Discrete velocities for solitary-wave solutions selected by self-induced transparency

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Theoretical studies of self-induced transparency have used the slowly varying envelope approximation with perturbative corrections. These analyses indicated that for a carrier wave of given frequency, there was a continuum of possible steady-state pulse velocities. We show that the *exact* steady-state solutions of the full Maxwell-Bloch equations do not share this property; instead, there is a nonperturbative selection mechanism: a carrier wave of given frequency gives rise to steady-state pulses with discrete values of the velocity only. We present analytical and numerical calculations, and suggest some experiments for the semiconductor CdS.

I. INTRODUCTION

The discovery of self-induced transparency (SIT) by McCall and Hahn[1] in 1969 focused attention on the very old problem of light propagation in dielectrics. The older investigations of Sommerfeld[2] and Brillouin[3] had given a complete description of the light-dielectric interaction in the framework of the classical Lorentz linear model of harmonically oscillating charges. However, this model is adequate only if the light intensity is low, or if the light frequency is far from any of the atomic resonances of the dielectric medium. Since the 1960s, intense and practically coherent monochromatic laser light has been available as a probe of optically resonant systems. The response of such systems near resonance is not satisfactorily described by the linear Lorentz model: nonlinearities produce a wide range of nonclassical effects (Allen and Eberly,[4]) such as SIT, photon echoes, optical nutation, saturation phenomena, etc. When incident light lies inside a frequency gap occurring near resonance, it is strongly absorbed. What McCall and Hahn discovered is that above an intensity threshold, a pulse light can propagate with anomalously low energy loss even near resonance, the highly absorptive medium becoming essentially transparent. The physical picture of SIT is that the front part of a short pulse coherently excites the atoms in the medium, e.g., up to the state of complete inversion. The macroscopic polarization formed in this process emits coherent radiation which joins the back part of the pulse. If the oscillators in the medium return to the ground state after this stimulation process, *steady-state* propagation is realized. This front to back energy exchange gives rise to pulse velocities which can be several orders of magnitude slower than that of light in a vacuum.

McCall and Hahn demonstrated[1] SIT theoretically and experimentally, generating much interest and further work.[5–9] Central to this and subsequent theoretical studies is the so-called slowly varying envelope approximation (SVEA) which is used in solving the Maxwell-Bloch equations. The purpose of this paper is to go beyond this approximation and beyond perturbative expansions about this limit to show that steady-state pulses do *not* exist for arbitrary pulse widths τ, but only for selected pulse widths. The experimental consequence of this fact is that steady-state pulses will propagate only at certain special velocities.

This article is organized as follows: we begin by reviewing the Maxwell-Bloch equations which are the standard starting point for SIT studies. This is followed by an explanation of the SVEA and McCall and Hahn's work. In Sec. IV, following Bialynicka-Birula[10] and Akimoto and Ikeda,[11] the complete Maxwell-Bloch equations for steady-state pulses are derived without assuming the SVEA, together with the dispersion relation of the carrier wave and the pulse velocity. After reviewing previous methods for approximating the solitary-wave solutions, we present in Sec. VI an electric-field amplitude expansion to solve the coupled Maxwell-Bloch equations perturbatively. General expressions for the pulse shapes and their phase modulations are derived as a function of the incident carrier frequency ω and the arbitrary pulse width τ. In particular, these expressions include Akimoto and Ikeda's results for short and long pulses inside and outside the gap. Setting ω to be on resonance in our method gives the expansion introduced by Marth, Holmes, and Eberly.[12]

The rest of the paper shows that the steady-state pulse width τ is not arbitrary, rather it can only take on certain "selected" discrete values. In Sec. VII, we explain why perturbation expansions are misleading: they are not uniformly valid on the whole domain $(-\infty, +\infty)$, and they *cannot* determine whether there exist solitary-wave solutions (steady-state pulses). We find that it is necessary to tune the pulse width τ for each given laser frequency to satisfy the boundary conditions on the pulse at $\pm\infty$. Numerical results from the integration of the Maxwell-Bloch equations are presented in Sec. VIII. We show that solitary-wave solutions exist only for restricted parameters, contrary to the implications of all previous work. Figure 1 gives the location of these selected solitary solutions. The mathematical essence of the selection mechanism is given in Sec. IX. Finally, in Sec. X we discuss experimental consequences.

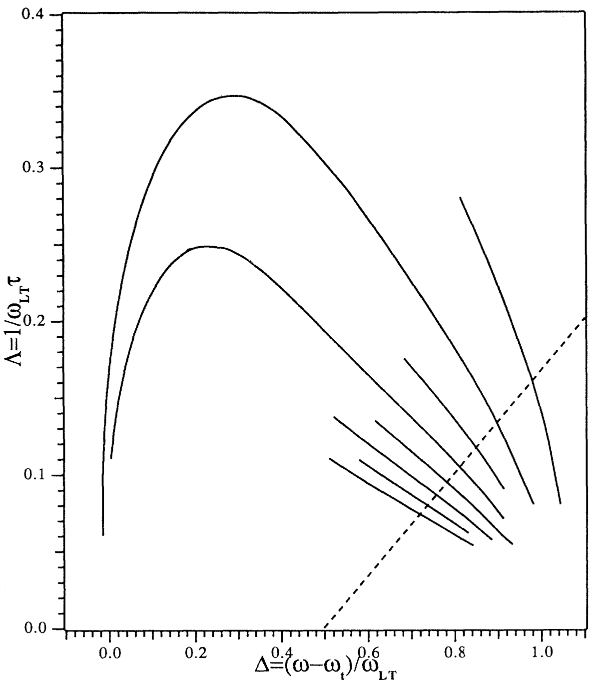

FIG. 1. Values of the parameters Δ,Λ for which steady-state solitary-wave solutions exist (solid curves). We have taken $\omega_t/\omega_{LT}=1000$.

II. MAXWELL-BLOCH EQUATIONS

SIT is the result of a coherent coupling of the em field to the resonant atoms of an absorptive medium. Any change in the state of the absorbers which is incoherent with the optical pulse will destroy SIT. One must prevent incoherent spontaneous radiation, and phonon or defect coupling to absorbers in solids. This places an upper limit on the pulse width which can give rise to SIT propagation, so that in particular the pulse width τ must be much shorter than the relaxation times T_1 and T_2. In general, the starting point of theoretical SIT analyses[4–9] is the semiclassical description given by the Maxwell-Bloch equations [Eqs. (2.1)–(2.4)]. The dielectric (gas, semiconductor, etc.) is modeled as an ensemble of noninteracting two-level systems or dipoles (atoms, excitons). Each dipole moment has strength d. A true quantum dipole (an atom in a gas, an exciton in a semiconductor) has a series of energy levels. For most problems of interest, however, one can restrict oneself to the ground state and first excited state (of energy $\hbar\omega_t$) for each dipole. (This is particularly true near resonance.) The em field is treated classically and the two-level systems semiclassically. There are no impurities, no spatial dispersion or finite temperature effects, nonresonant losses are absent, relaxation times are infinite (no damping), and no sample boundary effects are included. If one takes fields which are x and y independent then the (classical) Maxwell wave equation becomes

$$\left[\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\mathbf{E}(t,z)=\frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{P}(t,z)\ . \tag{2.1}$$

$\mathbf{E}$ is the electric field, and $\mathbf{P}$ is the polarization due to the dipoles of density N. Defining $\mathcal{E}$ to be the electric-field magnitude, there are functions $\Theta(t,z)$, $u(t,z)$, and $v(t,z)$ such that

$$\begin{aligned}&\mathbf{E}(t,z)=\mathcal{E}(t,z)\hat{\mathbf{a}}(t,z)\ ,\\ &\hat{\mathbf{a}}=\hat{\mathbf{x}}\cos\Theta(t,z)+\hat{\mathbf{y}}\sin\Theta(t,z)\ ,\\ &\Theta=\omega t-Kz+\phi(t,z)\ ,\end{aligned} \tag{2.2}$$

$$\begin{aligned}&\mathbf{P}(t,z)=\tfrac{1}{2}N\hbar\kappa[u(t,z)\hat{\mathbf{a}}+v(t,z)\hat{\mathbf{b}}]\ ,\\ &\hat{\mathbf{b}}=-\hat{\mathbf{x}}\sin\Theta(t,z)+\hat{\mathbf{y}}\cos\Theta(t,z)\ .\end{aligned} \tag{2.3}$$

The time dependence of $\mathbf{P}$ is given in the semiclassical limit by the Bloch equations[13,14] ($\kappa=2d/\hbar$):

$$\begin{aligned}&\frac{\partial u}{\partial t}=\left[\frac{\partial\Theta}{\partial t}-\omega_t\right]v\ ,\\ &\frac{\partial v}{\partial t}=-\left[\frac{\partial\Theta}{\partial t}-\omega_t\right]u+\kappa\mathcal{E}w\ ,\\ &\frac{\partial w}{\partial t}=-\kappa\mathcal{E}v\ .\end{aligned} \tag{2.4}$$

u and v are the in-phase (parallel to $\mathbf{E}$ or dispersive) and out-of-phase (orthogonal to $\mathbf{E}$ or absorptive) components of $\mathbf{P}$, and w is the population inversion of the medium; they satisfy $u^2+v^2+w^2=1$. By projecting Eq. (2.1) onto the unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$, the wave equation becomes the following pair of partial differential equations (PDE's) ($k_0=\omega/c$):

$$\left[\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\mathcal{E}-\left\{(K^2-k_0^2)-2\left[K\frac{\partial}{\partial z}+\frac{k_0}{c}\frac{\partial}{\partial t}\right]\phi+\left[\left[\frac{\partial\phi}{\partial z}\right]^2-\frac{1}{c^2}\left[\frac{\partial\phi}{\partial t}\right]^2\right]\right\}\mathcal{E}$$
$$=\frac{2\pi N\hbar\kappa}{c^2}\left[\frac{\partial^2 u}{\partial t^2}-\left[\omega+\frac{\partial\phi}{\partial t}\right]^2u-\frac{\partial^2\phi}{\partial t^2}v-2\left[\omega+\frac{\partial\phi}{\partial z}\right]\frac{\partial v}{\partial t}\right]\ , \tag{2.5}$$

$$\left[\left[\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\phi\right]\mathcal{E}+2\left\{-\left[K\frac{\partial}{\partial z}+\frac{k_0}{c}\frac{\partial}{\partial t}\right]\mathcal{E}+\left[\left[\frac{\partial \mathcal{E}}{\partial z}\right]\left[\frac{\partial \phi}{\partial z}\right]-\frac{1}{c^2}\left[\frac{\partial \mathcal{E}}{\partial t}\right]\left[\frac{\partial \phi}{\partial t}\right]\right]\right\}$$
$$=\frac{2\pi N\hbar\kappa}{c^2}\left[\frac{\partial^2 v}{\partial t^2}-\left[\omega+\frac{\partial\phi}{\partial t}\right]^2 v+\frac{\partial^2\phi}{\partial t^2}u+2\left[\omega+\frac{\partial\phi}{\partial z}\right]\frac{\partial u}{\partial t}\right] . \quad (2.6)$$

Note that the nonlinearities of the Maxwell-Bloch (MB) equations come in through Eq. (2.4). Hereafter, we only consider the case of no inhomogeneous broadening.

For very small electric fields (linear optics), $w \approx -1$, the atoms are predominantly in the ground state; one can recover the Lorentz limit by linearizing these equations. By translation invariance, all solutions can be expressed in terms of the Fourier modes which behave as $\exp[i(\omega t - Kz)]$. Such a plane-wave solution is a mixture of the photon and the electronic excitations in the localized two-level system (in the case of a semiconductor, the two-level system is an exciton and the mixed mode is called a polariton). The dispersion relation of these plane waves is given by

$$\left[\frac{cK}{\omega}\right]^2 = 1+\frac{2\pi N\hbar\kappa^2}{\omega_t-\omega} . \quad (2.7)$$

It varies steeply near the resonance frequency ω_t and is accompanied by a gap of size $\omega_{LT}=2\pi N\hbar\kappa^2$ in which propagation is forbidden (this is the polariton gap of semiconductors in the "local optics" picture). Light is strongly absorbed in the gap because K becomes imaginary there. For large electric field intensities, the nonlinearities of the Bloch equations become important, and SIT shows that propagation can in fact occur in the gap.

III. SLOWLY VARYING ENVELOPE APPROXIMATION

McCall and Hahn considered a circularly polarized pulse with a carrier wave of frequency ω and wave number K so that $\Theta(t,z)=\omega t - Kz$. On the time and length scale of this carrier wave, the envelope is typically slowly varying. The slowly varying envelope approximation[1] consists in taking an envelope function $\mathcal{E}(t,z)$ which satisfies

$$\left|\frac{\partial \mathcal{E}}{\partial z}\right| \ll |K\mathcal{E}| \quad \text{and} \quad \left|\frac{\partial \mathcal{E}}{\partial t}\right| \ll \omega|\mathcal{E}| \quad (3.1)$$

and neglecting all subleading terms in this limit. If one uses this approximation and forces the wave to be circularly polarized, most derivatives in Eqs. (2.5) and (2.6) can be dropped and one is left with the much simpler system

$$\left[\left[\frac{K}{k_0}\right]^2-1\right]\mathcal{E}=2\pi N\hbar\kappa u , \quad (3.2)$$

$$2\left[\left[\frac{cK}{k_0}\right]\frac{\partial}{\partial z}+\frac{\partial}{\partial t}\right]\mathcal{E}=2\pi N\hbar\kappa\omega v , \quad (3.3)$$

$$\frac{\partial u}{\partial t}=(\omega-\omega_t)v ,$$
$$\frac{\partial v}{\partial t}=-(\omega-\omega_t)u+\kappa\mathcal{E}w , \quad (3.4)$$
$$\frac{\partial w}{\partial t}=-\kappa\mathcal{E}v .$$

It is assumed that all absorbers are in the ground state prior to the passage of the radiation pulse. McCall and Hahn simulated these partial differential equations [Eqs. (3.2)–(3.4)] on the computer and found that after the pulse had propagated a few classical absorption lengths into the medium, the pulse evolved into the shape of a symmetric hyperbolic-secant traveling wave, the area under the pulse envelope being equal to 2π ("2π pulse"). This suggested looking for steady-state solutions, i.e., solutions which are time independent in a moving frame. Taking $\mathcal{E},u,v,w$ to be functions only of $\zeta=t-z/V$, Eqs. (3.2)–(3.4) become coupled ordinary differential equations (ODE's):

$$\left[\left[\frac{K}{k_0}\right]^2-1\right]\mathcal{E}=2\pi N\hbar\kappa u , \quad (3.5)$$

$$2\left[\frac{cK}{Vk_0}-1\right]\frac{d\mathcal{E}}{d\zeta}=-2\pi N\hbar\kappa\omega v , \quad (3.6)$$

$$\frac{du}{d\zeta}=(\omega-\omega_t)v ,$$
$$\frac{dv}{d\zeta}=-(\omega-\omega_t)u+\kappa\mathcal{E}w , \quad (3.7)$$
$$\frac{dw}{d\zeta}=-\kappa\mathcal{E}v .$$

As McCall and Hahn found, these equations have solutions:

$$\mathcal{E}(t,z)=\frac{2}{\kappa\tau}\operatorname{sech}\left[\frac{t-z/V}{\tau}\right]$$
$$\text{with } \kappa\int_{-\infty}^{+\infty}\mathcal{E}(t,z)dt=2\pi , \quad (3.8)$$

which correspond to a family of *solitary waves* of arbitrary pulse width τ with velocity $V=c/(1+2\pi\kappa\omega N d\tau^2)$. Solitary waves are waves which decay at infinity and which are steady-state, i.e., which depend upon z and t only through $\zeta=t-z/V$. (See Appendix A.) Such pulses have constant shapes and thus are transmitted with no loss (SIT). Actually McCall and Hahn allowed deviations from circular polarization (chirping) of the form $\Theta(t,z)=\omega t-Kz+\phi(z)$, but it turns out that this ansatz for ϕ does not lead to any new solutions. In the next section, we shall see how ϕ should be chosen for steady-state waves.

In 1971, Lamb[5] showed that the Maxwell-Bloch equations in the SVEA [Eqs. (3.2)–(3.4)] form an exactly integrable system. This meant that the propagating hyperbolic-secant pulses found by McCall and Hahn were in fact solitons. *Solitons* are solitary waves which preserve their form after collision.[15,16] What happens if one goes beyond the SVEA by keeping higher-order terms in the Maxwell wave equation? A generic perturbation destroys exact integrability. Thus one expects that as soon as one goes beyond the SVEA, the pulses are no longer solitons, but one might still expect there to be solitary waves. However, we will see that the solitary waves also disappear except for a discrete set of solutions, and that this phenomena cannot be seen at any finite order in a perturbative expansion about the SVEA.

IV. EQUATIONS OBEYED BY STEADY-STATE WAVES

Without assuming the slowly varying envelope approximation in dielectrics, we are looking for "traveling-wave" solutions for the electric field $\mathbf{E}(t,z)$ and the polarization density $\mathbf{P}(t,z)$, which suggests talking all fields to depend only on the variable $\zeta=(t-z/V)$, where V is a velocity to be determined. It turns out that this form does not lead to any solutions because the carrier wave and the envelope cannot both be of this form. To justify the procedure used in the later SIT studies, it is best to follow the standard prescription (Barenblatt,[17] Olver[18]) for finding "self-similar" solutions. Given a partial differential equation, one considers its symmetry group G. In general, G will be an m parameter Lie group. Self-similar solutions are those solutions which are invariant under a one-parameter subgroup of G. In the case of Maxwell-Bloch equations [Eqs. (2.1)–(2.4)], the symmetry group of interest consists of translations in space, translations in time, and rotations about the propagation axis, z. A self-similar solution will be characterized by a one parameter subgroup: $t\to t+A$, $z\to z+VA$, and $\Theta\to\Theta+(\omega-KV)A$. V and K are real parameters. Recently, it has been common practice in applied mathematics to call such parameters "nonlinear eigenvalues"[17] because for special values of V and K the associated boundary-value problem will have solutions. Choosing a pulse to be invariant under this subgroup amounts to asking for pulses which are time independent in a uniformly translating *and* rotating frame. Defining $\zeta=t-z/V$, this gives

$$\Theta(t,z)=\omega t-Kz+\phi(\zeta) \tag{4.1}$$

and imposes that $\mathcal{E}$, u, v, and w must be functions of ζ only. This form contains phase modulation (chirping) ϕ for the electric field so one is restricted to circularly polarized waves. K generally is not equal to the magnitude of the vacuum wave vector, $K\neq k_0=\omega/c$.

For the rest of this section, we follow Bialynicka-Birula[10] and Akimoto and Ikeda.[11] The wave number K and the pulse envelope velocity V can be derived by linearizing the set of Maxwell-Bloch equations [(2.4)–(2.6)] and setting $\dot\phi=0$ and $w=-1$. This corresponds to defining ω and K in the tail of the pulse where the excitation is very weak. Within the linear theory, the solutions are exponential, $\mathcal{E}\sim\exp(\pm\zeta/\tau)$. Even though such exponential solutions are divergent, they are meaningful in the tail of a steadily propagating pulse since the exponential form can describe a growing (or decaying) wave locally. In the presence of nonlinearity, the divergence will be suppressed and a pulse may be formed. τ introduces a time scale which allows one to write everything in dimensionless form. As shown in Ref. 11, K and V then satisfy

$$\left[\frac{K}{k_0}\right]^2=\frac{1}{2}\left[(1-s^2)-\frac{(1-s^2)\Delta+2s\Lambda}{\Delta^2+\Lambda^2}+(1+s^2)\left[\frac{(\Delta-1)^2+\Lambda^2}{\Delta^2+\Lambda^2}\right]^{1/2}\right], \tag{4.2}$$

$$\left[\frac{c}{V}\right]^2=\frac{s^{-2}}{2}\left[-(1-s^2)+\frac{(1-s^2)\Delta+2s\Lambda}{\Delta^2+\Lambda^2}+(1+s^2)\left[\frac{(\Delta-1)^2+\Lambda^2}{\Delta^2+\Lambda^2}\right]^{1/2}\right], \tag{4.3}$$

where $\Delta=(\omega-\omega_t)/\omega_{LT}$ is the dimensionless frequency detuning, $\omega_{LT}=2\pi N\hbar\kappa^2$, $\Lambda=1/\omega_{LT}\tau$ is the dimensionless reciprocal pulse width, and $s=1/\omega\tau=\Lambda/[\Delta+\omega_t/\omega_{LT}]$. Furthermore, we introduce the coefficients α, β, and γ

$$\alpha=\left[\frac{K}{k_0}\right]^2-1\,,\qquad \beta=2s\left[\frac{cK}{Vk_0}-1\right],\qquad \gamma=s^2\left[\left[\frac{c}{V}\right]^2-1\right], \tag{4.4}$$

which are known functions of the three dimensionless parameters Δ, Λ, and s. In particular, the coefficient β is given by

$$\beta=\frac{\Lambda(1-s^2)}{\Delta^2+\Lambda^2}-\frac{2s\Delta}{\Delta^2+\Lambda^2}\,. \tag{4.5}$$

(For later reference, remark that α, β, and γ do not follow from the SVEA but are defined as shown.) The gap where there is no propagation in the linear theory corresponds to the range $0<\Delta<1$. Now we can write the Maxwell-Bloch equations for a solitary wave in dimensionless form. Introduce the dimensionless electric-field amplitude $E=\kappa\mathcal{E}/\omega_{LT}$, and the dimensionless time $\xi=\zeta/\tau=(t-z/V)/\tau$ where τ is the above-defined time scale which will be essentially the pulse width. For a steady-state pulse, Eqs. (2.4)–(2.6) turn into coupled ODE's:

$$\gamma\ddot E-\left[\alpha+\beta\dot\phi+\gamma\dot\phi^2+\frac{s}{\Lambda}\left[\frac{s\Delta}{\Lambda}-2\right]w+\frac{s^2}{\Lambda}\dot\phi w\right]E=-\left[\frac{s\Delta}{\Lambda}-1\right]^2u\,, \tag{4.6}$$

$$\gamma\ddot{\phi}E+\left[\beta+2\gamma\dot{\phi}-\frac{s^2}{\Lambda}w\right]\dot{E}=-\left[\left[\frac{s\Delta}{\Lambda}-1\right]^2+\left[\frac{s}{\Lambda}\right]^2E^2\right]v\ , \quad (4.7)$$

$$\Lambda\dot{u}=(\Delta+\Lambda\dot{\phi})v\ , \quad (4.8)$$

$$\Lambda\dot{v}=-(\Delta+\Lambda\dot{\phi})u+Ew\ , \quad (4.9)$$

$$\Lambda\dot{w}=-Ev\ , \quad (4.10)$$

where hereafter dots mean $d/d\xi$ (i.e, $\dot{w}=dw/d\xi$).

The problem is to solve these differential equations under the boundary condition that in the limit $\xi\to\pm\infty$, the electric field vanishes, $E=0$, and all the dipoles of the dielectric are in the ground state, so that $w=-1$.

V. SURVEY OF PREVIOUS EXPANSIONS

Equations (4.6)–(4.10) are nonlinear and thus in general the best one can hope to do is derive perturbative expansions for the solutions. Consider beginning with Eqs. (4.6)–(4.10), and then drop the chirping and the higher derivatives, and set $s=0$ wherever it appears explicitly. This gives

$$\alpha E=u,\quad \beta\dot{E}=-v\ ,$$
$$\Lambda\dot{u}=\Delta v,\quad \Lambda\dot{v}=-\Delta u+Ew,\quad \Lambda\dot{w}=-Ev\ . \quad (5.1)$$

Note that, using the first three of these, one obtains the constraint $\Lambda\alpha=-\Delta\beta$ which is not satisfied by the values appearing in Eqs. (4.6)–(4.10) so the above system has no solutions. The reason for this is that α, β, and γ depend on s and Λ, so the constraint $\Lambda\alpha=-\Delta\beta$ is only satisfied in the limit $s\to0$ and $\Lambda\to0$. In that limit however, we do recover the SVEA equations.

Thus the correct starting point for Eqs. (4.6)–(4.10) is to take in Eqs. (5.1) the limiting values of α and β as determined by the SVEA. Then one can derive a perturbative expansion by including the terms dropped in Eqs. (4.6)–(4.10) simultaneously with the corrections to the limiting values of the coefficients α and β. Finally, one needs to find an expansion parameter which makes the perturbation expansion meaningful. In general, this means long pulses, i.e., $\Lambda\to0$. This is essentially the approach first introduced by Bialynica-Birula[10] whose expansion parameter was $s=1/\omega\tau$. Her work was systematically expanded upon by Akimoto and Ikeda[11] to include various expansions whose parameters depended on the nature of the pulse. Their results can be summarized as follows. Besides the usual SIT with full inversion, steady-state pulse solutions were also found with only partial inversion in the case when the pulse width is much longer than the reciprocal of the gap frequency ω_{LT}. For a semiconductor, a long pulse outside the gap (i.e., with frequency ω outside the gap) behaves as a weak polariton-soliton wave in the picosecond regime. A long pulse inside the gap propagates very slowly as a sort of standing wave of a nonlinear polariton. A short pulse of strong intensity realizes the usual SIT. The derived pulse shapes depend continuously on the pulse width τ which is an arbitrary parameter in the problem, just as McCall and Hahn found in the SVEA.

A slightly different approach for solving Eqs. (4.6)–(4.10) was developed by Marth, Holmes, and Eberly,[12] who went beyond the SVEA for the on-resonance case. Their method of approximation is based on a series expansion in powers of the electric field rather than in a parameter. It can be viewed as an amplitude expansion. In this way, they studied very short pulses finding corrections to the results of McCall and Hahn. This expansion will be generalized in the next section to the off-resonance case. A perhaps more ambitious attempt was made earlier by Matulic and Eberly[19] to find exact solutions within the SVEA with phase modulation included. This corresponds to dropping $\ddot{E}$, $\ddot{\phi}$, and s terms, but keeping the rest. They found multipulse chirped steady-state wave trains. However their ansatz was too restricted to lead to any solutions for single steady-state pulses.

There also have been investigations of SIT in spatially dispersive media. This case was studied by Agranovich and Rupasov[20] in the SVEA. Additional work by Belkin and co-workers[21,22] in the SVEA considered the effects of saturation and phase modulation on the form of steady-state solutions which lead to the spreading of pulses. Finally Ikeda and Akimoto developed a systematic perturbation expansion beyond the SVEA to include nonlocal effects in a spirit similar to what they had done for the local optics case.[23]

All these perturbative studies find steady-state pulse shapes which depend continuously on the pulse width τ—it is an arbitrary parameter. We shall see in Sec. VII that this is not true of the *exact* steady-state solutions of the MB equations: for a given carrier wave frequency, solitary pulses can propagate only at special parameter values and velocities (cf. Fig. 1), a phenomenon we call velocity selection. This has not been realized previously because it cannot be seen at any order in a perturbative expansion about the SVEA.

VI. A POWER-SERIES EXPANSION IN THE ELECTRIC FIELD

In this section, we show how to extend the amplitude expansion of Marth, Holmes, and Eberly[12] to the off-resonance case by using a method which generalizes the perturbation theory of previous studies. The phase equation (4.7) is a first-order linear differential equation for $\dot{\phi}$, from which one can derive the relation

$$\gamma\dot{\phi}E^2=-\frac{\beta}{2}E^2+\Lambda\left[\frac{s\Delta}{\Lambda}-1\right]^2(w+1)+\frac{s^2}{\Lambda}E^2w-\frac{s^2}{\Lambda}\int_0^E E'w(E')dE'\ . \quad (6.1)$$

This is the $\Delta\neq0$ generalization of the first integral derived in Ref. 12. Equation (6.1) shows that if one knows a functional relation $w=w(E)$ between the dimensionless energy of the dipoles w and the pulse amplitude E, then $\dot{\phi}(E)$ is determined. Furthermore, if both $w(E)$ and $\dot{\phi}(E)$ are known, then one can determine, at least formally by using the Bloch equations, the functional relation $u=u(E)$. This turns the amplitude Eq. (4.6) into a nonlinear second-order differential equation for E alone. The ansatz we use for $w=w(E)$ is a power-series expansion in E:

$$w(E)=\sum_{i=0}^{\infty} w_{2i}E^{2i} . \tag{6.2}$$

The pulse boundary condition gives $w_0=-1$ because the dipoles of the dielectric are in their ground state when the pulse amplitude is zero. With this ansatz, $\dot{\phi}$ and v are also power series (in E and $\dot{E}$). All the coefficients of these series can be determined recursively. The convergence of these series is a rather difficult mathematical question to decide, since the coefficients depend in complicated ways on the parameters Δ, Λ, and s. We will come back to this question in Sec. VII.

From Eq. (6.1) one obtains

$$\gamma\dot{\phi}=\left[-\frac{\beta}{2}-\frac{s^2}{2\Lambda}+\Lambda\left[\frac{s\Delta}{\Lambda}-1\right]^2 w_2\right] + \left[\Lambda\left[\frac{s\Delta}{\Lambda}-1\right]^2 w_4+\frac{3s^2}{4\Lambda}w_2\right]E^2+\cdots . \tag{6.3}$$

When $\xi\to\pm\infty$ there is no phase modulation so that the E-independent term of $\dot{\phi}$ is zero. This leads to an expression for w_2 and for ϕ_2 in terms of Δ, Λ, s, and w_4:

$$w_2=\frac{1}{2\Lambda}\frac{\left[\beta+\frac{s^2}{\Lambda}\right]}{\left[\frac{s\Delta}{\Lambda}-1\right]^2}=\frac{1}{2(\Delta^2+\Lambda^2)} , \tag{6.4}$$

$$\dot{\phi}=\phi_2E^2+\cdots \quad \text{where } \phi_2=\frac{\Lambda\left[\frac{s\Delta}{\Lambda}-1\right]^2 w_4+\frac{3s^2}{4\Lambda}w_2}{\gamma} . \tag{6.5}$$

The power series for u and v can be derived using the Bloch equations (4.8)–(4.10). This gives

$$u(E)=u_1E+u_3E^3+\cdots , \tag{6.6}$$

where

$$u_1=-2\Delta w_2 \quad \text{and} \quad u_3=-\tfrac{1}{3}(4\Delta w_4+2\Lambda\phi_2 w_2) \tag{6.7}$$

and

$$v=v_1\dot{E}+\cdots \quad \text{with } v_1=-2\Lambda w_2 . \tag{6.8}$$

Now, one can solve the amplitude equation (4.6) after using Eqs. (6.2) and (6.4)–(6.7) for $w(E)$, $\dot{\phi}(E)$, and $u(E)$. All the terms in Eq. (4.6) except $\dot{\phi}^2$ contribute at this order of the approximation. The result is the following ODE for the electric-field amplitude E:

$$\ddot{E}=E+\frac{\left[\beta+\frac{s^2}{\Lambda}\right]\phi_2+\frac{s}{\Lambda}\left[\frac{s\Delta}{\Lambda}-2\right]w_2-\left[\frac{s\Delta}{\Lambda}-1\right]^2 u_3}{\gamma}E^3+\cdots . \tag{6.9}$$

Dropping higher-order terms in E, the solution to this ODE is a hyperbolic secant. To derive w_4, another independent equation for E can be obtained from the conservation law $(u^2+v^2+w^2=1)$ by using the lowest-order terms for u, v, and w from Eqs. (6.2) and (6.6)–(6.8). This leads to another order ODE for the electric-field amplitude E with the unknown w_4:

$$\ddot{E}=E+2\left[\frac{2w_4-w_2^2-2u_1u_3}{v_1^2}\right]E^3+\cdots . \tag{6.10}$$

By equating the coefficients of E^3 in Eqs. (6.9) and (6.10) an equation of first degree for w_4 is obtained. The coefficient w_4 is a rather complicated function of the parameters Δ, Λ, and s. The formulas for w_4 and for the other lowest-order coefficients in the case $s\neq 0$ are given in Appendix B. Here we restrict ourselves to the case $s=0$ (since for gases $\omega_{LT}/\omega_t\sim 10^{-8}$, and for excitonic semiconductors $\omega_{LT}/\omega_t\sim 10^{-3}$, this is a good approximation).

$$w_4=\frac{-3\gamma^2w_2^2}{2[-3\gamma^2+8(\Delta\gamma)^2w_2+8\Delta\gamma(\Lambda w_2)^2+3\beta\Lambda^3w_2^2+2\Lambda^4w_2^3]} . \tag{6.11}$$

The solution of the nonlinear ODE's in Eqs. (6.9) and (6.10) is given by

$$E=C\,\mathrm{sech}(\xi)=C\,\mathrm{sech}\left[\frac{t-z/V}{\tau}\right] \quad \text{where } \ddot{E}=E-\frac{2}{C^2}E^3 \tag{6.12}$$

and

$$C^2=\frac{4[-3\gamma^2+8(\Delta\gamma)^2w_2+8\Delta\gamma(\Lambda w_2)^2+3\beta\Lambda^3w_2^2+2\Lambda^4w_2^3]}{w_2[4\Delta\gamma+3\beta\Lambda+2\Lambda^2w_2]} . \tag{6.13}$$

The other w_4-dependent coefficients ϕ_2 and u_3 can also be derived:

$$\phi_2=\frac{-3\gamma\Lambda w_2^2}{2[-3\gamma^2+8(\Delta\gamma)^2w_2+8\Delta\gamma(\Lambda w_2)^2+3\beta\Lambda^3w_2^2+2\Lambda^4w_2^3]} \tag{6.14}$$

and

$$u_3=\frac{\gamma w_2^2(2\Delta\gamma+\Lambda^2w_2)}{-3\gamma^2+8(\Delta\gamma)^2w_2+8\Delta\gamma(\Lambda w_2)^2+3\beta\Lambda^3w_2^2+2\Lambda^4w_2^3} . \tag{6.15}$$

Using this amplitude expansion, it is possible to derive the various limits of Akimoto and Ikeda in a unified way. Consider, for instance, a long pulse outside the gap. The following inequalities are then satisfied: $\Lambda \ll |\Delta|$ and $\Lambda \ll |\Delta-1|$ (in Akimoto and Ikeda's paper[11] the expansion parameter is $\epsilon=\Lambda/\Delta \ll 1$). Explicit forms of the lowest-order solutions are

$$E=\Lambda\left[\frac{4\Delta-3}{\Delta-1}\right]^{1/2}\operatorname{sech}\xi , \tag{6.16a}$$

$$\dot\phi=-\frac{3\Lambda}{2\Delta(4\Delta-3)}\operatorname{sech}^2\xi , \tag{6.16b}$$

$$u=-\frac{\Lambda}{\Delta}\left[\frac{4\Delta-3}{\Delta-1}\right]^{1/2}\operatorname{sech}\xi+\frac{\Lambda^3}{2\Delta^2[(4\Delta-3)(\Delta-1)]^{1/2}(\Delta-1)}\operatorname{sech}^3\xi , \tag{6.16c}$$

$$v=\left[\frac{\Lambda}{\Delta}\right]^2\left[\frac{4\Delta-3}{\Delta-1}\right]^{1/2}\operatorname{sech}\xi\tanh\xi , \tag{6.16d}$$

$$w=-1+\left[\frac{\Lambda}{\Delta}\right]^2\frac{(4\Delta-3)}{2(\Delta-1)}\operatorname{sech}^2\xi-\frac{3}{8}\left[\frac{\Lambda}{\Delta}\right]^4\frac{1}{(\Delta-1)^2}\operatorname{sech}^4\xi . \tag{6.16e}$$

The other limits (such as long pulses inside the gap) are given in Appendix C. We find that the phase coefficient ϕ_2 has negative values for long pulses and tends to zero for short pulses (no phase modulation) in agreement with McCall and Hahn's results. The product $\phi_2 C$ is proportional to Λ for long pulses and to $1/\Lambda$ for short pulses. For short pulses ($\Lambda\rightarrow\infty$), the coefficient w_4 goes to zero, and leads to total population inversion. For long pulses outside and inside the gap ($\Lambda\rightarrow 0$), full population inversion cannot be obtained. This concludes our generalization of the amplitude expansion method.

VII. NONEXISTENCE OF SOLITARY-WAVE SOLUTIONS

We saw that within the SVEA and all the perturbation methods, there is a continuous family of solitary pulses parametrized by their width τ or equivalently by their velocity V. As we will see soon, these expansions do not guarantee the existence of solitary-wave solutions of the full MB equations. In the remainder of this paper we will show that the perturbative expansions of the previous sections are misleading, and that in general solitary-wave solutions of SIT do *not* exist. There are two reasons for doubting the validity of the perturbative expansions. First, the SVEA as used by McCall and Hanh is a highly degenerate limit as explained below. Second, the higher derivatives in Eqs. (4.6)–(4.10) form a singular perturbation which makes the solitary-wave boundary value problem ill-posed.

Consider first the five SVEA equations with no chirping [Eqs. (3.5)–(3.7)]. These are four first-order ODE's for the four functions, $\mathcal{E},u,v,w$ and in addition a constraint, Eq. (3.5). For general ODE's, this constraint cannot be satisfied even locally unless it is a conservation law of the ODE's. In the case of Eqs. (3.6) and (3.7), Eq. (3.5) is indeed a consequence of the others if K is chosen properly. But this conservation property is not structurally stable: under a small change in the equations, there will be no solutions to Eqs. (3.5)–(3.7) in any finite interval. One example of this was given in Sec. V. For a more subtle example, consider the "small" change consisting in dropping, $\dot\phi$, $\ddot\phi$, and $\ddot E$, but forcing $s\neq 0$ in Eqs. (4.6)–(4.10). The constraint corresponding to the modified Eq. (4.6) is no longer conserved by the ODE's and there are no solutions at all. This shows that the SVEA equations are very special, so a perturbative expansion about them may be misleading.

The second source of possible problems is the higher derivatives dropped in the SVEA. Are there solutions on the whole real line which satisfy the boundary conditions at $\xi=\pm\infty$ when these terms are included? The expansions described in the previous sections are locally valid, but they need not be valid on the whole domain $(-\infty,+\infty)$. In order to clarify this statement, let us first present a mode counting argument. Equations (4.6)–(4.10) are equivalent to a system of six first-order ODE's in E, $\dot E$, u, v, w, and $\dot\phi$. These equations have a six-dimensional solution space. One dimension of this space corresponds to the solution (unique up to translations in ξ, and some discrete symmetries) which comes out of the $E=0$ and $w=-1$ point at $\xi=-\infty$. The five other dimensions correspond to modes which are "bad," i.e., which do not satisfy the boundary conditions as $\xi\rightarrow-\infty$. The same kinds of modes occur for $\xi=+\infty$. In general, the continuation of the good solution from $\xi=-\infty$ will have some of the bad modes as $\xi\rightarrow+\infty$. This is generically the case, so in general there are no solitary-wave solutions. To remove the bad modes at $\xi=+\infty$, one has to tune the parameters Λ, Δ, and s in order to come back to the $E=0$ and $w=-1$ point at $\xi=+\infty$. This is the condition for a solitary solution to exist. As will be described in the next section, Eqs. (4.6)–(4.10) are symmetric under $\xi\rightarrow-\xi$; $\dot E$ and v should be odd, and the other functions should be even. Thus one has to satisfy one constraint to have a solitary wave: $\dot{E}$ and v must simultaneously vanish. For the case of the SVEA equations, if $\dot{E}$ vanishes, v does also, so there is no additional constraint which need to be satisfied: the solution which satisfies the boundary conditions at $\xi \to -\infty$ also satisfies them at $\xi = +\infty$.

The derivatives dropped in going from the full equations to the SVEA change the nature of the space of solutions, and thus these derivatives form a singular perturbation. A consequence of this is that perturbative expansions starting with the lower-order equations miss some nonperturbative terms which in general are nonzero. This kind of behavior can be illustrated with two simple examples from singular perturbation theory. Consider the first degree equation: $Ax+B=0$. The solution is $x=-B/A$. Now introduce a small perturbation: for $\epsilon \ll 1$, consider the equation of second degree $\epsilon x^2 + Ax + B = 0$. The extra solution, $2A/\epsilon + O(\epsilon)$, is nonanalytic (singular) in ϵ at $\epsilon = 0$: ϵx^2 is a singular perturbation. As a second example, consider the first-order ODE $\dot{y}+y=0$ with solution $y=C\exp(-x)$. Introduce a small perturbation: for $\epsilon \ll 1$, consider the higher-order ODE

$$\epsilon \ddot{y} + \dot{y} + y = 0 \ . \tag{7.1}$$

Its solutions are

$$y = A\exp\{[-1+O(\epsilon^2)]x\} + B\exp\{-[\epsilon^{-1}+O(\epsilon)]x\} \ . \tag{7.2}$$

The new family of solutions is nonanalytic in ϵ at $\epsilon=0$. Again, $\epsilon\ddot{y}$ is a singular perturbation. If one solves such equations by expanding the unknown function y in powers of ϵ, one misses the part which is nonperturbative in ϵ.

This singular perturbation argument can be applied to the case of the Maxwell-Bloch equations [in particular, the first term in Eq. (7.3)]. Take, for example, a long pulse case outside the gap; the expansion parameter is $\epsilon = \Lambda/\Delta \ll 1$. The system of Maxwell-Bloch equations for steady-state solutions can in principle be transformed to a sixth-order ODE for the amplitude E. Schematically, this would give

$$\epsilon^{(\ )}(E^{(6)} + \cdots) + \ddot{E} - E + \frac{2}{C^2}E^3 = 0 \ . \tag{7.3}$$

For $\epsilon=0$, the equation has a conservation law that reduces the ODE to quadratures. An analogy for this case is a ball rolling down a potential $V(E)$ for which the conserved quantity is

$$U_{\text{total}} = \frac{\dot{E}^2}{2} + V(E) \ . \tag{7.4}$$

A solitary pulse corresponds to a trajectory from $E(\xi=-\infty)=0$ to $E(\xi=+\infty)=0$ which has total energy $U_{\text{total}}=0$. Because of this conservation law, the equation is reduced to evaluating an integral and the trajectory leaving $E=\dot{E}=0$ is guaranteed to return to this point. However, as soon as $\epsilon \neq 0$, there is no conservation law, and in general, the trajectory coming out of $\xi=-\infty$ misses the $E=\dot{E}=0$ point when $\xi=+\infty$. This property cannot be seen by expanding the unknown function E in powers of ϵ: it is precisely the ϵ nonanalytic part of E (which does not appear in any ϵ expansion) which spoils the boundary conditions.

In Eqs. (4.6)–(4.10), $\epsilon=0$ corresponds to the SVEA. For general Δ, Λ, and s, the nonperturbative terms in ϵ will spoil the boundary conditions, and there will be no solitary-wave solutions. In the next section, we will see that a single condition needs to be imposed for the solution from $-\infty$ to have none of the bad modes as $\xi \to +\infty$. This means that one can have solitary waves if Δ, Λ, and s satisfy one constraint. In the next section, we will determine this constraint numerically.

To explicitly see that perturbative expansions of the type discussed in Secs. V and VI give rise to approximate solutions which satisfy the boundary conditions at each order, consider for instance the expansion of Akimoto and Ikeda. At each other of their iteration $(E=\sum_n \epsilon^n E_n)$, for $n \geq 1$, the electric-field amplitude E_n satisfies a certain linear second-order ODE. For those ODE's, there is always at least one solution which satisfies the boundary conditions $E_n \to 0$ at $\xi = \pm\infty$. The terms E_n can be calculated iteratively and never suggest any problem. That such a series can be constructed does not imply that there exist solitary-wave solutions. This explains why Akimoto and Ikeda never realized that the expansion was misleading and that solitary waves generally do not exist. Similar remarks apply to our expansion of the population inversion $w(E)$ in powers of E, given in Sec. VI. All these expansions are locally valid, but they are not uniformly valid on the whole domain $(-\infty, +\infty)$; in general, the nonperturbative parts violate the boundary conditions.

This section has shown that singular perturbations have dramatic effects: there is selection of steady-state solutions because the boundary conditions are overspecified. Do these small perturbations also affect more general time-dependent solutions? When going from the SVEA [Eqs. (3.2)–(3.4)] to the full MB equations [Eqs. (2.4)–(2.6)], the initial value problem is modified: one has to specify in addition to the fields the time derivative of the electric-field vector. Once this is done, the initial value problem is well posed; thus we expect no selection principle to be applicable to general time-dependent solutions. However, the perturbations dropped in the SVEA are still singular: the solution develops nonanalyticities in the corresponding small parameters. One consequence is that standard perturbative treatments should not be expected to provide approximations which are uniformly valid at all times.

VIII. NUMERICAL RESULTS

To find for which values of Λ, Δ, and s there are solitary waves, we first make the problem well-posed. We follow the procedure developed for similar boundary-value problems in other fields.[24,25] Equations (4.6)–(4.10) have translational symmetry in ξ and ϕ, and are invariant under $\xi \to -\xi$, $v \to -v$. Also, changing the sign of E, u and v is a symmetry. Since the boundary conditions at

$\xi=-\infty$ define the solution everywhere modulo the above translations and modulo the sign symmetry, it is not difficult to see that solitary-wave solutions (after shifting ξ so that the pulse peak is at $\xi=0$) have w and $\dot{\phi}$ even in ξ, and E must be either even or odd. The standard hyperbolic-secant 2π pulses are even, so we will restrict ourselves to E even. (The analysis for odd pulses would proceed similarly.) Then u is even and v is odd. Let us thus consider the ODE's on the interval $(-\infty,0]$ with the same boundary conditions at $\xi=-\infty$ and the condition $v=0$ at $\xi=0$. This new boundary value problem is well posed. If $\dot{E}=0$ at $\xi=0$ also, it is easy to see that one can construct a solitary-wave solution on the whole ξ axis by reflection of the solution on the half line with a change of sign for v. However, in general, $\dot{E}\neq 0$ at $\xi=0$, corresponding to a solution which has some amount of growing (bad) modes as $\xi\to+\infty$. The amount of these bad modes is zero to all orders in perturbation theory, but nonetheless can be nonzero. Using the above, we see that the condition for existence of a solitary-wave solution is that $\dot{E}$ and v vanish simultaneously; this condition can be interpreted as forbidding any cusp in E.

We numerically integrated the system of Eqs. (4.6)–(4.10) from $\xi\to-\infty$. For ξ in the tail, the solution can be obtained from the perturbative expressions for E, $\dot{E}$, $\dot{\phi}$, u, v, and w; these are used as initial conditions on the fields. We evolve forward and find the ξ or ξ's where $v=0$, and determine $\dot{E}$ there. Call this value $\dot{E}_{\rm tip}(\Delta,\Lambda,s)$. As expected on the basis of the arguments in the preceding section, in general $\dot{E}_{\rm tip}\neq 0$, and solitary waves do not exist for those values of Δ,Λ,s. However, we find that $\dot{E}_{\rm tip}$ changes sign when the parameters are varied, and that there are surfaces in the (Δ,Λ,s) space on which $\dot{E}_{\rm tip}=0$. These are the surfaces for which solitary-wave solutions exist; they determine the selected velocities for steady-state pulse propagation.

The numerical analysis shows that one can tune the parameters Λ, Δ, and s so as to obtain $\dot{E}_{\rm tip}=0$ at least in the gap $0\le\Delta\le 1$. Using a root solver, we determined the locus of the curves (Λ,Δ) for which there are solitary solutions at fixed ω_{LT}/ω_t. As can be seen in Fig. 1, various branches of solutions rise from $\Delta=0$ and set at $\Delta=1$; note that these two points are also special in the linear theory. We have drawn some branches as stopping inside the gap because there appear multiple solutions to $v=0$. As one decreases Δ, the pulse shapes along these branches continue to develop more and more oscillations, eventually looking nothing like the original hyperbolic-secant shapes. In Fig. 2, we plot the mismatch function, i.e., the magnitude of the cusp of the solution, $\dot{E}_{\rm tip}$ for the parameters given by the dashed line of Fig. 1. The oscillations are difficult to resolve all the way down to small Λ, but it is likely that there are an infinite number of branches coming out of the point $\Delta=1,\Lambda=0$.

In summary, the solutions of the full MB equations differ qualitatively from the solutions in the SVEA: steady-state pulses do not exist for arbitrary pulse width τ (or equivalently velocities), but only for selected values of τ which in turn determine the propagation velocity. Furthermore, such solutions are only solitary waves, not solitons since the system of Maxwell-Bloch equations beyond the SVEA is not exactly integrable. (For instance, in the presence of chirping, the identity between the pulse area and the dipole turning angle in Bloch space is no longer true: the area theorem fails.) Also, since the pulse shape depends on Δ, contrary to the case of the SVEA, there should not be true solitary-wave pulses when the absorption line is inhomogeneously broadened. This is already evident at the level of the perturbation expansions since the pulse shapes depend on Δ at each order.

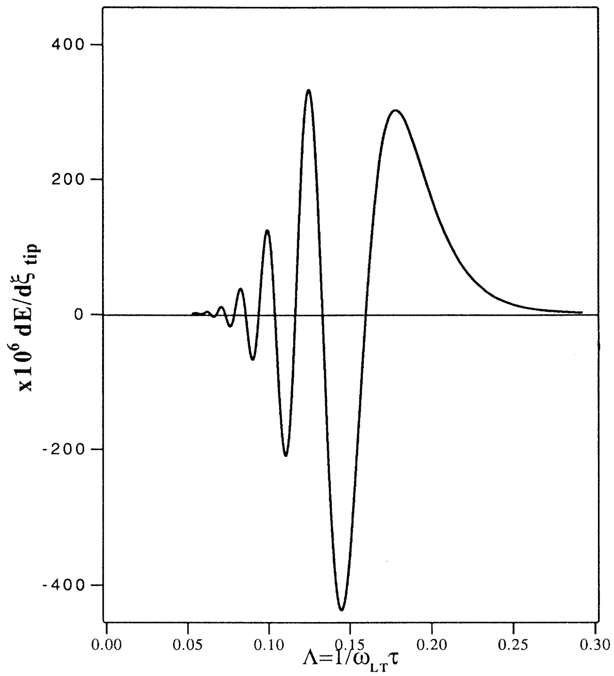

FIG. 2. Mismatch function for (Δ,Λ,s) given by the dashed line of Fig. 1.

IX. ASYMPTOTICS BEYOND ALL ORDERS

In Sec. VII we argued that generically there should be velocity selection of steady-state pulses, and in Sec. VIII we showed numerically that this "generic" behavior does in fact occur in our physics problem. In this section we show how the selection can be studied analytically using boundary layer methods. These are techniques often practiced in applied mathematics, and after the pioneering work of Kruskal and Segur,[24] they have become the main analytical tools in studying selection mechanisms.[24,25] They allow one to calculate the mismatch function when a parameter in the problem becomes small: in this sense, they are examples of asymptotic analysis methods. Kruskal and Segur showed that their method was capable of obtaining the asymptotics "beyond all orders of perturbation theory," thus the title of this section.

There are three parameters in Eqs. (4.6)–(4.10). s does not represent a singular perturbation, so the asymptotics as $s\to 0$ can simply be obtained by a perturbative expansion in powers of s. Δ and Λ play much more interesting roles as they give rise to nonanalyticities. The Kruskal-Segur analysis can be applied to the case where the small parameter multiples the highest derivatives. It is readily seen in our problem that this requires Δ to be outside the gap and $\Lambda \to 0$ (long pulse). Thus this is the case we shall consider. Our result is that when $\Lambda \to 0$, the mismatch function behaves as $\dot{E}_{\text{tip}} \approx \exp[-\lambda(\Delta,s)/\Lambda]$ times power corrections in Λ. We were not able to carry the analysis to the point of obtaining a closed-form formula for the function $\lambda(\Delta,s)$, but in Fig. 3 we show the result of the numerical calculation. We see that λ depends rather weakly on Δ far away from $\Delta=0$ or 1, and that the asymptotic behavior sets in rather quickly. Note that the behavior we have derived for $\dot{E}_{\text{tip}}$ is nonanalytic in Λ as argued in Sec. VII. There we said that $\dot{E}_{\text{tip}}$ should be zero to all orders in perturbation theory in Λ, and this is indeed the case. Our result shows that there are no long solitary pulse solutions outside the gap, as $\dot{E}_{\text{tip}}$ does not vanish.

Here, we outline the calculation of $\dot{E}_{\text{tip}}$, the mismatch function. As $\Lambda \to 0$, one expands all parameters and functions in $\epsilon = \Lambda/\Delta$. Akimoto and Ikeda have obtained the first term in this expansion. The first step in the Kruskal and Segur analysis is to extend the independent variable ξ to the complex plane. The first-order solution of Akimoto and Ikeda has a singularity at $\xi = i\pi/2$. Imagine obtaining the entire series in ϵ. It will turn out that this series is only asymptotic. One can argue that the various terms in the series have the leading behavior $[\epsilon/\cosh(\xi)]^n$ as $\xi \to i\pi/2$, and in fact this behavior can be shown rigorously by taking $\sinh(\xi)$ as the new independent variable. The series in ϵ is thus well behaved except in the region of the complex plane consisting of the disk of radius ϵ centered at $\xi = i\pi/2$. This disk is called the "inner region" and the boundary layer is the surrounding region also of size ϵ.

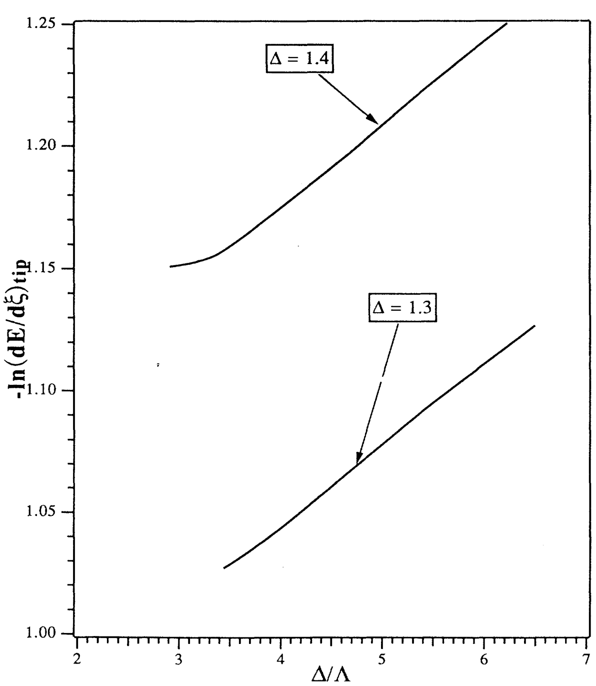

FIG. 3. Exponential dependence of the mismatch function on Λ as $\Lambda \to 0$ outside the gap.

The second step in the Kruskal-Segur analysis consists in investigating the behavior of the fields on the imaginary axis. For our steady-state pulse solution, the fields v and $\dot{E}$ are odd. Thus they have only odd terms in their Taylor series about $\xi=0$, and this shows that they are pure imaginary on the imaginary axis. Similarly, since all the other fields are even, they must be pure real on the imaginary ξ axis. However, if the mismatch is nonzero, these fields will not have this property. As can be seen recursively, each term in the ϵ series does satisfy the even or odd property, and thus the nonzero value of the mismatch function cannot be seen using this series. Let us thus write each field as its asymptotic series plus a nonperturbative term. For instance,

$$E(\xi) = \epsilon E_1(\xi) + \epsilon^3 E_3(\xi) + \cdots + E_{\text{NP}}(\xi) \; . \tag{9.1}$$

Then derive the set of ODE's which these nonperturbative quantities satisfy. On the imaginary axis, and supposing we are not too close to the boundary layer, the asymptotic series is a very good approximation to the exact fields and the nonperturbative terms are small. Thus it is sufficient to linearize this latest set of ODE's: the result is a set of six first-order homogeneous ODE's. They are homogeneous because if the nonperturbative parts vanish at $\xi=0$, they vanish all along the imaginary axis because the condition for a solitary wave is satisfied. One finds that the nonperturbative functions decrease exponentially as one comes down the imaginary axis towards $\xi=0$. For a complete solution, it is necessary to find the initial conditions for these ODE's which means obtaining the functions $E, u, v, \ldots$ in the boundary layer.

The third and last part of the Kruskal-Segur analysis is to find the solution at the edge of the boundary layer. In practice this is done by integrating along a line parallel to the x axis from $-\infty$ to the imaginary axis. In general, this cannot be done analytically and one must resort to numerical integration. Far away from the boundary layer, the asymptotic series can be taken as an initial condition on the fields. On the imaginary axis, one will obtain in general a value for, e.g., E_{NP} which is $O(1)$. [The inner problem has no ϵ dependence, so this matching problem has no small parameter. Either E_{NP} is identically zero (a nongeneric case) or it is $O(1)$ at the edge of the inner region.] The values on the imaginary axis are then used as the initial conditions for the homogeneous ODE's discussed in the second step. Now integrate these ODE's down the imaginary axis to $\xi=0$. Suppose for simplicity that the system of homogeneous ODE's is ξ independent. Then the value of the mismatch function at $\xi=0$ can be written as a linear combination of exponentials $\exp(-\lambda\pi/2\Lambda)$ where $-i\lambda/\Lambda$ is an eigenvalue of the matrix describing the ODE's. The mismatch function is then exponentially small as $\Lambda \to 0$. In our case, the matrix is not ξ independent, but for any ξ it has eigenvalues which scale as $1/\Lambda$ and so essentially one still gets an exponential dependence. This is very similar to the WKB approach where the eigenvalues are also ξ dependent, and this gives rise to power corrections to the pure exponentials. Thus, up to such slowly varying corrections, one obtains an exponential dependence on $1/\Lambda$. To go beyond this leading order requires integrating the ξ-dependent ODE's, but as can be seen in Fig. 3, the corrections are very small.

X. EXPERIMENTAL CONSEQUENCES

SIT was first observed by McCall and Hahn[1] in a liquid-helium-cooled ruby rod. Further experimental confirmations of SIT were reported by Patel and Slusher,[7] and by Slusher and Gibbs[8] in gaseous absorbers. Gibbs and Slusher[9] later described detailed experiments on the propagation of coherent optical pulses in dilute rubidium vapor in a magnetic field. They observed pulses with time delays and with low energy low consistent with the theoretical predictions.[1] SIT effects in semiconductors (first studied theoretically by Poluektov and Popov,[26] and by Tzoar and Gersten[27]) have also been investigated experimentally.[28–31] The two-level systems in these materials are the excitons. Experiments on materials like CdS_xSe_{1-x} and GaAs (Refs. 28–31) showed that powerful light pulses (~ 100 MW/cm^2) near the absorption resonance can propagate above a threshold intensity with anomalously small losses; significant delays (~ 100 psec) in time (larger than the pulse widths) were observed. These features of nonlinear absorption of ultrashort optical pulses were identified with SIT for excitons in semiconductors.

It would be of interest to verify our results experimentally for both gases and solid-state systems. Our most important prediction, that of pulse shape and velocity selection, requires that one be able to resolve the gap. Thus line broadening must be kept at a minimum. In addition, one must have a system of several Beer absorption lengths wide before one can expect pulse propagation to become steady state. There have been to date no experimental verification of steady-state shapes, nor careful determinations of velocities. Similarly the perturbative corrections to the SVEA have not been investigated experimentally. The first step is thus to do velocity measurements to see if the velocity reaches an asymptotic value as the sample thickness increases. Steady-state velocities can drop to 10^3-10^4 times less than the velocity of the light in the medium away from resonance, so this may be a feasible measurement. If one can show that there is steady-state propagation, the velocity should correspond to the selected one as determined by our theory, given ω and the material parameters. In addition, if the pulse shape is measurable, it can be compared with the theoretical prediction, providing a further check on the steady-state nature of the propagation. Since steady-state pulses exist only inside the gap, one needs to be able to experimentally resolve the gap rather well; in particular, any line broadening must be small compared to $\omega_{\rm LT}$.

This constraint effectively rules out doing experiments with gases [e.g., Rb (Refs. 8 and 9)]. However, the gap is large enough in many semiconductors to permit an experiment to test our theory. Consider, for instance, a local optics ($m^* = \infty$) semiconductor with parameters like CdS. Take $\hbar\omega_t = 2.55$ eV and $\hbar\omega_{\rm LT} = 2.0$ meV, and assume (as in the case of CdS) that the line broadenings due to the finite relaxation times T_1 and T_2 are small enough so that the structure inside the gap is not washed out. Then we find that the top branch at $\Delta = 0.3$ corresponds to a pulse of width $\tau = 0.95$ psec, leading to a pulse velocity $V/c = 3.4 \times 10^{-4}$. On the same branch in Fig. 1 at $\Delta = 0.9$, we find $\tau = 2.5$ psec, and $V/c = 3.65 \times 10^{-4}$. These velocities should be measurable. In practice, spatial dispersion effects ($m^* < \infty$) need to be considered. The change in results should be small if selected velocities are not too small, but a separate investigation is required to get quantitative results.

Finally, let us point out another possibility for experimental observation of SIT. Within the SVEA, an arbitrary pulse will break up into a sequence of steady-state pulses. This is likely to be true for the full Maxwell-Bloch equations also, so that a pulse eventually becomes a sequence of solitary waves. However, it is also possible that a pulse may propagate with no loss, may not spread out, and yet never become steady state. It could resemble, for instance, a sech pulse with a time-dependent ripple superposed on it. This might occur for instance if, as ω is increased, the steady-state pulse goes unstable according to a Hopf bifurcation. To determine theoretically whether this happens, one would need to do a linear stability analysis of the selected pulses. However, we think that a *non-steady-state* SIT solution is very unlikely to exist. The reason is that a SIT pulse must return the pseudospins to the down state. If the pulse is steady state, all pseudospins have the same rotation history, so the SIT constraint reduces to one constraint on the envelope. If the pulse is not steady state, different pseudospins have different rotation histories. There is thus one new constraint for each dipole: $w = -1$ before and after the pulse has passed the point of interest. The envelope function would have to realize these constraints in a very nontrivial way; since our system is not exactly integrable, we do not think this would occur. Also note that in the SVEA, all SIT solutions are steady state. This makes the Hopf bifurcation scenario mentioned above rather improbable. On the other hand, *non-steady-state* SIT pulses would be demonstrated experimentally by seeing SIT in thick samples outside the gap since there are no steady-state pulses there at all according to our theory.

XI. CONCLUSIONS

A perturbation expansion for the population inversion w in powers of E was used to solve the coupled Maxwell-Bloch equations in the lowest orders of approximation. General expressions for solitary-wave shapes were derived for an arbitrary incident carrier frequency ω and an arbitrary pulse width τ. The expressions derived confirmed Akimoto and Ikeda's perturbation results but used a more uniform approach. The derived solutions for frequencies inside and outside the gap indicated that only in the case of very short pulses ($\Lambda \to \infty$) is the phase modulation zero, corresponding to complete inversion

("2π pulse"). In the other cases, the inversion of the atoms is not complete. In the short pulse limit one obtains McCall and Hahn's solutions and the area theorem.

However, we showed that such perturbation series generally do *not* converge to the exact solutions. For instance, our amplitude expansion and Akimoto and Ikeda's perturbation expansions are only locally valid: they are not uniformly valid on the whole domain $(-\infty,+\infty)$, and thus they cannot be used to determine whether solitary-wave solutions exist. We showed that the existence of solitary waves requires the fine tuning of the pulse width τ for a given frequency detuning $\omega-\omega_t$. Numerical results from the integration of the Maxwell-Bloch equations determined that branches of selected solitary-wave solutions exist inside the gap but not outside. This selection principle dramatically changes the previous picture of SIT. Also, even when there are steady-state pulse solutions, they are not solitons as for SIT in the SVEA. Rather they are solitary waves since the system of Maxwell-Bloch equations beyond the SVEA is not exactly integrable.

A possible extension of this work is to investigate intense pulse propagation in local media with finite relaxation times (damping $\Gamma\neq 0$) to include the relaxation times T_1 and T_2. Because of the energy loss, there will not be any true steady-state waves, but there will be pulse shapes which are nearly steady state. As T_1 and T_2 diverge, these shapes will go to the ones calculated in this paper. The major question is how fast does the dissipation effectively absorb the pseudosolitary pulses and can the losses be compensated by focusing of the beam. It is also possible to extend our analysis to spatially dispersive media. A generalization of Ikeda and Akimoto's work[23] for spatially dispersive media along the lines of our work will determine which of their "soliton" pulses are selected by the laser frequency. Furthermore, a calculation of the stability of the selected pulses is desirable.

Finally, in practice, one has to deal with the fact that there has to be an incident pulse from vacuum into the medium. This requires solving for the boundary effects due to the vacuum-dielectric interface. It would be worthwhile to investigate the shape of the incident pulses which efficiently generate solitary waves inside the medium after a few absorption lengths. As the incident pulse shape changes, one should see a discontinuous jump in the velocity of the steady-state pulse transmitted into the medium as one switches from one branch of solitary waves to another (cf. Fig. 1).

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APPENDIX A: SOLITARY WAVES AND SOLITONS

We restrict ourselves here to plane waves, i.e., waves which are x and y independent. A steady-state wave is a disturbance which depends on z and t only through $\zeta=t-z/V$, where V is a fixed constant, the velocity of the wave. Thus it preserves its shape with time. Generally, disturbances obey a partial differential equation; for a steady-state wave, the PDE reduces to an ordinary differential equation in ζ.

A solitary wave is a localized steady-state wave, i.e., it decays to zero as $\zeta\to\pm\infty$. A special subclass of solitary waves is the soliton solutions of certain nonlinear dispersive wave equations. A working definition for solitons is the following one: A soliton is a solitary-wave solution of a wave equation which asymptotically preserves its shape and velocity upon collisions with other solitons.[16]

A simple example of waves with these properties is a pulselike steady-state wave solution of the dispersionless linear wave equation. Because of linearity, the solitary waves can collide and still return to their initial shape: thus they are in fact solitons. Introducing dispersion without nonlinearity into the wave equation destroys the possibility of solitary waves because the various Fourier components of any initial conditions will propagate at different velocities. By the same token the introduction of nonlinearity without dispersion usually removes the possibility of solitary waves also because the pulse energy is continuously injected (via harmonic generation) into higher frequency modes. In the time domain this often appears as the formation of a shock wave. But with both dispersion and nonlinearity, solitary waves can arise. The solitary wave can be qualitatively understood as representing a balance between the effect of nonlinearity and that of dispersion. In very special cases, the PDE describing the wave dynamics will be exactly integrable, and the solitary waves will then be solitons.

In 1971, Lamb[5] showed that the propagating isolated hyperbolic-secant pulses found by McCall and Hahn are in fact solitons of the Maxwell-Bloch equations in the SVEA. These equations are exactly integrable and possess an infinite number of conservation laws, preventing solitons from disintegrating during collisions. A general initial condition will break up into a sequence of isolated coherent optical pulses as McCall and Hahn had found numerically and later studied by Dolfi and Hahn.[32] The amplitude of each of these solitons can be predicted from the conservation laws derived by Lamb. The analytical technique[33] for carrying out these calculations is known as the inverse scattering transform (IST). Lamb[15] showed using a series of variable transformations that the SIT equations reduce to one of the standard equations of IST, the Zakharov-Shabat equations[34] and Ablowitz, Kaup, and Newell showed how to solve the general initial value problem.[35] A general review of the IST as applied to SIT in the SVEA was given by Hauss.[36]

APPENDIX B: GENERAL EXPRESSIONS FOR THE COEFFICIENTS ($s\neq 0$) IN THE ELECTRIC-FIELD AMPLITUDE EXPANSION

In this appendix, the general formulas for w_2, w_4, and the other lowest-order coefficients C^2, ϕ_2, u_1, u_2, and v_1 for the case $s\neq 0$ are presented. All the algebraic calculations were done by using the symbolic calculation software program Mathematica on an SE-Macintosh personal computer:

$$w_2=\frac{1}{2(\Delta^2+\Lambda^2)}\ , \tag{B1}$$

$$\begin{aligned}w_4=\{&-3w_2^2[2\gamma^2-8\gamma\Lambda s w_2+8\Delta\gamma s^2w_2+3\beta\Lambda s^2w_2+3s^4w_2+2(\Lambda s w_2)^2-4\Delta\Lambda s^3w_2^2+2(\Lambda s^2w_2)^2]\}\\&\times\tfrac{1}{4}[-3\gamma^2+8(\Delta\gamma)^2w_2+8\Delta\gamma(\Lambda w_2)^2+3\beta\Lambda^3w_2^2+2\Lambda^4w_2^3-16\gamma\Lambda s(\Delta w_2)^2-6\beta\Delta s(\Lambda w_2)^2+8\gamma\Delta^3(sw_2)^2\\&\quad+3\beta\Lambda(\Delta s w_2)^2+3(\Lambda s w_2)^2-6\Delta\Lambda s^3w_2^2+3(\Delta s^2w_2)^2-8\Delta s(\Lambda w_2)^3+12(\Delta\Lambda s)^2w_2^3-8\Lambda(\Delta s w_2)^3\\&\quad+2w_2^3(\Delta s)^4]^{-1}\ ,\end{aligned} \tag{B2}$$

$$v_1=-\frac{\Lambda}{(\Delta^2+\Lambda^2)}\ , \tag{B3}$$

$$\begin{aligned}C^2=8\Lambda^2&[-3\gamma^2+8(\Delta\gamma)^2w_2+8\Delta\gamma(\Lambda w_2)^2+3\beta\Lambda^3w_2^2+2\Lambda^4w_2^3-16\gamma\Lambda s(\Delta w_2)^2-6\beta\Delta s(\Lambda w_2)^2\\&+8\gamma\Delta^3(sw_2)^2+3\beta\Lambda(\Delta s w_2)^2+3(\Lambda s w_2)^2-6\Delta\Lambda s^3w_2^2+3(\Delta s^2w_2)^2-8\Delta s(\Lambda w_2)^3+12(\Delta\Lambda s)^2w_2^3\\&-8\Lambda(\Delta s w_2)^3+2w_2^3(\Delta s)^4]\\&\times w_2^{-1}[-24\gamma\Lambda s+12\Delta\gamma s^2+9\beta\Lambda s^2+9s^4+8\Delta\gamma\Lambda^2w_2+6\beta\Lambda^3w_2+48\Delta^2\gamma\Lambda s w_2-12\beta\Delta\Lambda^2 s w_2\\&\quad-24\Delta^3\gamma s^2w_2-18\beta\Delta^2\Lambda s^2w_2+12(\Lambda s)^2w_2-24\Delta\Lambda s^3w_2-12\Delta^2s^4w_2+4\Lambda^4w_2^2+16\Delta\Lambda^3 s w_2^2\\&\quad-56(\Delta\Lambda s w_2)^2+48\Lambda w_2^2(\Delta s)^3-12w_2^2(\Delta s)^4]^{-1}\ ,\end{aligned} \tag{B4}$$

$$\begin{aligned}\phi_2=3w_2&[-3\gamma s^2-2\gamma w_2\Lambda^2+4\Delta\gamma\Lambda s w_2+6\gamma w_2(\Delta s)^2+8s\Lambda^3w_2^2-16\Delta(\Lambda s w_2)^2+8\Lambda(\Delta w_2)^2s^3]\\&\times(4\Lambda)^{-1}[-3\gamma^2+8(\Delta\gamma)^2w_2+8\Delta\gamma(\Lambda w_2)^2+3\beta\Lambda^3w_2^2+2\Lambda^4w_2^3-16\gamma\Lambda s(\Delta w_2)^2-6\beta\Delta s(\Lambda w_2)^2+8\gamma\Delta^3(sw_2)^2\\&\quad+3\beta\Lambda(\Delta s w_2)^2+3(\Lambda s w_2)^2-6\Delta\Lambda s^3w_2^2+3(\Delta s^2w_2)^2-8\Delta s(\Lambda w_2)^3+12(\Delta\Lambda s)^2w_2^3-8\Lambda(\Delta s w_2)^3\\&\quad+2w_2^3(\Delta s)^4]^{-1}\ ,\end{aligned} \tag{B5}$$

$$u_1=-\frac{\Delta}{(\Delta^2+\Lambda^2)}\ , \tag{B6}$$

$$\begin{aligned}u_3=w_2^2&[+4\Delta\gamma^2+3\gamma s^2+2\gamma w_2\Lambda^2-20\Delta\gamma\Lambda s w_2+10\gamma w_2(\Delta s)^2+6\beta\Delta\Lambda s^2w_2+6\Delta s^4w_2-8s\Lambda^3w_2^2+20\Delta(\Lambda s w_2)^2\\&-16\Lambda(\Delta w_2)^2s^3+4\Delta^3s^4w_2^2]\\&\times\tfrac{1}{2}[-3\gamma^2+8(\Delta\gamma)^2w_2+8\Delta\gamma(\Lambda w_2)^2+3\beta\Lambda^3w_2^2+2\Lambda^4w_2^3-16\gamma\Lambda s(\Delta w_2)^2-6\beta\Delta s(\Lambda w_2)^2+8\gamma\Delta^3(sw_2)^2\\&\quad+3\beta\Lambda(\Delta s w_2)^2+3(\Lambda s w_2)^2-6\Delta\Lambda s^3w_2^2+3(\Delta s^2w_2)^2-8\Delta s(\Lambda w_2)^3+12(\Delta\Lambda s)^2w_2^3-8\Lambda(\Delta s w_2)^3\\&\quad+2w_2^3(\Delta s)^4]^{-1}\ .\end{aligned} \tag{B7}$$

APPENDIX C: LIMITING CASES OF THE ELECTRIC-FIELD AMPLITUDE EXPANSION

We have the following cases.

(i) *Long pulses inside the gap.* The expansion parameter $\epsilon=\Lambda/\Delta$ is the same inside and outside the gap but the terms α, β, and γ have different expressions depending on which case is treated. In Sec. VI, we gave results for long pulses outside the gap. For pulses inside the gap, we find for the lowest-order solutions

$$E=2\Delta\sqrt{1-\Delta}\,\mathrm{sech}\xi\ , \tag{C1a}$$

$$\dot{\phi}=-\frac{3}{2}\frac{\Lambda}{\Delta}\mathrm{sech}^2\xi\ , \tag{C1b}$$

$$u=-2\sqrt{1-\Delta}\,\mathrm{sech}\xi+4(1-\Delta)^{3/2}\mathrm{sech}^3\xi\ , \tag{C1c}$$

$$v=\frac{2\Lambda}{\Delta}\sqrt{1-\Delta}\,\mathrm{sech}\xi\tanh\xi\ , \tag{C1d}$$

$$w=-1+2(1-\Delta)\mathrm{sech}^2\xi-6(1-\Delta)^2\mathrm{sech}^4\xi\ . \tag{C1e}$$

(ii) *Long pulse at resonance, i.e., $\Delta=0$ or $\omega=\omega_t$.* The expansion parameter is $\epsilon=\Lambda$ and the results are

$$E=\Lambda\,\mathrm{sech}\xi\ , \tag{C2a}$$

$$\dot{\phi}=-\tfrac{3}{4}\mathrm{sech}^2\xi\ , \tag{C2b}$$

$$u=\tfrac{1}{4}\mathrm{sech}^3\xi\ , \tag{C2c}$$

$$v=\mathrm{sech}\xi\tanh\xi\ , \tag{C2d}$$

$$w=-1+\tfrac{1}{2}\mathrm{sech}^2\xi-\tfrac{3}{8}\mathrm{sech}^4\xi\ . \tag{C2e}$$

(iii) *Long pulses at $\Delta=1$, or $\omega=\omega_t+\omega_{LT}$.* In this case, the expansion parameter is $\epsilon=\Lambda$ and the results are

$$E=\sqrt{2\Lambda}\,\mathrm{sech}\xi\ , \tag{C3a}$$

$$\dot{\phi}=-\frac{3\Lambda}{2}\mathrm{sech}^2\xi\ , \tag{C3b}$$

$$u=-\sqrt{2\Lambda}\,\mathrm{sech}\xi+(2\Lambda^3)^{1/2}\mathrm{sech}^3\xi\ , \tag{C3c}$$

$$v=(2\Lambda^3)^{1/2}\text{sech}\xi\tanh\xi\ , \tag{C3d}$$

$$w=-1+\Lambda\,\text{sech}^2\xi-\frac{3\Lambda^2}{2}\text{sech}^4\xi\ . \tag{C3e}$$

(iv) *Short pulses.* The following conditions are satisfied: $\Lambda \gg 1$ and $\Lambda \gg \Delta$ (in Akimoto and Ikeda's paper,[11] $\epsilon=1/\Lambda \ll 1$). The solutions are given as follows:

$$E=2\Lambda\,\text{sech}\xi\ , \tag{C4a}$$

$$\dot{\phi}=-\frac{3}{8\Lambda}\text{sech}^2\xi\ , \tag{C4b}$$

$$u=-2\frac{\Delta}{\Lambda}\text{sech}\xi+\frac{1}{4\Lambda}\text{sech}^3\xi\ , \tag{C4c}$$

$$v=2\,\text{sech}\xi\tanh\xi\ , \tag{C4d}$$

$$w=-1+2\,\text{sech}^2\xi-\frac{3}{8\Lambda^2}\text{sech}^4\xi\ . \tag{C4e}$$

For $\Lambda\rightarrow\infty$, one obtains complete inversion and no phase modulation, as found by McCall and Hahn.[1]

APPENDIX D: NUMERICAL INTEGRATION OF THE MAXWELL-BLOCH EQUATIONS

In this appendix, details for the numerical integration of the Maxwell-Bloch equations are presented. The Maxwell-Bloch equations for the steady-state pulse form a system of ODE's [Eqs. (4.6)–(4.10)] which can be written in a more convenient way for computational purposes as a system of first-order equations. Defining $\dot{E}=e$, and $\dot{\phi}=\Phi$, one has

$$\dot{E}=e\ , \tag{D1}$$

$$\dot{e}=\frac{\alpha}{\gamma}E+\frac{\beta}{\gamma}\Phi+\Phi^2E+\left[\frac{s}{\gamma\Lambda}\right]\left[\frac{s\Delta}{\Lambda}-2\right]wE-\left[\frac{s^2}{\gamma\Lambda}\right]\Phi wE-\frac{1}{\gamma}\left[\frac{s\Delta}{\Lambda}-1\right]^2u\ , \tag{D2}$$

$$\dot{\Phi}=-\left[\frac{\beta}{\gamma}\right]\frac{e}{E}-2\frac{\Phi e}{E}+\left[\frac{s^2}{\gamma\Lambda}\right]\frac{we}{E}-\frac{1}{\gamma}\left[\frac{s\Delta}{\Lambda}-1\right]^2\frac{v}{E}-\left[\frac{s^2}{\gamma\Lambda^2}\right]Ev\ , \tag{D3}$$

$$\dot{u}=\frac{\Delta}{\Lambda}v+\Phi v\ , \tag{D4}$$

$$\dot{v}=-\frac{\Delta}{\Lambda}u-\Phi u+\frac{1}{\Lambda}Ew\ , \tag{D5}$$

$$\dot{w}=-\frac{1}{\Lambda}Ev\ . \tag{D6}$$

In order to integrate the system numerically, we use the expressions for E, $\dot{\phi}$, u, v, and w derived in Sec. VI as asymptotic initial conditions:

$$\begin{aligned}&E=C\,\text{sech}\xi,\quad \dot{E}=-C\,\text{sech}\xi\tanh\xi\ ,\\&\Phi=\phi_2E^2,\quad u=u_1E+u_3E^3\ ,\\&v=v_1\dot{E},\quad w=-1+w_2E^2+w_4E^4\ .\end{aligned} \tag{D7}$$

The general form of the coefficients C, ϕ_2, u_1, u_3, v_1, w_2, and w_4 ($s\neq0$) is given in Appendix B [Eqs. (B1)–(B7)].

The transformed system of ODE's [Eqs. (D1)–(D6)] is a stiff set of equations. Stiffness occurs in ODE's when there are two or more relevant scales of the independent variable ξ. In our case [Eqs. (D1)–(D6)] the large scale is the pulse width τ, which corresponds to the overall structure of the solitary-wave solutions. The short scale is of size ϵ, where ϵ is a small parameter associated with rapidly oscillating modes. These can be seen explicitly by linearizing the ODE's in the tail: two of the eigenvalues of the linearized system are equal to $\pm i/\epsilon$, and thus rapidly oscillating. For stiff ODE's most integration methods (such as Runge-Kutta, Bulirsch-Stoer, and predictor-correctors[37]) fail, because the stability of these integration schemes is controlled by the most rapidly varying component. Therefore, to follow the long-time behavior (the overall structure of the solitary wave solutions), one must choose time steps smaller than the shortest time scale. Then the large number of steps leads to laborious integrations plus potentially large accumulation of errors. Fortunately, algorithms for stiff ODE's have been developed which do not have this problem. One of the most efficient ones due is to Kaps and Rentrop,[38] and it is a generalization of the Runge-Kutta scheme that monitors the local truncation error to adjust stepsize. An introduction to this stiff algorithm is given by Press and Teukolsky.[39] We used the subroutine STIFF from Ref. 39. It requires two subroutines JACOBI and DERIVS which contain all the information regarding the system of six ODE's. STIFF uses the subroutines ODEINT, LUDCM, and LUBKSB from the *Numerical Recipes* book.[37]

The overall program[40,41] searches for solitary solutions by solving Eqs. (D1)–D5) with initial conditions given by Eq. (D7). The input parameters are the frequency detuning $\Delta=(\omega-\omega_t)/\omega_{LT}$, the inverse pulse width $\Lambda=1/\omega_{LT}\tau$, and the material parameter ω_t/ω_{LT}. For different values of Δ and Λ, the program scans in the domain of (Δ,Λ) for solitary-wave solutions. The solvability condition (cf. Sec. VIII) for the existence of steady-state solitary waves is that both the derivative of the electric-field amplitude $dE/d\xi$ and the out-of-phase component of the polarization v vanish simultaneously. The program integrates from some asymptotic value ξ_{initial} near $\xi=-\infty$ and defines ξ_{tip} by the location of $v=0$. Then one does a *regula falsi* method on Δ to find when $\dot{E}_{\text{tip}}=0$ at the tip. The program outputs the set of points in (Δ,Λ) space for which the solvability condition is satisfied.

Various tests on the program have been conducted. The initial value of ξ used in Eq. (D7) was varied. The results of the integration showed a fast convergence as one took large negative values for ξ_{initial}. The results also were seen to converge as the accuracy at each step was increased. The integration was checked against Akimoto and Ikeda's solutions and the perturbative solutions of Sec. VI. In both cases, the numerical results agreed very well with the analytic formulas for ξ away from zero. We also checked that the condition $u^2+v^2+w^2=1$ was preserved under integration to the expected accuracy.

Another check consists in using the conservation law for the Bloch components u, v, and w ($u^2+v^2+w^2=1$) to transform the system of six ODE's to a system of five ODE's in E, e, Φ, u, and v. We coded this five-dimensional system and the agreement between the two programs for the selected values of (Δ,Λ) was to the accuracy expected.

Finally, one can subtract out from the vector of six functions $\mathbf{Y}$ the analytically derived part: $\mathbf{Y}=\mathbf{Y}_{\text{Ikeda}}+\epsilon\delta\mathbf{Y}$, where ϵ is a small expansion parameter related to the pulse width for Akimoto and Ikeda's analysis. We linearized Eqs. (D1)–(D6) in $\delta\mathbf{Y}$ and integrated numerically the corresponding linear inhomogeneous system. As $\epsilon\to 0$, the program for the linearized equations and for the original nonlinear equations agreed to the expected order in ϵ.

The programs were run on the Celerity and VAX computer machines at the City College of New York, Department of Physics.

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