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Analytic solution of a two-dimensional hydrogen atom. I. Nonrelativistic theory

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The two-dimensional hydrogen problem is solved analytically. In the nonrelativistic case, exact formulas for energy eigenvalues and eigenfunctions for both the discrete and continuous parts of the spectrum, dipole matrix elements, dc Stark effect, single- and two-photon transition rates, and fine and hyperfine structures are obtained. Comparison is made between the two- and the three-dimensional cases. Some interesting aspects of the solution unique to the two-dimensional case are discussed.

I. INTRODUCTION TO TWO-DIMENSIONAL HYDROGEN ATOM

The hydrogen atom is the name given to the system composed of an electron with mass m_e and charge $-e$ and a positively charged nucleus (Ze) located at the origin of the coordinate system. The central force between the electron and the nucleus is determined by the attractive Coulomb potential function,

$$V(r)=-\frac{Ze^2}{r}\ . \tag{1.1}$$

The three-dimensional (3D) hydrogen atom played a central role in the early formulation and development of quantum mechanics and is now part of the standard curriculum in modern undergraduate physics.

If the motion of the electron around the nucleus is constrained in a plane by certain boundary conditions, then such a system is called the two-dimensional hydrogen atom. We would like to point out that "2D" in the name "2D hydrogen atom" only emphasizes that the motion of the electron around a positive point charge (not a line charge) is constrained in a plane. This system is not 2D in a strict sense that all fields including electromagnetic fields, photon emission, angular momentum, and spin are not confined to a plane.

The quantum-mechanical properties of the 2D hydrogen atom are the subject of this investigation. In this paper, paper I, we give a full account of the nonrelativistic description of a two-dimensional hydrogen atom. A detailed presentation[1] of the complete theory including the relativistic case and the Chern-Simons term will appear in paper II (we will refer to it as II) of this series. In the following section, we will present the solutions to the nonrelativistic case while in Sec. III some concluding remarks are made. Possible implications and potential applications to other branches of physics are pointed out.

Atomic units will be used unless otherwise stated.

II. NONRELATIVISTIC THEORY

The eigenstates of a 2D hydrogen atom are described by the Schrödinger equation, in polar coordinates,

$$\left[-\frac{\hbar^2}{2m_e}\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right]-\frac{Ze^2}{r}\right]\psi(r,\varphi)=E\psi(r,\varphi)\ . \tag{2.1}$$

Using separation of variables

$$\psi(r,\varphi)=R(r)\Phi(\varphi)\ , \tag{2.2}$$

we obtain

$$\Phi(\varphi)=\frac{1}{(2\pi)^{1/2}}e^{il\varphi},\quad l=0,\pm1,\pm2,\ldots \tag{2.3}$$

and

$$\frac{d^2}{dr^2}R(r)+\frac{1}{r}\frac{d}{dr}R(r)+\left[\frac{2m_e}{\hbar^2}\left[E+\frac{Ze^2}{r}\right]-\frac{l^2}{r^2}\right]R(r)=0\ . \tag{2.4}$$

Equation (2.4) is the 2D radial Schrödinger equation; its solution $R(r)$, depends only on $|l|$. Solution (2.3) is also the eigenfunction of the angular momentum along the z direction

$$\hat{L}_z=-i\hbar\frac{\partial}{\partial\varphi}\ , \tag{2.5}$$

which commutes with the Hamiltonian. Hence l is a good quantum number.

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A. Solutions of the radial Schrödinger equation

The Schrödinger equation contains both bound discrete states and unbound continuous states. Let us consider these separately.

1. Discrete spectrum $(E<0)$

Let us define

$$E=-\frac{Z^2}{2N^2}\left[\frac{m_e e^4}{\hbar^2}\right] \tag{2.6}$$

and

$$r=\frac{N\hbar^2}{2m_e Ze^2}x=\frac{1}{\beta_N}x \tag{2.7}$$

and

$$R(x)=x^{|l|}e^{-x/2}G(x) . \tag{2.8}$$

Equation (2.4) then becomes

$$x\frac{d^2G}{dx^2}+[(2|l|+1)-x]\frac{dG}{dx}-(-N+|l|+\tfrac{1}{2})G=0 . \tag{2.9}$$

The solution of Eq. (2.9), which is regular at $x=0$, is the confluent hypergeometric function[2]

$$G(x)={}_1F_1(-N+|l|+\tfrac{1}{2},2|l|+1,x) . \tag{2.10}$$

A solution which satisfies the condition at infinity is obtained only for negative integral (or zero) values of $-N+|l|+\frac{1}{2}$, then the function (2.10) reduces to a finite polynomial. Otherwise it diverges at $r\rightarrow\infty$. Thus we arrive at the condition

$$N=\tfrac{1}{2},\tfrac{3}{2},\tfrac{5}{2},\ldots . \tag{2.11}$$

Let

$$n=N+\tfrac{1}{2}=1,2,3,\ldots , \tag{2.12}$$

where n, an integer, is called the principal quantum number. For a given n, $|l|$ can take the values

$$|l|=0,1,2,\ldots,n-1 . \tag{2.13}$$

Recalling the definitions (2.6) and (2.12), we obtain the 2D energy eigenvalue

$$E_n=-\frac{Z^2}{2(n-\frac{1}{2})^2}\frac{m_e e^4}{\hbar^2} . \tag{2.14}$$

The normalized radial eigenfunction is given by

$$\begin{aligned}R_{nl}(r)=&\frac{\beta_n}{(2|l|)!}\left[\frac{(n+|l|-1)!}{(2n-1)(n-|l|-1)!}\right]^{1/2}\\&\times(\beta_n r)^{|l|}\exp(-\beta_n r/2)\\&\times{}_1F_1(-n+|l|+1,2|l|+1,\beta_n r) ,\end{aligned} \tag{2.15}$$

where

$$\beta_n=\frac{2Z}{n-\frac{1}{2}}\frac{m_e e^2}{\hbar^2} . \tag{2.16}$$

For the purpose of future reference, we give the first few functions R_{nl} explicitly:

$$\begin{aligned}&R_{10}=\beta_1 e^{-\beta_1 r/2} ,\\&R_{20}=(\beta_2/3^{1/2})(1-\beta_2 r)e^{-\beta_2 r/2} ,\\&R_{21}=(\beta_2^2/6^{1/2})re^{-\beta_2 r/2} ,\\&R_{30}=[\beta_3/2(5)^{1/2}](2-4\beta_3 r+\beta_3^2 r^2)e^{-\beta_3 r/2} ,\\&R_{31}=(\beta_3^2/30^{1/2})r(3-\beta_3 r)e^{-\beta_3 r/2} ,\\&R_{32}=[\beta_3^3/(5!)^{1/2}]r^2 e^{-\beta_3 r/2} .\end{aligned} \tag{2.17}$$

The average values of various powers of r are useful in many different physical problems to be discussed later and can be calculated from the formula

$$(\overline{r^k})_{nl}=\int_0^\infty r^{k+1}(R_{nl})^2 dr . \tag{2.18}$$

The general expression for $(\overline{r^k})_{nl}$ can be obtained by means of formula (f7) of Ref. 2. Here, we shall give the first few values of $(\overline{r^k})_{nl}$:

$$\begin{aligned}&(\overline{r})_{nl}=\tfrac{1}{2}[3n(n-1)-|l|^2+1](a_0/Z) ,\\&(\overline{r^2})_{nl}=\tfrac{1}{8}(2n-1)[n(10n^2-15n+11)\\&\qquad\qquad-3|l|^2(2n-1)-3](a_0/Z)^2 ,\\&(\overline{r^{-1}})_{nl}=\frac{1}{(n-\frac{1}{2})^2}(Z/a_0) ,\\&(\overline{r^{-2}})_{nl}=\frac{1}{(n-\frac{1}{2})^3|l|}(Z/a_0)^2 \quad\text{for } l\neq 0 ,\\&(r_{\text{most probable}})_{n,l=n-1}=(n-\tfrac{1}{2})^2(a_0/Z) ,\end{aligned} \tag{2.19}$$

where $a_0=\hbar^2/(m_e e^2)$ is the Bohr radius.

Following Kohn and Luttinger,[3] the 2D hydrogen ground-state energy, $E_1=-2$ a.u., is four times that of the 3D counterpart. More general and detailed analysis was carried out by Alliluev[4] in momentum space, by Zaslow and Zandler[5] in polar coordinates, and by Cisneros and McIntosh[6] in parabolic coordinates. In particular, the eigenfunction R_{nl} in terms of associated Laguerre polynomials was given in Ref. 5 (there are a few misprints in Table I of that work). We would like to point out that by expressing the eigenfunctions in terms of confluent hypergeometric functions we can treat the discrete and continuous, nonrelativistic and relativistic cases all on the same footing. In addition, the computation and derivation involving confluent hypergeometric functions are simple and well known.[7]

It is interesting to observe that by replacing $n-\frac{1}{2}$ by n (n is the principle quantum number), we retrieve many relevant formulas of the 3D hydrogenic atom [such as formula (2.14) and the last one of formula (2.19)]. Since the 3D energy spectrum was first derived from the Bohr quantization condition, a few words on such a "replacement" are now in order. In contrast to the 3D case, the two linearly independent solutions of the 2D relativistic Dirac equation are *disjoint* to each other (see paper II). And, in the nonrelativistic limit, one represents spin up, the other spin down, such that both have a definite z-component angular momentum. Mathematically, this means a ket for an electron state is a tensor product of a ket in spin space, $|s_z\rangle$, and one in orbital space, $|l_z\rangle$. Now consider a rotation by a finite angle ϕ about the z axis. If the ket before rotation is given by $|\alpha\rangle$, the ket after rotation, $|\alpha\rangle_R$, is given by[8]

$$|\alpha\rangle=|s_z\rangle\otimes|l_z\rangle\rightarrow|\alpha\rangle_R=\exp(-i\hat{s}_z\phi/\hbar)|s_z\rangle \otimes\exp(-i\hat{l}_z\phi/\hbar)|l_z\rangle \; . \tag{2.20}$$

A rotation of $\phi=2\pi$ (one round), gives $|\alpha\rangle_R=-|\alpha\rangle$ (the minus sign comes from the spin part). Thus we need a $\phi=4\pi$ (two rounds) rotation to get back to the same ket with a plus sign. In the intuitive language of de Broglie, the condition for standing wave implies

$$2(2\pi r)=(2n-1)\lambda, \quad n=1,2,3,\ldots \; . \tag{2.21}$$

Note that even-number multiples of wavelength on the right-hand side of Eq. (2.21) are excluded; otherwise $2(2\pi r)=(2n)\lambda$ means that one round rotation is possible. Relation (2.21) leads to the following condition:

$$rp=(n-\tfrac{1}{2})\hbar, \quad n=1,2,\ldots \; . \tag{2.22}$$

We may refer to (2.22) as the 2D Bohr quantization condition, which together with the Coulomb law leads to the 2D energy spectrum (2.14). For the 3D case, instead of Eq. (2.20), we have

$$|\alpha\rangle\rightarrow|\alpha\rangle_R=\exp(-i\hat{l}_z\phi/\hbar)|\alpha\rangle \; . \tag{2.23}$$

[Since spin can change direction in 3D case, it no longer appears in Eq. (2.23).] Therefore we only need a $\phi=2\pi$ (one round) rotation to get $|\alpha\rangle_R=|\alpha\rangle$, which gives the usual Bohr quantization condition. Thus the loss of a single space dimension has a profound effect on a quantum system.

2. *Continuous spectrum* $(E>0)$

Let us define

$$\begin{aligned} k &\equiv \left[\frac{2E}{Z^2}\frac{\hbar^2}{m_e e^4}\right]^{1/2} , \\ \beta &\equiv Z\left[\frac{m_e e^2}{\hbar^2}\right] , \\ N' &= -i/k \; , \\ x &= i2k\beta r \; , \end{aligned} \tag{2.24}$$

then Eq. (2.4) becomes

$$\frac{d^2R(x)}{dx^2}+\frac{1}{x}\frac{dR(x)}{dx}+[(-\tfrac{1}{4}+N'/x)-l^2/x^2]R(x)=0 \; , \tag{2.25}$$

and its regular solution is

$$R_{kl}(r)=C_{kl}(2k\beta r)^{|l|}\exp(-ik\beta r) \times {}_1F_1(i/k+|l|+\tfrac{1}{2},2|l|+1,i2k\beta r) \; . \tag{2.26}$$

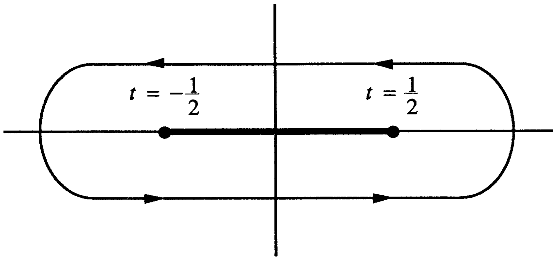


FIG. 1. Contour of the complex integral (2.29).

The constant C_{kl} is determined from the δ-function normalization condition

$$\int_0^\infty R_{kl}(r)R_{k'l}(r)r\,dr=\delta(k-k') \; , \tag{2.27}$$

and the result gives

$$C_{kl}=\beta\left[\frac{2k}{1+e^{-2\pi/k}}\right]^{1/2}\prod_{s=0}^{|l|-1}[(s+\tfrac{1}{2})^2+1/k^2]^{1/2} \; ; \tag{2.28}$$

for $l=0$, the product is replaced by unity. Expression (2.28) appears in a recent paper[9] without derivation. Our derivation will be presented elsewhere.[10]

Using the integral representation of $F(\alpha,\gamma,z)$,[11] the continuous eigenfunction R_{kl} can be represented as a complex integral:

$$R_{kl}=C_{kl}\frac{(-2k\beta r)^{-|l|}}{2\pi i} \times\oint\exp(i2k\beta rt)(t+\tfrac{1}{2})^{i/k-|l|-1/2} \times(t-\tfrac{1}{2})^{-i/k-|l|-1/2}dt \; , \tag{2.29}$$

which is taken along the contour as shown in Fig. 1.

B. Dipole matrix elements

The analytic expressions of the dipole matrix elements are useful in evaluating optical transition rates, Stark shift, light shift, oscillator strength, etc. The following two matrix elements involving the ground state are the most important ones:

$$\begin{aligned} \langle R_{10}|r|R_{n1}\rangle &= \int_0^\infty R_{10}(r)rR_{n1}(r)r\,dr \\ &= \frac{1}{4}\frac{(2n-1)^{5/2}(n-1)^{n-5/2}}{n^{n+3/2}} \\ &\sim \frac{1}{n^{3/2}} \text{ for large } n \; , \end{aligned} \tag{2.30}$$

$$\langle R_{10}|r|R_{k1}\rangle=\int_0^{\infty}R_{10}(r)rR_{k1}(r)r\,dr$$
$$=16\left[\frac{2k}{1+e^{-2\pi/k}}\right]^{1/2}$$
$$\times\frac{\exp[-(2/k)\tan^{-1}(k/2)]}{(k^2+4)^2}\ . \qquad (2.31)$$

In deriving formulas (2.30) and (2.31), we have utilized formula (f1) of Ref. 2. We would like to mention that the matrix elements $\langle R_{10}|r|R_{n1}\rangle$ and $\langle R_{10}|r|R_{k1}\rangle$ in a 3D H atom are given in Refs. 12 and 13 and have the form

$$\langle R_{10}|r|R_{n1}\rangle=\frac{2^4(n-1)^{n-5/2}n^{7/2}}{(n+1)^{n+5/2}}$$
$$\sim\frac{1}{n^{3/2}}\ , \qquad (2.32)$$

$$\langle R_{10}|r|R_{k1}\rangle=16\left[\frac{k}{1-e^{-2\pi/k}}\right]^{1/2}$$
$$\times\frac{\exp[(-2\tan^{-1}k)/k]}{(k^2+1)^{5/2}}\ . \qquad (2.33)$$

C. Coulomb Green function

In the second-order perturbation theory, one has to deal with a difficult problem of how to evaluate exactly the infinite sums extended to the whole atomic spectrum. The Green-function method is designed to handle such exact calculations.

We define the partial-wave Coulomb Green function via

$$\mathcal{G}_l(E;r,r')=\mathop{\mathsf{S}}_{n\ (>|l|)}\frac{R_{nl}(r)R_{nl}(r')}{E_n-E}\ , \qquad (2.34)$$

where E is a parameter and S runs over all radial wave functions, including *both* bound and continuous states. An analytic expression for $\mathcal{G}_l$ was derived by Hostler[14] in 1970:

$$\mathcal{G}_l(E;r,r')=\frac{\nu}{(rr')^{1/2}}\Gamma(\tfrac{1}{2}+|l|-\nu)W_{\nu,|l|}$$
$$\times\left[\frac{2r_>}{\nu}\right]M_{\nu,|l|}\left[\frac{2r_<}{\nu}\right]\ , \qquad (2.35)$$

where $E=-[1/(2\nu^2)]$. In expression (2.35), $r_<$ ($r_>$) is the smaller (larger) of r and r', and W and M are the Whittaker functions.[15]

We now consider an integral involving $\mathcal{G}_l$:

$$J_l(\nu,\beta,\beta')=\tfrac{1}{2}\int_0^{\infty}dr\int_0^{\infty}dr'(rr')^{|l|}$$
$$\times e^{-(1/\nu)(\beta r+\beta' r')}\mathcal{G}_l(E;r,r')\ .$$
$$(2.36)$$

(The closed-form expression for J_l is given in Ref. 10). The result is rather simple:

$$J_l(\nu,\beta,\beta')=\frac{2^{2|l|}(2|l|)!\nu^{2|l|+2}}{(|l|-\nu+\frac{1}{2})[(\beta+1)(\beta'+1)]^{2|l|+1}}\,{}_2F_1\left[2|l|+1,|l|+\tfrac{1}{2}-\nu;|l|+\tfrac{3}{2}-\nu;\frac{(\beta-1)(\beta'-1)}{(\beta+1)(\beta'+1)}\right]\ , \qquad (2.37)$$

where ${}_2F_1$ is the usual hypergeometric function. Since integrals involving powers of r (or r') can be obtained from J_l by the proper differentiation with respect to β (or β'), formula (2.36) has many computational applications (see Secs. II D and II E). Note that the corresponding 3D formula for J_l was first derived by Rapoport and Zon[16] and has been applied to the calculation of the two-photon transition rates[17–19] and the dynamic multipole polarizability.[20]

For completeness, we will also briefly discuss an alternate representation of the Coulomb Green function involving a sum over a discrete set of states only.[21,14,22] In the 3D coordinate space, such a representation[14,22] is referred to as "Sturmian" since it makes use of the so-called Coulomb "Sturmian" functions introduced by Rotenberg[23] in 1962. The advantage of using this representation has been demonstrated by Maquet[24] in his work on light shift of hydrogenic (highly excited) states. We have studied the 2D Coulomb "Sturmian" functions and derived a corresponding 2D representation for the Coulomb Green function. Again we refer the reader to Ref. 10 for the details and only quote the final result

$$\mathcal{G}_l(E;r,r')=-\frac{2^{2|l|+1}\nu^{-2|l|}}{[(2|l|)!]^2}e^{-(1/\nu)(r+r')}(rr')^{|l|}$$
$$\times\sum_{n=|l|+1}^{\infty}[(n-\tfrac{1}{2})-\nu]^{-1}\frac{(n+|l|-1)!}{(n-|l|-1)!}\,{}_1F_1(|l|+1-n,2|l|+1,2r/\nu)\,{}_1F_1(|l|+1-n,2|l|+1,2r'/\nu)\ . \qquad (2.38)$$

D. dc Stark effect

In the Stark effect, the 2D hydrogen atom is perturbed by a uniform external electric field $\mathbf{E}=E_0\hat{\mathbf{x}}$. Since the fields applied in laboratory experiments are usually weak in comparison with atomic fields, the perturbation in the Hamiltonian is (in a.u.)

$$H'=-\mathbf{r}\cdot(q\mathbf{E})=E_0x\ . \qquad (2.39)$$

1. Stark shift in the ground-state energy

The perturbation expansion for the nondegenerate ground-state energy is given as a series

$$E_{\mathcal{G}}=\mathcal{E}_0+\mathcal{E}_1+\mathcal{E}_2+\cdots\ , \qquad (2.40)$$

where $\mathcal{E}_0=-2$ (unperturbed energy), and $\mathcal{E}_1=\mathcal{E}_3$

$=\mathcal{E}_5=\cdots=0$ due to the odd parity of H'. We rewrite expansion (2.40) as

$$E_{\mathcal{g}}=-a_0E_0^0-a_2E_0^2-a_4E_0^4-a_6E_0^6-\cdots\ , \tag{2.41}$$

with $a_0=2$.

(i) *Evaluation of* a_2. Standard second-order perturbation theory gives

$$a_2=\tfrac{1}{2}\mathop{\mathsf{S}}\limits_{n=2}\frac{|\langle R_{10}|r|R_{n1}\rangle|^2}{E_n^0-E_1^0}\ , \tag{2.42}$$

where we have used $x=r\cos\varphi$ and evaluated the angular part. We shall show the evaluation of a_2 via three independent methods.

(a) *Coulomb Green-function method.* With $E_1^0=-2=-1/(2\nu^2)$, i.e., $\nu=\frac{1}{2}$, in terms of the definitions (2.34) and (2.36), we can rewrite expression (2.42) as

$$\begin{aligned}a_2&=16\left[\nu^2\frac{\partial}{\partial\beta}\frac{\partial}{\partial\beta'}J_1(\nu,\beta,\beta')\right]_{\nu=1/2,\beta=\beta'=1}\\&=\frac{21}{2^8}\ .\end{aligned} \tag{2.43}$$

This result is exact. Alternatively, we may employ the Sturmian representation (2.38); thus

$$a_2=4\sum_{n=2}^{\infty}n\left[\frac{\Gamma(4)}{4^3}{}_2F_1(2-n,4;3;1)\right]^2\ . \tag{2.44}$$

Using formula 9.122.1 of Ref. 15, only the first two terms in the sum contribute, which again leads to result (2.43).

(b) *Implicit technique.* The main point of the implicit technique introduced by Dalgarno and Lewis in 1955[25,26] is to replace the evaluation of the infinite summations in the second-order matrix elements by the solution of an inhomogeneous differential equation. This procedure is much simpler.

Let us define an auxiliary function

$$u(r)=\mathop{\mathsf{S}}\limits_{n=2}\frac{R_{n1}(r)\int_0^\infty R_{n1}(r')r'R_{10}(r')dr'}{E_1^0-E_n^0}\ , \tag{2.45}$$

then

$$a_2=-\tfrac{1}{2}\int_0^\infty R_{10}(r)ru(r)r\,dr=-2\int_0^\infty e^{-2r}r^2u(r)dr\ . \tag{2.46}$$

Using the radial Schrödinger equation and the closure relation for $R_{n1}(r)$ gives us the differential equation satisfied by $u(r)$:

$$r^2\frac{d^2u}{dr^2}+r\frac{du}{dr}+(-4r^2+2r-1)u=8r^3e^{-2r}\ . \tag{2.47}$$

Its solution is easy to find via the series expansion method;[27] thus

$$u(r)=-[(3r/4)+r^2]e^{-2r}\ . \tag{2.48}$$

If we feed this back into expression (2.46), we have

$$a_2=\tfrac{1}{2}\int_0^\infty e^{-4r}(3r^3+4r^4)dr=\frac{21}{2^8}\ . \tag{2.49}$$

It is instructive to describe an alternative solution[28] to Eq. (2.47), which avoids solving the equation totally. We note that if we define the Laplace transform of $u(r)$:

$$v(p)=\int_0^\infty u(r)e^{-pr}dr\ , \tag{2.50}$$

then

$$a_2=-2\left.\frac{d^2v}{dp^2}\right|_{p=2}\ . \tag{2.51}$$

Now we find the Laplace transform of Eq. (2.47) is

$$(p^2-4)\frac{d^2v}{dp^2}+(3p-2)\frac{dv}{dp}=\frac{48}{(p+2)^4}\ . \tag{2.52}$$

If we set $p=2$ in Eq. (2.52), we find

$$\left.\frac{dv}{dp}\right|_{p=2}=\frac{3}{2^6}\ . \tag{2.53}$$

Differentiating Eq. (2.52) with respect to p, then setting $p=2$, and using Eq. (2.53) gives

$$\left.\frac{d^2v}{dp^2}\right|_{p=2}=-\frac{21}{2^9}\ , \tag{2.54}$$

which yields a_2 as expected. We would like to remark that for more general problems (such as the two-photon decay problem of Sec. II E), the differential equation for $u(r)$ which replaces Eq. (2.47) may be less tractable than in the present problems, while the Laplace transformed Eq. (2.52), as a *first-order* equation in dv/dp, is always solvable.

(c) *Direct numerical summation method.* Using the analytic expression for the dipole matrix elements (2.30) and (2.31), one has

$$\begin{aligned}a_2=&\frac{1}{2^8}\sum_{n=2}^{\infty}\frac{(2n-1)^7}{n^4(n-1)^6}\left[\frac{n-1}{n}\right]^{2n}\\&+2^9\int_0^\infty dk\frac{k\exp[-(4/k)\tan^{-1}(k/2)]}{(1+e^{-2\pi/k})(k^2+4)^5}\ .\end{aligned} \tag{2.55}$$

The sum (which needs sufficient terms to converge) and the integral (its integrand is regular enough that a 20-point Gaussian quadrature is adequate) are calculated on an IBM XT personal computer with little effort. The numerical values are

$$\begin{aligned}a_2&=4.241\,698\times10^{-2}+3.961\,374\times10^{-2}\\&=0.082\,031\ ,\end{aligned} \tag{2.56}$$

which is very close to the exact result $21/(2^8)$ with an error $\sim10^{-5}$.

From all the above methods, we see clearly that the continuous part contributes a fair share and must be taken into account in calculations involving the second-order matrix elements.

In summary, we have used three independent methods to evaluate a_2. There is one method that has not yet been discussed which requires no calculation at all. Using proper substitution, Tanaka *et al.*[29] recently obtained a_2 from the expression for $a_2^{3\text{D}}$ and treated it as a model of an impurity state localized in a thin layer. However, one must be cautious since there is no guarantee that such a substituting procedure will always work. Surprise and exception are sometimes possible.

Before proceeding to study higher-order Stark effects, we remark that second-order matrix elements involving summation over intermediate states appear in various physical problems: Stark effect, Rayleigh scattering, Compton scattering, two-photon transitions, second Born approximation, dynamic polarizability, long-range interaction of atoms, logarithmic mean excitation energies, etc. In the past 30 or so years, methods based on (a), (b), and (c) and their variants have been employed to study all these problems. We believe the calculations on a_2 described here make these methods more transparent.

(ii) *Evaluation of* a_4 *and* a_6. For $\mathcal{E}_4$, $\mathcal{E}_6, \ldots,$ direct evaluations on expressions involving multiple sums over intermediate states are no longer practical. Here we use an iteration method[30,31] to obtain the exact results for a_4 and a_6.

Standard perturbation expansion[32]

$$\begin{aligned} H\psi_{\mathcal{g}} &= E_{\mathcal{g}}\psi_{\mathcal{g}} \ , \\ H &= H_0 + H' \ , \\ E_{\mathcal{g}} &= \mathcal{E}_0 + \mathcal{E}_2 + \mathcal{E}_4 + \mathcal{E}_6 + \cdots \\ &= -a_0E_0^0 - a_2E_0^2 - a_4E_0^4 - a_6E_0^6 - \cdots \ , \\ \psi_{\mathcal{g}} &= \psi_0 + \psi_1 + \psi_2 + \psi_3 + \cdots \ , \end{aligned} \tag{2.57}$$

gives the following equations:

$$\begin{aligned} &(H_0 - \mathcal{E}_0)\psi_0 = 0 \ , \\ &(H_0 - \mathcal{E}_0)\psi_1 = -H'\psi_0 \ , \\ &(H_0 - \mathcal{E}_0)\psi_2 = -H'\psi_1 + \mathcal{E}_2\psi_0 \ , \\ &(H_0 - \mathcal{E}_0)\psi_3 = -H'\psi_2 + \mathcal{E}_2\psi_1 \ , \\ &\cdots \ . \end{aligned} \tag{2.58}$$

Solutions of ψ_i are obtained by solving equations in (2.58) successively; thus

$$\begin{aligned} \psi_0 &= 4e^{-2r}/(2\pi)^{1/2} \ , \\ \psi_1 &= -E_0(\tfrac{3}{4}r + r^2)e^{-2r}\cos\varphi/(2\pi)^{1/2} \ , \\ \psi_2 &= E_0^2[f_0(r) + f_2(r)\cos 2\varphi]e^{-2r}/(2\pi)^{1/2} \ , \\ \psi_3 &= E_0^3[f_1(r)\cos\varphi + f_3(r)\cos 3\varphi]e^{-2r}/(2\pi)^{1/2} \ , \end{aligned} \tag{2.59}$$

where

$$\begin{aligned} f_0(r) &= \tfrac{1}{16}(-\tfrac{149}{64} + \tfrac{21}{8}r^2 + \tfrac{7}{3}r^3 + r^4) \ , \\ f_1(r) &= -\tfrac{1}{128}(\tfrac{381}{128}r + \tfrac{127}{32}r^2 + \tfrac{339}{32}r^3 + \tfrac{79}{8}r^4 + \tfrac{53}{12}r^5 + r^6) \ , \\ f_2(r) &= \tfrac{1}{8}(\tfrac{5}{8}r^2 + r^3 + \tfrac{1}{2}r^4) \ , \\ f_3(r) &= -\tfrac{1}{128}(\tfrac{35}{32}r^3 + \tfrac{15}{8}r^4 + \tfrac{5}{4}r^5 + \tfrac{1}{3}r^6) \ . \end{aligned}$$

From Eq. (2.58) and expression (2.59), we find

$$\begin{aligned} \mathcal{E}_4 &= \langle \psi_0 | H' | \psi_3 \rangle = -(22\,947/2^{20})E_0^4 \ , \\ a_4 &= 22\,947/2^{20} \ . \end{aligned} \tag{2.60}$$

Furthermore, by means of the second and the fourth equations in (2.58), we have an alternative expression for $\mathcal{E}_4$:

$$\mathcal{E}_4 = \langle \psi_1 | H' | \psi_2 \rangle - \mathcal{E}_2 \langle \psi_1 | \psi_1 \rangle \ , \tag{2.61}$$

which again gives the same result [(2.60)]. This in turn gives us confidence in ψ_2 and ψ_3, which are needed to evaluate $\mathcal{E}_6$,

$$\begin{aligned} \mathcal{E}_6 &= \langle \psi_0 | H' | \psi_5 \rangle \\ &= \langle \psi_2 | H' | \psi_3 \rangle - \mathcal{E}_2(\langle \psi_1 | \psi_3 \rangle + \langle \psi_2 | \psi_2 \rangle) - \mathcal{E}_4 \langle \psi_1 | \psi_1 \rangle \ , \\ &= -\frac{48\,653\,931}{2^{31}}E_0^6 \ , \end{aligned} \tag{2.62}$$

or

$$a_6 = \frac{48\,653\,931}{2^{31}} \ . \tag{2.63}$$

Therefore the ground-state energy of a 2D hydrogen atom correct to sixth order in E_0 is

$$E_{\mathcal{g}}^{2\mathrm{D}} = -2 - \frac{21}{2^8}E_0^2 - \frac{22\,947}{2^{20}}E_0^4 - \frac{48\,653\,931}{2^{31}}E_0^6 - \cdots \ . \tag{2.64}$$

For a 3D hydrogen atom, the corresponding result is[30,31,33–38]

$$E_{\mathcal{g}}^{3\mathrm{D}} = -\frac{1}{2} - \frac{9}{2^2}E_0^2 - \frac{3555}{2^6}E_0^4 - \frac{2\,512\,779}{2^9}E_0^6 - \cdots \ . \tag{2.65}$$

The evaluation of $a_2^{3\mathrm{D}} = \frac{9}{4}$ was the first quantum-mechanical problem[33] to be treated by perturbation theory in 1926; the results for $a_4^{3\mathrm{D}}$ and $a_6^{3\mathrm{D}}$ were obtained rather recently in 1965 and 1976, respectively; and for a $a_n^{3\mathrm{D}}$ $(n \geq 8)$, only numerical results has been reported.[36–38] Both 2D and 3D a_n are depicted in Fig. 2.

Several comments are now in order. We note that both series (2.64) and (2.65) are expected to be asymptotic,[39] i.e., both are divergent even as $E_0 \to 0$. The physics underlying the breakdown of the perturbation expansion is the quantum-mechanical tunneling effect,[40] by which an electron starting from a stable ground state has some probability of reaching a position far away from the origin. The divergence of the 3D series is quite obvious as seen in Fig. 2. Since the ground-state binding energy of a 2D hydrogen atom is four times larger, it has a smaller response to an external field than does a 3D atom. As a result, the first few $a_n^{2\mathrm{D}}$ do not grow as fast. Nevertheless, the 2D series eventually also diverges. Therefore it is natural to ask whether the series expansion representing enormous efforts from many physicists over the period of more than half a century makes any sense. In a series of papers,[41] Simon and his associates proposed new techniques other than the perturbation approach. They showed that if one begins with a complex electric field **E**, sums the series via Borel summations, and then lets Im(**E**) tend to zero, the complex energy for the ground

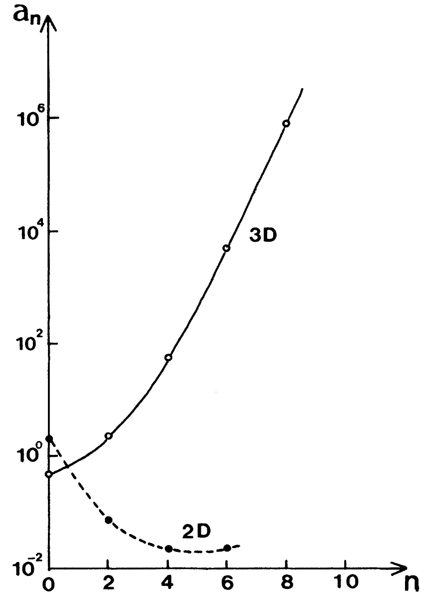


FIG. 2. Ground-state energy of a hydrogen atom in external electric field intensity E_0, $E_g = -a_0 E_0^0 - a_2 E_0^2 - a_4 E_0^4 - a_6 E_0^6 - \cdots$. ——, 3D hydrogen atom; — — —, 2D hydrogen atom.

state can be found exactly. Furthermore, if the field is not too large, then the time taken for the tunneling is so long that the effect can be neglected in any conceivable experiment. For a given electric field intensity E_0, there is an optimum number of terms in the series (2.65) which gives the best approximation to the actual energy shift. Thus perturbation expansion still provides useful information and the expressions for a_n^{2D} $(n \geq 8)$ are needed. However, analytic solution to a_n^{2D} with $n \geq 8$ is very tedious and a numerical approach involving recurrence formulas may be more practical. We are presently carrying out such an investigation.

2. *Stark effect on the n = 2 level*

The $n = 2$ level is threefold degenerate: $\{ |\psi_{2,-1}\rangle, |\psi_{2,0}\rangle, |\psi_{2,1}\rangle \}$. The matrix elements between these degenerate states are

$$a \equiv \langle \psi_{2,-1} | H' | \psi_{20} \rangle = \langle \psi_{21} | H' | \psi_{20} \rangle = -\frac{9E_0}{4(2)^{1/2}} \ . \tag{2.66}$$

The energy shift λ may be found immediately via perturbation theory for a degenerate state,

$$\begin{vmatrix} -\lambda & a & 0 \\ a & -\lambda & a \\ 0 & a & -\lambda \end{vmatrix} = 0 \ . \tag{2.67}$$

From Eq. (2.67) we can deduce the energy shift to first order in E_0 and the eigenstates to zeroth order:

Eigenstates	Energy correction
$(\lvert\psi_{2,1}\rangle - \lvert\psi_{21}\rangle)/2^{1/2}$	0
$(\lvert\psi_{2,-1}\rangle + 2^{1/2}\lvert\psi_{20}\rangle + \lvert\psi_{21}\rangle)/2$	$-\frac{9}{4}E_0$
$(-\lvert\psi_{2,-1}\rangle + 2^{1/2}\lvert\psi_{20}\rangle - \lvert\psi_{21}\rangle)/2$	$\frac{9}{4}E_0$

(2.68)

The degeneracy of the $n = 2$ level is thus *completely* removed, whereas the degeneracy of the $n = 2$ level of a 3D atom is only partially removed.[42]

E. Optical transitions

Due to the confinement of the electron motion in a plane, only planar polarizations are possible. Thus in the computation of the optical transition rate, one must take this into consideration.

1. *Single-photon transition*

In the dipole approximation, the transition rate $A_{k \to k'}$ (and the spontaneous lifetime t_{spon}) is similar to the 3D case given by Ref. 43 with the proper modification of the prefactor from $\frac{1}{3}$ to $\frac{1}{2}$:

$$A_{k \to k'} = \frac{1}{t_{\text{spon}}} = \frac{4\omega_{kk'}^3}{2c^3} |\langle k' | \mathbf{r} | k \rangle|^2 \ , \tag{2.69}$$

with the usual dipole selection rule,

$$\Delta l = \pm 1 \ . \tag{2.70}$$

For transition to the ground state, we have, via the dipole matrix elements (2.30),

$$A_{np \to 1s}^{2D} = \frac{1}{t_{np \to 1s}^{2D}} = \frac{32Z^4}{c^3} \frac{(n-1)^{2n-2}}{n^{2n}(2n-1)} \ . \tag{2.71}$$

In particular, $t_{2p \to 1s}^{2D} \cong 0.93 \times 10^{-10}$ sec, which indicates that the 2D transition rate is faster by a factor of 6 than that of a 3D atom, $t_{2p \to 1s}^{3D} \cong 1.6 \times 10^{-9}$ sec.

2. *Two-photon decay*

The possibility of a two-photon process, which proceeds via intermediate states, was first pointed out by Geoppert-Mayer[44] dating as far back as the early days of quantum mechanics. In 1940, Breit and Teller[13] applied this theory to the case of $2s \to 1s$ transition, which is the most probable radiative decay mode. Later, more detailed calculations[45–48] predicted the total transition rate $A_{2s \to 1s}^{3D}$ for a 3D hydrogenlike atom, to be $(8.226-8.228)Z^6 \text{ sec}^{-1}$.

Let us consider the $2s \to 1s$ two-photon transition of a 2D hydrogen atom. The formula for the transition probability per second of spontaneous two-photon emission with one photon in the frequency range $d\nu'$ at ν' (with unit polarization vector $\hat{\epsilon}'$ and photon momentum $\mathbf{k}'$) in the dipole approximation is[44,45,49]

$$A_{2s\to 1s}(\nu')d\nu'=\frac{2^{10}\pi^6\nu'^3\nu''^3}{c^6}\times\left|\mathop{\mathrm{S}}_{n\,(>2)}\left[\frac{\langle\psi_{10}|\hat{\epsilon}'\cdot\mathbf{r}|\psi_{n1}\rangle\langle\psi_{n1}|\hat{\epsilon}''\cdot\mathbf{r}|\psi_{20}\rangle}{E_n-(E_2-E_{\nu'})}+\frac{\langle\psi_{10}|\hat{\epsilon}''\cdot\mathbf{r}|\psi_{n1}\rangle\langle\psi_{n1}|\hat{\epsilon}'\cdot\mathbf{r}|\psi_{20}\rangle}{E_n-(E_2-E_{\nu''})}\right]\right|^2_{\mathrm{av}}d\nu'\ , \tag{2.72}$$

where ν'' is the frequency (with $\hat{\epsilon}'',\mathbf{k}''$) of the second photon which is, in turn related to ν' by

$$\nu''+\nu'=\frac{E_2-E_1}{2\pi}=\frac{8}{9\pi}\ . \tag{2.73}$$

In expression (2.72), $E_{\nu'}$ ($E_{\nu''}$) is the photon energy and "av" is the average of relative angle of $\hat{\epsilon}'$ and $\hat{\epsilon}''$ over all orientations if the polarization is not detected. The total transition rate is

$$A^T_{2s\to 1s}=\int_0^{(E_2-E_1)/4\pi}A_{2s\to 1s}(\nu')d\nu'\ . \tag{2.74}$$

Let

$$\nu'=\frac{8}{9\pi}x\ ,\qquad \nu''=\frac{8}{9\pi}(1-x)\ , \tag{2.75}$$

where x is a scaled variable with $0\le x\le 1$. After carrying out the angular and "av" parts, we find

$$A^T_{2s\to 1s}=\int_0^{0.5}A_{2s\to 1s}(x)dx \tag{2.76}$$

where $A_{2s\to 1s}(x)$ is the spectral distribution function,

$$A_{2s\to 1s}(x)=\frac{2^{28}\alpha^6}{3^{14}\pi}x^3(1-x)^3[P^{20}_{10}(E_{\nu'},1)+P^{20}_{10}(E_{\nu''},1]^2\ . \tag{2.77}$$

In expression (2.77), $\alpha=e^2/\hbar c$ is the fine-structure constant, and $P^{20}_{10}(E_\nu,1)$ is

$$P^{20}_{10}(E_\nu,1)=\mathop{\mathrm{S}}_{n\,(\ge 2)}\frac{\langle R_{10}|r|R_{n1}\rangle\langle R_{n1}|r|R_{20}\rangle}{E_n-(E_2-E_\nu)}\ . \tag{2.78}$$

Expression (2.78) is very similar to the second-order Stark effect (2.42). Thus the methods of Coulomb Green function, the implicit technique, and the direct summation can all be employed to evaluate expression (2.78) without much further effort. We have used all three of these methods to evaluate $P^{20}_{10}(E_\nu,1)$ and hence $A_{2s\to 1s}(x)$. For comparison we present both 2D and 3D $A_{2s\to 1s}(x)$ in Fig. 3. We have also calculated $A^T_{2s\to 1s}$ via numerical integration (2.76) with the following results:

	Coulomb Green function	Implicit technique	Direct summation
$A^T_{2s\to 1s}$	1149.84 sec^{-1}	1149.87 sec^{-1}	1149.78 sec^{-1}

Although the agreement is excellent, we believe that the Coulomb Green-function method is the most accurate one. Note that $A^{T,2D}_{2s\to 1s}$ is about two orders of magnitude faster than those of a 3D atom. Because $A^{T,2D}_{2s\to 1s}$ is also proportional to Z^6, we find that 2D hydrogen transition rate for $Z=1$ is the same order as the 3D helium ion, $Z=2$ transition rate which is experimentally detectable.

F. The fine structure of the $n=2$ level

In Sec. II of paper II, we shall give a full account of the exact solution of the Dirac equation for a 2D hydrogen atom. But for the purpose of discussing the fine-structure problem here, it is sufficient to isolate the α^4 term from the expansion formula (2.32) of II for the exact energy eigenvalue in powers of α, we get the fine-structure term

$$(E_{n,l,s_z})_f=-\frac{1}{2(n-\frac{1}{2})^4}\left[\frac{n-\frac{1}{2}}{|l+s_z|}-\frac{3}{4}\right]m_ec^2\alpha^4\ . \tag{2.79}$$

If we let $k=l+s_z$, the quantum number of the total angular momentum along the z axis, then $(E_{n,l,s_z})_f$ becomes

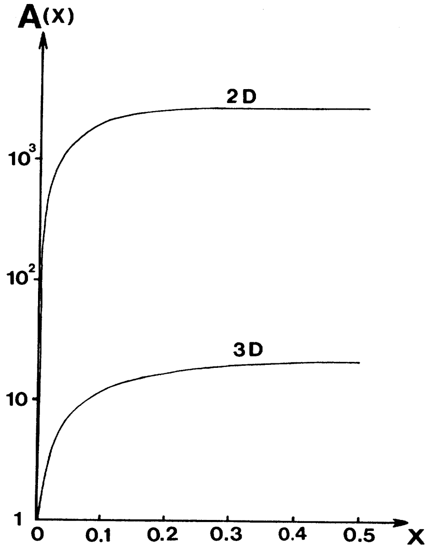


FIG. 3. 2D and 3D two-photon spectral distribution function $A(x)$.

$(E_{n,k})_f$. Under the effect of the fine-structure correction, the $n=2$ level splits into three fine levels written as $2s_{1/2}$, $2p_{1/2}$, and $2p_{3/2}$. The shifts are the same for the $2s_{1/2}$ and $2p_{1/2}$ with $n=2$, $k=\frac{1}{2}$. For the $2p_{1/2}$ state, the shift via formula (2.79) is

$$(E_{2p_{1/2}})=-\tfrac{2}{9}m_ec^2\alpha^4 \ . \tag{2.80}$$

Now let us consider the fine-structure Hamiltonian[50]

$$W_f=W_{mv}+W_{\text{s.o.}}+W_D \ . \tag{2.81}$$

The first term, W_{mv}, is due to the relativistic variation of the mass with velocity,

$$W_{mv}=-\frac{p^4}{8m_e^3c^2}=-\frac{\hbar^4}{8m_e^3c^2}(\nabla^2)^2 \ , \tag{2.82}$$

which contributes for all states. The second term, $W_{\text{s.o.}}$, is the usual spin-orbit coupling

$$W_{\text{s.o.}}=\frac{1}{2m_e^2c^2}\frac{1}{r}\frac{dV}{dr}\hat{\mathbf{l}}_z\cdot\hat{\mathbf{s}}_z=\frac{e^2}{4m_e^2c^2}\frac{1}{r^3}\left[\hat{\mathbf{j}}_z^2-\hat{\mathbf{l}}_z^2-\frac{\hbar^2}{4}\right] , \tag{2.83}$$

which is zero for s electrons. The last term in expression (2.81) W_D, is the Darwin term due to the nonlocalized interaction between the electron and the field,

$$W_D=\frac{\hbar^2}{8m_e^2c^2}\nabla^2V=-\frac{e^2\hbar^2}{8m_e^2c^2}\frac{1}{r^3} \ . \tag{2.84}$$

For the 3D case, since $\nabla^2V\propto\delta(\mathbf{r})$, the Darwin term is nonzero only for the s electron. However, this is no longer true for the 2D hydrogen atom, the Darwin term *affects all* the electrons.

As an illustration, we consider the $2p_{1/2}$ state again. With the nonrelativistic wave function (2.17), simple perturbation calculation yields

$$\langle\psi_{21}|W_{mv}|\psi_{21}\rangle=-\frac{1}{8m_e^3c^2}\langle p^2\psi_{21}|p^2\psi_{21}\rangle$$

$$=-\tfrac{1}{8}\tfrac{16}{27}m_ec^2\alpha^4 \ , \tag{2.85}$$

$$\langle\psi_{21}|W_{\text{s.o.}}|\psi_{21}\rangle=-\tfrac{8}{81}m_ec^2\alpha^4 \ , \tag{2.86}$$

and

$$\langle\psi_{21}|W_D|\psi_{21}\rangle=-\tfrac{1}{8}\tfrac{32}{81}m_ec^2\alpha^4 \ . \tag{2.87}$$

Grouping these results together, we find

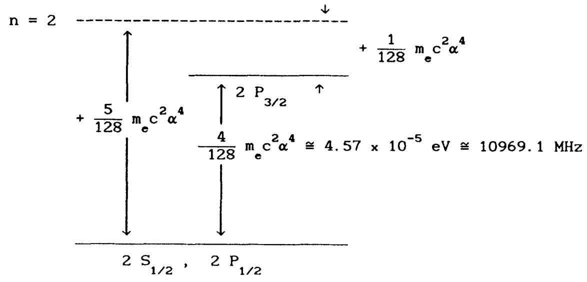
$$\langle\psi_{21}|(W_{mv}+W_{\text{s.o.}}+W_D)|\psi_{21}\rangle=-\tfrac{2}{9}m_ec^2\alpha^4 \ , \tag{2.88}$$

which is exactly the same as predicted from the Dirac solution (2.80). Note that in arriving at result (2.85), we employ[51] $p^2=2m_e(H_0-V)$ in $p^2\psi_{21}$ to facilitate computation.

According to formula (2.79), the fine structure of the $n=2$ level is

$$(E_{2p_{3/2}}-E_{2p_{1/2}})_f=(-\tfrac{2}{81}+\tfrac{2}{9})m_ec^2\alpha^4=\tfrac{16}{81}m_ec^2\alpha^4 \ . \tag{2.89}$$

(a) 3 D hydrogen atom



(b) 2 D hydrogen atom

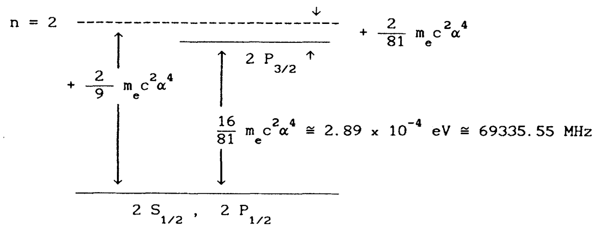


FIG. 4. Fine structure of the $n=2$ level of the hydrogen atom. (a) 3D hydrogen atom, (b) 2D hydrogen atom. Note the scale in (a) and (b) is not the same. In fact, the 2D splitting is larger by a factor of 6 than that of a 3D atom.

For comparison, the fine structure of the $n=2$ level of the 2D and 3D hydrogen atoms is presented in Fig. 4.

G. The hyperfine structure of the $n=1$ level

The fine-structure correction merely shifts the 1s level as a whole by a quantity equal to

$$(E_{1s})_f=-2m_ec^2\alpha^4 \ , \tag{2.90}$$

where we utilize formula (2.79) with $n=1$, $l=0$, and $s_z=\pm\frac{1}{2}$.

For the $n=1$ level, only the Fermi contact term[52] contributes to the hyperfine splitting,

$$W_{\text{hf}}=\frac{16\pi}{3}\mu_B\mu 2\hat{\mathbf{i}}_z\cdot\hat{\mathbf{s}}_z\psi_{10}^2(0) \ . \tag{2.91}$$

In expression (2.91) μ_B is the Bohr magneton, μ is the proton magnetic moment, and $\hat{\mathbf{i}}_z$ is the proton spin along the z axis. With $\psi_{10}^2(0)=16/2\pi$ and

$$2\hat{\mathbf{i}}_z\cdot\hat{\mathbf{s}}_z=\hat{\mathbf{F}}_z^2-\hat{\mathbf{i}}_z^2-\hat{\mathbf{s}}_z^2 \ , \tag{2.92}$$

where $\hat{\mathbf{F}}_z$ is the total spin along the z axis, perturbation

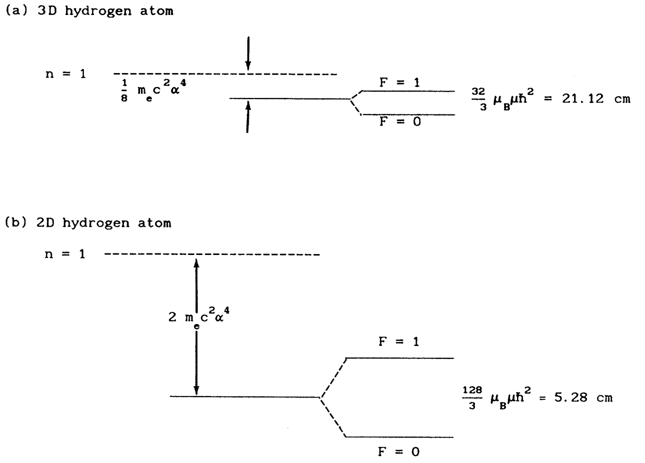


FIG. 5. Hyperfine structure of the $n=1$ level of the hydrogen atom. (a) 3D hydrogen atom, (b) 2D hydrogen atom.

bation theory gives

$$(E_{1s,F=1})_{\rm hf}=\tfrac{64}{3}\mu_B\mu\hbar^2 \;, \tag{2.93}$$

and

$$(E_{1s,F=0})_{\rm hf}=-\tfrac{64}{3}\mu_B\mu\hbar^2 \;. \tag{2.94}$$

The hyperfine splitting therefore is

$$(E_{1s,F=1})_{\rm h}-(E_{1s,F=0})_{\rm hf}=\tfrac{128}{3}\mu_B\mu\hbar^2 \;, \tag{2.95}$$

which is four times the corresponding splitting of a 3D atom. Instead of the famous 21.12-cm line in a 3D H-atom, we will have a 5.28-cm line for a 2D hydrogen atom. The hyperfine structure of the $n=1$ level is illustrated in Fig. 5.

III. CONCLUDING REMARKS

In spite of the fundamental importance of the hydrogen problem in terms of basic theory and in various practical applications, there have been only a few attempts to apply the nonrelativistic Schrödinger equation to a 2D hydrogen atom. This is probably due to the fact that a single hydrogen atom must be treated as a 3D problem. In this paper we have presented in detail the analytical solutions to the Schrödinger equation for a 2D hydrogen atom. On the theoretical side, these analytical results give insight into the quantization conditions, the difference, as well as the similarity between the 2D and 3D cases. Among the theoretical results, the Darwin term in the fine-structure Hamiltonian is of particular interest. For the 3D case, the Darwin term contributes only for the s electrons. However, it affects all the electrons in the 2D case. The physical origin of this phenomenon is still not clear. In the dc Stark problem, we may not need many terms to give a reasonably accurate result because of the large binding energy of a 2D atom. A rigorous proof along the line of Simon's work[41] is still needed. The ac Stark effect is another interesting problem[53] which we have not considered in this paper.

Historically, the study of this 2D hydrogen problem was introduced as a leading approximation for the electron motion in a highly anisotropic crystal.[3] For example, let us consider an anisotropic mass tensor of the electron inside the crystal field, such that $m_x=m_y=m_e$ and $m_z=\alpha m_e$, where $\alpha \gg 1$. Then it follows that by scaling the z variable to $\alpha^{1/2}z$, we can transform the Coulomb potential $V(r)$ into a 2D Coulomb potential $V(\rho)$, where $\rho=(x^2+y^2)^{1/2}$ plus a term which is of order $(1/\alpha)V'(z,\rho)$. Thus in the limit $\alpha \rightarrow \infty$, this electron will be confined to planar motion. Therefore the solution of the 2D hydrogen is of interest to the study of anisotropic effect in solids.[54]

Note added in proof. Exact analysis results for 2D and 3D Stark shifts up to 20th order have recently been obtained by C. K. Au and K. L. Pom, using logarithmic theory (unpublished). We wish to thank Professor Kenneth Young for informing us of the new result.

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