond to the angular-momentum states $|J;jm\rangle$ as follows:

$$\begin{aligned} |N_1,N_2\rangle&=|J;j=\tfrac{1}{2}(N_1+N_2),m=\tfrac{1}{2}(N_1-N_2)\rangle ,\\ |N_1=j+m,N_2=j-m\rangle&=|J;jm\rangle . \end{aligned}\tag{72}$$

The Hilbert space H_{2j+1} is spanned by the states

$$H_{2j+1}=\{|N_1,N_2=2j-N_1\rangle|N_1=0,\ldots(2j)\} . \tag{73}$$

It is now clear that the $H=\sum H_{2j+1}$ is isomorphic to $H_A\times H_B$. In this model

$$\begin{aligned} |J;\alpha\beta\rangle&=\sum_{j,m}Y_{jm}^*(\alpha,\beta)|N_1=j+m,N_2=j-m\rangle ,\\ |\theta;\alpha\beta\rangle&=\sum_{j,m}X_{j,-m}^*(\alpha,\beta)|N_1=j+m,N_2=j-m\rangle , \end{aligned}\tag{74}$$

and the J and θ representations of a state $|f\rangle$ are defined as

$$\begin{aligned} |f\rangle&=\sum_{N,M}f_{NM}|N,M\rangle ,\\ f_J(\alpha,\beta)&=\langle J;\alpha\beta|f\rangle=\sum_{N,M}f_{NM}Y_{jm}(\alpha,\beta) ,\\ f_\theta(\alpha,\beta)&=\langle\theta;\alpha\beta|f\rangle=\sum_{N,M}f_{NM}X_{j,-m}(\alpha,\beta) ,\\ j&=\tfrac{1}{2}(N+M) ,\\ m&=\tfrac{1}{2}(N-M) . \end{aligned}\tag{75}$$

The J representation is suitable for the study of systems with the Hamiltonian (3), which we rewrite here as

$$\begin{aligned} h_1&=\Omega n_s+2\varepsilon J_z+\lambda J_++\lambda^*J_- ,\\ n_s&=a_1^\dagger a_1+a_2^\dagger a_2 ,\\ \Omega&=\tfrac{1}{2}(\omega_1+\omega_2) ,\\ \varepsilon&=\tfrac{1}{2}(\omega_1-\omega_2) . \end{aligned}\tag{76}$$

Note that

$$\begin{aligned} [n_s,J_z]&=[n_s,J_+]=[n_s,J_-]=0 ,\\ J^2&=\tfrac{1}{2}n_s(\tfrac{1}{2}n_s+1) . \end{aligned}\tag{77}$$

We assume that the two-mode system with the Hamiltonian (3), is initially ($t=0$) in the state

$$|f\rangle=\sum_j\pi_{2j+1}|f\rangle\equiv\sum_j|f_{2j+1}\rangle , \tag{78}$$

where $|f_{2j+1}\rangle$ belongs in the Hilbert space H_{2j+1}. The evolution of this system in time is given by

$$\exp(ih_1 t)|f\rangle=\exp(ih_1 t)\sum_j |f_{2j+1}\rangle$$
$$=\sum_j \exp[i(2j\Omega t)]W_1(\varepsilon t,\lambda t)|f_{2j+1}\rangle\ , \quad (79)$$

$$W_1(\varepsilon t,\lambda t)=\exp[it(2\varepsilon J_z+\lambda J_+ +\lambda^* J_-)]\ . \quad (80)$$

The operator W_1 describes SU(2) rotations. Consequently the state $|f_{2j+1}\rangle$ evolves into states which are within the Hilbert space H_{2j+1}. In the spherical-J representation we rewrite (78) as

$$f_J(\alpha,\beta)=\sum_j \phi_{2j+1}(\alpha,\beta)\ ,$$
$$\phi_{2j+1}(\alpha,\beta)=\langle J;\alpha\beta|f_{2j+1}\rangle \quad (81)$$
$$=\sum_m Y_{jm}(\alpha,\beta)\int f_J(\gamma,\delta)Y^*_{jm}(\gamma,\delta)d\cos\gamma\, d\delta\ ,$$

and the time evolution is given by

$$\langle J;\alpha\beta|\exp(ih_1 t)|f\rangle=\sum_j \exp[i(2j\Omega t)]\langle J;\alpha\beta|W_1(\varepsilon t,\lambda t)|f_{2j+1}\rangle$$
$$=\sum_j \exp[i(2j\Omega t)]\langle J;\alpha(t),\beta(t)|f_{2j+1}\rangle$$
$$=\sum_j \exp[i(2j\Omega t)]\phi_{2j+1}(\alpha(t),\beta(t))\ , \quad (82)$$

where the rotation operator $W_1(\varepsilon t,\lambda t)$ rotates the point (a,β) of the J sphere, into the point $(\alpha(t),\beta(t))$.

This technique, based on the Schwinger representation of SU(2), describes the evolution of the system h_1 with rotations in the J sphere. This picture leads to a full understanding of the behavior of such systems. The limitation of this method lies in the fact that it can only be used for the Hamiltonian h_1 and it cannot accommodate higher-order nonlinear terms like $g(a_2^\dagger)^2a_2^2, g(a_1^\dagger)^2a_2^2$, etc. Such terms are, however, very interesting in various models in quantum optics[13,14] in many-body theory, in chaotic systems, etc.

The fact that the terms of a Hamiltonian obey a finite Lie algebra [e.g., the terms of the Hamiltonian h_1 obey the SU(2) algebra] is very important in the practical calculations of the evolution of the relevant system. Usually the Baker-Hausdorff or other similar type of calculation between noncommuting operators is needed in such problems and the Lie algebra is helpful in the exact evaluation of such relations. In the case of Hamiltonians with higher-order terms [like $g(a_2^\dagger)^2a_2^2$], which do not obey a finite Lie algebra, such an exact calculation is very difficult. The usual method is to do some approximation in the small coupling regime and use it in numerical calculations. The infinite-dimensional algebra of Eq. (65) might help to do something deeper than this, if some extension of the Schwinger representation could be found that expresses the J_{jm} in terms of the $a_1,a_1^\dagger,a_2,a_2^\dagger$. It will be much easier to work with a Hamiltonian which contains J_{jm} type of terms rather than a Hamiltonian which contains high-order $a_1^N a_2^M$ terms. More generally we feel that the whole subject of infinite-dimensional Lie algebras, currently under intensive study in mathematical physics, could be useful in such problems.

VI. CONCLUSIONS

We have studied various types of transformations in the angle–angular-momentum phase space. In the $Z_{2j+1}\times Z_{2j+1}$ phase space associated with a system with angular momentum j the analogue of squeezing is $SL(2,Z_{2j+1})$ transformations. We have constructed explicitly the special case of contraction-dilation transformations and we have seen in detail the meaning of these concepts, in the case of a discrete phase space. For example, the operator $J_{\lambda+}$ translates the J states by λ steps and the $\theta_{\lambda+}$ translates the θ states by $\tilde{\lambda}$ steps, where $\tilde{\lambda}$ is the "inverse" integer of λ. We have also constructed explicitly finite Fourier transforms that map the J states and operators into the θ states and operators. These ideas together with the Wigner techniques given in Ref. 8 provide a full study of the angle–angular-momentum phase space.

We then considered the Hilbert space H of Eq. (29) because there are problems, such as the two-mode harmonic oscillator, where this Hilbert space is relevant. In this Hilbert space we have considered the J and θ bases, and also the spherical-J and spherical-θ bases. Representations with respect to these bases and also transformations that connect these representations have been given.

The spherical-J bases can be transformed into other ones with similar properties, with the use of area-preserving diffeomorphisms on the J sphere. The relevant operators J_{jm} have been defined and shown to obey the infinite-dimensional algebra of Eq. (65). The standard angular-momentum operators are special cases of these more general operators. The use of so rich transformations in quantum phase-space methods opens new directions in this area and might lead to a deeper understanding of the behavior of systems with complicated Hamiltonians.

In Sec. V the evolution of systems with the Hamiltonian h_1 [of Eq. (3)] has been studied with the use of the spherical-J representation. The possibility of extending this method into Hamiltonians with higher-order terms, using the transformations of Sec. IV, has been discussed. Progress in the study of nonlinear systems, beyond the usual perturbative methods, requires an understanding of the often hidden symmetries of these systems, and the transformations of Sec. IV might be useful in such studies.

We feel that the transformations studied in this paper contribute in the understanding of the angle–angular-momentum phase space and we hope that they will be useful in developing further the quantum phase-space methods.

APPENDIX

We present here some properties of the operators J_g.[15] Let λ,μ,ν be real numbers and f_1,f_2,g_1,g_2,g functions of α,β. We can easily prove

$$J_g(\mu f_1+\nu f_2)=\mu J_g f_1+\nu J_g f_2 \ , \tag{A1}$$

$$J_g(f_1 f_2)=f_1 J_g f_2+f_2 J_g f_1 \ , \tag{A2}$$

$$J_g\{f_1,f_2\}=\{J_g f_1,f_2\}+\{f_1,J_g f_2\} \ , \tag{A3}$$

$$J_g^N(\mu f_1+\nu f_2)=\mu J_g^N f_1+\nu J_g^N f_2 \ , \tag{A4}$$

$$J_g^N(f_1 f_2)=\sum_M \binom{N}{M}(J_g^M f_1)(J_g^{N-M} f_2) \ , \tag{A5}$$

$$J_g^N\{f_1,f_2\}=\sum_M \binom{N}{M}\{J_g^M f_1,J_g^{N-M} f_2\} \ , \tag{A6}$$

$$[\exp(\lambda J_g)](\mu f_1+\nu f_2)=\mu[\exp(\lambda J_g)]f_1+\nu[\exp(\lambda J_g)]f_2 \ , \tag{A7}$$

$$[\exp(\lambda J_g)](f_1 f_2)=[\exp(\lambda J_g)f_1][\exp(\lambda J_g)f_2] \ , \tag{A8}$$

$$[\exp(\lambda J_g)]\{f_1,f_2\}=\{[\exp(\lambda J_g)f_1],[\exp(\lambda J_g)f_2]\} \ . \tag{A9}$$

[1]A. Vourdas Phys. Rev. A **41**, 1653 (1990).
[2]R. Loudon and P. L. Knight, J. Mod. Opt. **34**, 709 (1987); M. C. Teich and B. E. A. Saleh, Quantum Opt. **1**, 153 (1989).
[3]R. Balian and G. Itzykson, C. R. Acad. Sci. Paris **303**, 773 (1986).
[4]E. G. Floratos and J. Iliopoulos, Phys. Lett. B **201**, 237 (1988); J. Hoppe, *ibid.* **215**, 706 (1988); E. G. Floratos, J. Iliopoulos, and G. Tiktopoulos, *ibid.* **217**, 285 (1989); C. N. Pope and K. S. Stelle, *ibid.* **226**, 257 (1989); E. G. Floratos *ibid.* **228**, 335 (1989); B. de Wit, J. Hoppe, and H. Nicolai, Nucl. Phys. B **305** [FS23], 545 (1988); B. deWit, V. Marquard, and H. Nicolai, Commun. Math. Phys. **128**, 39 (1990); E. Bergshoeff, M. P. Blencowe, and K. S. Stelle, *ibid.* **128**, 213 (1990).
[5]T. A. Arakelyan and G. K. Savvidy, Phys. Lett. B **214**, 350 (1988).
[6]F. A. Berezin, Commun. Math. Phys. **40**, 153 (1975); F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer Ann. Phys. (N.Y.) **111**, 61 (1978); **111**, 111 (1978).
[7]W. H. Louisell, *Radiation and Noise in Quantum Electronics* (Krieger, Huntington, New York, 1977); K. Wodkiewicz and J. H. Eberly, J. Opt. Soc. Am B **2**, 458 (1985); B. Yurke, S. L. McCall, and J. R. Klauder, Phys. Rev. A **33**, 4033 (1986); R. F. Bishop and A. Vourdas, Z. Phys. B **71**, 527 (1988); R. A. Campos, B. E. A. Saleh, and M. C. Teich, Phys. Rev. A **40**, 1371 (1989); H. Fearn and R. Loudon, J. Opt. Soc. Am B **6**, 917 (1989).
[8]W. K. Wootters, Ann. Phys. (N.Y.) **176**, 1 (1987); J. C. Varilly, and J. M. Gracia-Bondia *ibid.* **190**, 107 (1989).
[9]D. B. Fairlie, P. Fletcher, and C. K. Zachos, Phys. Lett. B **218**, 203 (1989); D. B. Fairlie and C. K. Zachos, *ibid.* **224**, 101 (1989); D. B. Fairlie, P. Fletcher, and C. K. Zachos, J. Math. Phys. **31**, 1088 (1990).
[10]L. Auslander and R. Tolimieri, Bull. Am. Math. Soc. **1**, 847 (1979); B. W. Dickinson and K. Steiglitz, IEEE Trans. Acous. Speech. Signal Process **ASSP-30**, 25 (1982).
[11]V. I. Arnold, Ann. Inst. Fourier **16**, 319 (1966).
[12]J. Schwinger, in *Quantum Theory of Angular Momentum*, edited by L. C. Biedenharn and H. Van Dam (Academic, New York, 1965).
[13]G. S. Agarwal and R. P. Puri, Phys. Rev. A **39**, 2969 (1989).
[14]A. Vourdas, J. Mod. Opt. **37**, 1661 (1990).
[15]A. Deprit, Cel. Mech. **1**, 12 (1969).