Modified Ricatti approach to partially solvable quantum Hamiltonians: Finite Laurent-type potentials

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Partial solubility in quantum mechanics is investigated by studying the logarithmic derivative of the wave function. By explicitly isolating the singularities of the logarithmic derivative, a modified Ricatti equation for the regular component is obtained. For the finite Laurent series potentials considered, we derive the constraints on the coupling constants to obtain closed-form solutions for a subset of eigenstates. With an appropriate change of variables the method is generalized in order to cope with more general potentials. In this way, new families of partially solvable potentials related to the already known exactly solvable ones are identified.

I. INTRODUCTION

Exactly solvable potentials are of great interest in quantum mechanics, both for their intrinsic meaning and for testing the quality of perturbative, numerical, or semiclassical approximations. The usual concept of a solvable Hamiltonian entails the possibility of knowing exactly its whole set of eigenfunctions and the corresponding spectrum. The potentials that satisfy such a criterion are comparatively few, and are mostly those contained in the classification of Infeld and Hull[1,2] plus the ones related to them by the method of Abraham and Moses.[3] With the advent of supersymmetric theories[4] extensive use of their methods has been made to study solubility in quantum mechanics as a consequence of shape invariance.[5] However, it has been proven[6] that this approach is equivalent to the factorization method.[1,2]

In the last few years there has been a growing interest in a wider category of solvable potentials. Their characteristic is that they have an exactly solvable subset of eigenfunctions. Recently several papers have been published dealing with this problem.[7–20] These potentials fall into two categories: those which have only one exactly solvable state (from now on we will call them *partially solvable potentials),* and (much more interesting) those which have a family of exactly solvable states *(quasiexactly solvable potentials).* A well-known example of this last class is the sextic symmetric anharmonic polynomial oscillator.

In this work we consider an alternative approach to investigate partial solubility based on a modified Ricatti equation for the regular component of the wave-function logarithmic derivative, where the necessary conditions for exact solutions to exist arise naturally. The simplest version of the method is specially suited to cope with finite Laurent-series-type potentials. They include among others the anharmonic polynomial oscillators, intensively considered in the context of quantum field theory, the three-dimensional anharmonic polynomial oscillators, the Coulomb plus polynomial terms potentials (both of relevance in the problem of quark confinement in QCD) and the generalized Lennard-Jones-type potentials used to describe molecular interactions.

Furthermore, a generalization of the formalism, induced by simple mappings, allows us to analyze *partial* and *quasiexact* solubility in a very much larger class of potentials. These last can be thought of as members of families related to the well-known exactly solvable ones. A detailed analysis of this generalization and further developments are planned to be presented in a forthcoming article.[21]

This paper is organized as follows. In Sec. II we introduce the main features of the modified Ricatti approach for finite Laurent-series-type potentials. In Secs. III, IV, and V we analyze the anharmonic polynomial oscillators, the fractional polynomial potentials, and the general potentials with polynomial plus fractional terms, respectively. In Sec. VI we sketch the generalized formalism which suggests new families of partially solvable potentials related to the exactly solvable ones. Finally, Sec. VII is devoted to concluding remarks.

II. THE MODIFIED RICATTI APPROACH

In the present work we will restrict ourselves to potentials that admit bounded eigenfunctions, having in consequence a discrete energy spectrum. Let us consider the logarithmic derivative of the mth excited-state wave function,

$$g_m(x)=\frac{d}{dx}\ln\psi_m(x)\ , \tag{2.1}$$

which has proven to be very useful in classifying most of the exactly solvable Hamiltonians.[2] In terms of it we can write

$$\psi_m(x)\propto\exp\left[\int^x dy\, g_m(y)\right]\ . \tag{2.2}$$

When inserted into the Schrödinger equation

$$\left[\frac{d^2}{dx^2}+E_m-V(x)\right]\psi_m(x)=0\ , \tag{2.3}$$

it gives place to the Ricatti equation for $g_m(x)$:

$$g_m^2(x)+g_m'(x)=V(x)-E_m\ . \tag{2.4}$$

As follows from Eq. (2.2), a wave function will have a closed expression if and only if $g_m(x)$ has a primitive expressible in closed form. If, in addition, the resulting $g_m(x)$ function is such that $g_m^2+g_m'$ and $g_m^2-g_m'$ have the same functional dependence on x, after come reparametrization,[5,6] the problem will be strictly solvable in the sense of Infeld and Hull.[2]

The solubility problem has been traditionally addressed by making an ansatz for the wave function of the form

$$\begin{aligned}\psi_m(x)&=B_m(x)\bar{\psi}(x)\\&=B_m(x)\exp\left[\int^x dy\,\bar{g}(y)\right]\ .\end{aligned} \tag{2.5}$$

where $\bar{g}(x)=(\ln\bar{\psi})'$ and $\bar{\psi}(x)$ gives the leading behavior of the wave function at the boundaries. As the wave function is normalizable we can loosely assume that it will vanish at the boundary of the physical domain [the real interval (x_-,x_+)], i.e., $\lim_{x\to x_\mp}\bar{\psi}(x)=0$ and so

$$\lim_{x\to x_\mp}\int^x dy\,\bar{g}(y)=-\infty\ . \tag{2.6}$$

In the case of solubility $\bar{\psi}(x)$ can be written in closed form so that $B_m(x)$ becomes a finite power series. The key in this process is certainly the selection of $\bar{\psi}(x)$ [or equivalently $\bar{g}(x)$].

For all the very well-known exactly solvable Hamiltonians[2] $\bar{\psi}(x)$ can be constructed by solving the Schrödinger equation (2.3) in the limit $x\to x_\pm$. Conversely, the same can be said about $\bar{g}(x)$ which can be obtained by solving the Ricatti equation (2.4) in the limit $x\to x_\pm$.

For an arbitrary potential we lack a general procedure to identify the adequate decomposition as given in Eq. (2.5). However, for a particular, still large, class of potentials which generalize the exactly solvable ones, we propose a systematic method to identify $\bar{g}(x)$ through the modified Ricatti approach [Eq. (2.9) below].

As a first approach we will only require that the potentials will be regular *almost everywhere* in the complex plane; by these words we mean that if $V(x)$ has singularities they are located at the boundaries $x=x_\pm$ or at infinity ($|x|\to\infty$). We will come back to this point in Sec. VI. Removing this restriction will lead to other cases left out in this work.[21,22]

From the Sturm-Liouville theory,[23] the normalizable wave function $\psi_m(x)$ representing the mth excited state has m simple zeros $\{x_j\}_{j=1,\ldots,m}$ on the real axis in $x_-<x<x_+$. Moreover, $V(x)$ being regular *almost everywhere* in the complex plane, all the zeros of the wave function must be simple *almost everywhere*,[24] and we can explicitly isolate them:

$$\begin{aligned}\psi_m(x;n)&\equiv\bar{\psi}(x)\prod_{j=1}^{n}(x-x_j)\\&\equiv\bar{\psi}(x)B_m(x;n),\quad m\le n\ .\end{aligned} \tag{2.7}$$

We have introduced a notation that discriminates between the total number of zeros, n, and the $m\le n$ nodes of the wave function [the zeros in the interval (x_-,x_+)]. The so defined factor $\bar{\psi}(x)$ is regular almost everywhere.

In terms of the logarithmic derivative Eq. (2.7) becomes

$$g_m(x,n)=\bar{g}(x)+\sum_{j=1}^{n}\frac{1}{x-x_j}\ , \tag{2.8}$$

where $\bar{g}(x)=d(\ln\bar{\psi})/dx$ is regular almost everywhere in the complex plane.

Inserting Eq. (2.8) into Eq. (2.4) and by decomposition in partial fractions of the resulting expressions, we obtain a modified Ricatti equation for $\bar{g}(x)$

$$\bar{g}^2(x)+\bar{g}'(x)+2F_m(x;n)=V(x)-E_m\ , \tag{2.9}$$

where

$$F_m(x;n)=\sum_{j=1}^{n}\frac{\bar{g}(x)-\gamma_{n,j}}{(x-x_j)} \tag{2.10}$$

and $\gamma_{n,j}\equiv\sum_{k=1,k\neq j}^{n}(x_k-x_j)^{-1}$ for $j=1,2,\ldots,n$, together with $\gamma_{0,0}\equiv\gamma_{1,0}\equiv0$ and $F_0(x;0)\equiv0$. A useful property is $\sum_{j=1}^{n}\gamma_{n,j}=0$.

The potential $V(x)$ being regular almost everywhere the left-hand side of Eq. (2.9) cannot have singularities at $x=x_j$, so $F_m(x;n)$ cannot have them either. Therefore $\bar{g}(x)-\gamma_{n,j}$ *must* be proportional to $(x-x_j)$, or

$$\bar{g}(x_j)=\gamma_{n,j}(x_1,x_2,\ldots,x_n),\quad j=1,2,\ldots,n \tag{2.11}$$

strongly relating the zeros of the wave function to the regular component of its logarithmic derivative. Now we can rewrite $F_m(x;n)=\sum_{j=1}^{n}(x-x_j)^{-1}[\bar{g}(x)-\bar{g}(x_j)]$. We should remark at this point that, for a given n, the nonlinear system of Eqs. (2.11) can have more than one solution set $\{x_j,\ j=1,2,\ldots,n\}$ and correspondingly, more than one wave function with n zeros can be obtained.

Let us consider that there is more than one solution set to Eqs. (2.11). In principle, each set $\{x_j,\ j=1,2,\ldots,n\}$, will lead to a different $F_m(x;n)$ and thus the right-hand side of Eq. (2.9) must accordingly change. For a fixed value of n there are two possibilities.

(a) If the $F_m(x;n)$ corresponding to distinct solution sets $\{x_j\}$ differ from each other in their x dependence, the resulting wave functions correspond to different potentials (partially solvable potentials).

(b) Another much more interesting possibility is that the $F_m(x;n)$ corresponding to distinct solution set $\{x_j\}$ differ from each other by an additive constant. In this case, the different wave functions correspond to eigenstates of the same Hamiltonian, with different energies. The solution sets differ from each other in the number m of nodes (zeros in the physical domain). We have a quasiexactly solvable potential.

The set of Eqs. (2.11) is crucial for our formal analysis.

Once $\bar{g}(x)$ is known for a given n, a family of wave functions with n zeros, in principle, can be completely specified. This is because the remaining polynomial factor $B_m(x;n)=\prod_{j=1}^{n}(x-x_j)\equiv\sum_{j=0}^{n}b_j x^j$ can be determined at least in two ways.

(a) We can solve the nonlinear system (2.11) for the zeros of the wave function and from them construct $B_m(x;n)$ [see Eq. (2.7)]. However, except for low values of n this can be very difficult to implement due to the structure of Eqs. (2.11).

(b) An alternative to Eqs. (2.11) is to solve the set of Eqs. (B5) (in general also nonlinear) for the coefficients $\{b_j\}_{j=0,\ldots,n-1}$ of the polynomial $B_m(x;n)$ (see the derivation in Appendix B).

This being pointed out, let us study the conditions under which it can be easy to determine $\bar{g}(x)$ through Eq. (2.9).

A. The general Laurent-type potential

The modified Ricatti Equation (2.9) is clearly well suited to treat finite Laurent-type potentials

$$V(x)=\sum_{k=-2L}^{2M} v_k x^k, \quad M,L\geq 0 \tag{2.12}$$

as it allows an almost algebraic determination of $\bar{g}(x)$. The simplest example is $V(x)=\omega^2x^2/4$, for which just by inspection of Eq. (2.9) we derive $\bar{g}(x)=-\omega x/2$.

For $L=0$, $M>0$ the potential becomes a polynomial and the physical domain spans the whole real axis, i.e., $(x_-,x_+)=(-\infty,+\infty)$. For $L>0$ we will consider $(x_-,x_+)=(0,+\infty)$ either because $v_{-2L}>0$ or alternatively because we are thinking of Eq. (2.12) corresponding to a spherically symmetric three-dimensional (3D) potential.

We are looking for $\bar{g}(x)$ which is regular almost everywhere (if it has singularities they lie at infinity in the complex plane) with its primitive expressible in closed form. An obvious solution of Eq. (2.9) satisfying such requirements is the finite Laurent series

$$\bar{g}(x)=\sum_{k=-L}^{M} a_k x^k , \tag{2.13}$$

and thus

$$\bar{\psi}(x)\propto x^{a-1}\exp\left[\sum_{\substack{k=-L\\k\neq -1}}\frac{a_k}{k+1}x^{k+1}\right] . \tag{2.14}$$

B. The symmetric case

Reflection symmetric potentials, i.e., $V(x)=V(-x)$, of Laurent type deserve special attention. Introducing the mapping $u\equiv x^2$, a more economic formulation of the problem can be presented:

$$W(u)=\sum_{k=-L}^{M} w_k u^k . \tag{2.15}$$

We will restrict the analysis to L and M both odd positive integers. In analogy to Eq. (2.7) the eigenfunctions can be written

$$\psi_m(x;n)=\Psi_\mu(u;\nu)\equiv C_\mu(u;\nu)\bar{\Psi}(u) , \tag{2.16}$$

where $C_\mu(u;\nu)=\prod_{j=1}^{\nu}(u-u_j)=\sum_{j=0}^{\nu}c_j u^j$ is a polynomial of degree $\nu=[n/2]$ ([ ]) means "integer value of") and $\mu=[m/2]$ is the number of zeros of $C_\mu(u;\nu)$ in the real positive axis. When $L=0$, the well-defined parity $(-1)^m$ is accounted for by the remaining factor $\bar{\Psi}$. The logarithmic derivative of the wave function is according to (2.1)

$$g_m(x;n)=\frac{du}{dx}\frac{d}{du}\ln\Psi_\mu(u;\nu)\equiv\frac{du}{dx}G_\mu(u;\nu) . \tag{2.17}$$

Thus, in the symmetric case, the Ricatti equation (2.4) modifies to

$$4u[G_\mu^2(u;\nu)+\dot{G}_\mu(u;\nu)]+2G_\mu(u;\nu)=W(u)-E_m , \tag{2.18}$$

where the dot denotes the derivative with respect to u. Decomposing as before $G_\mu(u;\nu)$ in its regular and singular components,

$$G_\mu(u;\nu)=\bar{G}(u)+\sum_{j=1}^{\nu}\frac{1}{u-u_j} , \tag{2.19}$$

where $\bar{G}(u)=d(\ln\bar{\Psi})/du$, and replacing this expression in Eq. (2.18), a modified Ricatti equation for the symmetric case results:

$$4u[\bar{G}^2(u)+\dot{\bar{G}}(u)]+2\bar{G}(u)+2\Phi_\mu(u;\nu)=W(u)-E_m , \tag{2.20}$$

where

$$\Phi_\mu(u;\nu)=\sum_{j=1}^{\nu}\frac{4u[\bar{G}(u)-\Gamma_{\nu,j}]+1}{u-u_j} \tag{2.21}$$

and $\Gamma_{\nu,j}\equiv\sum_{k=1,k\neq j}^{\nu}(u_k-u_j)^{-1}$ for $j=1,2,\ldots,n$, together with $\Gamma_{0,0}\equiv\Gamma_{1,0}\equiv 0$ and $\Phi_0(x;0)\equiv 0$.

If the right-hand side of Eq. (2.20) is regular almost everywhere, $\Phi_\mu(u;\nu)$ *must* be regular too and consequently

$$\bar{G}(u_j)+\frac{1}{4u_j}=\Gamma_{\nu,j}, \quad j=1,2,\ldots,\nu \tag{2.22}$$

which are certainly a similar but simpler alternative to Eqs. (2.11), ([$n/2$] equations instead of n), relating the zeros of the wave function to $\bar{G}(u)$. As in the general nonsymmetric case the knowledge of $\bar{G}(u)$ completely specifies the wave function. Moreover, the recognition of partial, quasiexact, or exact solubility derives from the analysis of $\Phi_\mu(u;\nu)$ (constant or u dependent).

For the symmetric Laurent-type potentials (2.15) it is immediate to state that

$$\bar{G}(u)=\sum_{k=-(L-1)/2}^{(M-1)/2} A_k u^k . \tag{2.23}$$

The connection with the general case treated in Sec. II A is given by the identification $a_{2k}=0$, $a_{2k+1}=2A_k$. It is

convenient to rename the coefficients $A_{-1}=\delta/2$; for $L=0$, $\delta=n-2\nu=0,1$ is the parity of the wave function, while for $L>0$, $\delta=a_{-1}$.

Until now we have discussed the cases for which the modified Ricatti equation is suitable to determine the regular components $\bar{g}$ (in the general case) and $\bar{G}$ (in the symmetric case). Next we apply the formalism here presented to analyze partial solubility for the potential given in Eq. (2.12) in the cases ($L=0, M\neq 0$), ($L\neq 0, M=0$), and ($L\neq 0, M\neq 0$) in both its general and reflection symmetric forms.

III. POLYNOMIAL POTENTIAL ($L=0, M\neq 0$)

A. The general case

In order to ensure the existence of bound states we require $v_{2M}>0$. For even values of M, Eq. (2.14) violates the normalizability condition (2.6) and therefore we will further restrict it only to odd values and $a_M<0$. The function $F_m(x;n)$ defined in Eq. (2.10) is then a polynomial of degree $M-1$

$$F_m(x;n)=\sum_{k=0}^{M-1} x^k f_k^n , \tag{3.1}$$

where the coefficients f_k^n are derived in Appendix,

$$f_k^n=\sum_{q=k+1}^{M} a_q s_{q-k-1} , \tag{3.2}$$

with

$$s_q=\sum_{j=1}^{n} (x_j)^q . \tag{3.3}$$

Notice that $s_0=n$ and consequently

$$f_k^n=na_{k+1}+\sum_{q=k+2}^{M} a_q s_{q-k-1} . \tag{3.4}$$

From Eq. (2.9) we get for the potential

$$V(x)=E_m+\left[a_0^2+(2n+1)a_1+2\sum_{m=2}^{M} a_m s_{m-1}\right] + \left[\sum_{k=1}^{M-1} x^k\left[(2n+k+1)a_{k+1}+\sum_{q=0}^{k} a_q a_{k-q}+2\sum_{m=k+2}^{M} a_m s_{m-k-1}\right]\right]+\left[\sum_{k=M}^{2M} x^k\left[\sum_{q=0}^{k} a_q a_{k-q}\right]\right] . \tag{3.5}$$

This implies the following relations hold:

$$v_k=\begin{cases}(2n+k+1)a_{k+1}+\sum_{j=0}^{k} a_j a_{k-j}+2\sum_{m=k+2}^{M} a_m s_{m-k-1}, & k=1,2,\ldots,M-1 \quad (3.6a)\\ \sum_{j=k-M}^{M} a_j a_{k-j}, & k=M,M+1,\ldots,2M \quad (3.6b)\end{cases}$$

so the wave function and the energy are given by

$$\psi_m(x;n)=B_m(x;n)\exp\left[\sum_{k=0}^{M}\frac{a_k}{k+1}x^{k+1}\right] , \tag{3.7}$$

$$E_m=-\left[a_0^2+(2n+1)a_1+2\sum_{m=2}^{M} a_m s_{m-1}\right] .$$

By construction $\bar{g}(x)$ depends on $M+1$ parameters: the a_k's. Moreover, from Eq. (2.11)

$$x_j=x_j(a_0,a_1,\ldots,a_M), \quad j=1,2,\ldots,n \tag{3.8}$$

and so the wave function $\psi_m(x;n)$ depends only on $M+1$ independent parameters. On the other hand, the potential (2.12) is defined by $2M$ coefficients. It is clear that not all the polynomial potentials (i.e., with arbitrary coefficients $\{v_k\}$) will have a corresponding $\psi_m(x;n)$ expressible in the closed form given above. For this to be the case the coefficients $\{v_k\}_{k=1,2,\ldots,2M}$ must satisfy $M-1$ consistency (in general nonlinear) equations.

If $M=1$ there are no constraint relations and we are confronted with a truly *exactly solvable* potential, the displaced harmonic oscillator. In this case $F_m(x;n)=na_1$ does not depend on the position of the zeros of the wave function and consequently the equations (B5) for the coefficients $\{b_0,b_1,\ldots,b_{n-1}\}$ of the wave function become linear. All the wave functions of the form given above are solutions of the potential, and $m=n$ label the state. For this potential Eqs. (2.11) give us an interesting relation between the zeros of the Hermite polynomials, already found within a different perspective by Szegö. In Ref. 25 he demonstrates an analogous formula for Jacobi polynomials and sketches the proof for Laguerre and Hermite polynomials. Equations (2.11) actually generalize these results.

A general systematic procedure to obtain exact eigenfunctions $\psi_m(x;n)$ for the anharmonic oscillator (2.12) can be developed. From the $M+1$ Eqs. (3.6b) it is straightforward to derive $\{a_k\}_{k=0,1,\ldots,M}$ in terms of $\{v_k\}_{k=M,M+1,\ldots,2M}$ by means of the following backward recurrence relations:

$$
\begin{aligned}
&a_M=-\sqrt{v_{2M}}=a_M(v_{2M})\ ,\\
&a_{M-1}=v_{2M-1}/2a_M=a_{M-1}(v_{2M-1},v_{2M})\ ,\\
&a_k=\frac{1}{2a_{2M}}\left[v_{k+M}-\sum_{j=k+1}^{M}a_j a_{k+M-j}\right]\\
&\quad=a_k(v_{k+M},v_{k+M+1},\ldots,v_{2M})\\
&\qquad\qquad k=M-2,M-3,\ldots,0\ ,
\end{aligned}
\tag{3.9}
$$

We have rejected the solution $a_M>0$ as it would have resulted in a non-normalizable wave function. The scheme here proposed univocally determines all the coefficients $\{a_k\}$ in terms of the higher coupling constants $\{v_k\}_{k=M,M+1,\ldots,2M}$. Thus from Eq. (2.14) we obtain $\bar{\psi}(x)$.

The remaining polynomial factor $B_m(x;n)=\prod_{j=1}^{n}(x-x_j)\equiv\sum_{j=0}^{n}b_j x^j$ can be determined as discussed in Sec. II and the energy of the state can then be evaluated from Eq. (3.7).

This systematic sequence ends up with wave functions and energies completely specified only by the $M+1$ higher couplings of the potential $\{v_M,\ldots,v_{2M}\}$. Different solutions for $B_m(x;n)$ do not necessarily correspond to the same potential, as the remaining couplings are subject to the constraints given by Eq. (3.6a) that depend on each particular solution. The only exception is for $v_{M-1}=v_{M-1}(v_M,\ldots,v_{2M},n)$ which does not depend on the position of the zeros but on the total number of them.

A very interesting situation exceptionally arises when the constraints only depend on n: the different $\psi_m(x;n)$ will correspond to the same potential that we will call $V(x;n)$. For each solution set the number of zeros in the physical domain must be different (for 1D problems the bound states are nondegenerate due to time-reversal symmetry). Thus for a given arbitrary n we would have found a Hamiltonian $\hat{H}^{(n)}$ with an exactly solvable subset, with at most n elements, of its Hilbert space.

Let us illustrate the method for the simplest nontrivial case, i.e., for $M=3$. The general sextic polynomial oscillator reads[26]

$$
V(x)=v_1x+v_2x^2+v_3x^3+v_4x^4+v_5x^5+v_6x^6\ . \tag{3.10}
$$

This potential has been treated in the literature only in its *symmetric* version (with the particular choice $v_1=v_3=v_5=0$). Proceeding as described above we obtain

$$
\begin{aligned}
&a_3=-\sqrt{v_6}\ ,\\
&a_2=-v_5/2\sqrt{v_6}\ ,\\
&a_1=\left[\frac{v_5^2}{4v_6}-v_4\right]\Big/2\sqrt{v_6}\ ,\\
&a_0=-\left[\frac{v_5}{2v_6}\left[\frac{v_5^2}{4v_6}-v_4\right]+v_3\right]\Big/2\sqrt{v_6}\ ,
\end{aligned}
\tag{3.11}
$$

and the resulting constraints

$$
\begin{aligned}
&v_1=(2n+2)a_2+2a_0a_1+2a_3s_1(\{x_j\})\ ,\\
&v_2=(2n+3)a_3+2a_0a_2+a_1^2\ .
\end{aligned}
\tag{3.12}
$$

Let us consider the case $n=1$, i.e., we are going to look for eigenfunctions with only one zero in the complex plane. Having $s_1=x_1$ and the coefficient v_1 real, from Eq. (3.12) x_1 must be one real solution of Eq. (2.11), in this case the cubic equation:

$$
a_0+a_1x_1+a_2x_1^2+a_3x_1^3=0\ .
$$

Upon the particular values of v_3, v_4, v_5, and v_6 we can obtain (a) one real solution x_a and two complex conjugate roots or (b) three real solutions $\{x_{b1},x_{b2},x_{b3}\}$.

The wave functions and their energies can be immediately constructed from Eqs. (3.11), (2.14), and (3.7),

$$
\begin{aligned}
&\psi_1(x;1)\propto(x-x_\alpha)\bar{\psi}(x)\ ,\\
&E_1=\left[a_0^2+3a_1+2a_2x_\alpha+2a_3x_\alpha^2\right]\ ,
\end{aligned}
\tag{3.13}
$$

where x_α generically denotes solutions of type (a) or (b). Every solution found is an eigenfunction of a different Hamiltonian, as $v_1=v_1(x_\alpha)$ according to Eq. (3.12), and has a different eigenvalue from Eq. (3.7).

B. The symmetric case

For the symmetric polynomial potentials

$$
V(x)=W(u=x^2)=\sum_{k=0}^{M}w_k u^k \tag{3.14}
$$

the method of the preceding section is far from being the most synthetic one because it explicitly includes several null parameters ($\{v_{2k+1}\}$, $\{a_{2j}\}$, $\{s_{2q+1}\}$, etc.). The more appropriate discussion of the modified Ricatti approach for symmetric potentials in Sec. II applies. For instance, from Eq. (2.20) we immediately realize that

$$
\bar{G}(u)=\sum_{k=0}^{(M-1)/2}A_k u^k+\frac{\delta/2}{u}\ . \tag{3.15}
$$

In order to avoid unnecessary duplications, we notice that the previous reasoning remains unaltered by substituting $x^2\to u$, $V\to W$, $v_{2k}\to w_k$, $g\to 2xG$, $a_{2k+1}\to 2A_k$, $B_m\to C_\mu$, and so on.

An interesting nontrivial example is the sextic symmetric oscillator [$M=3$ in Eq. (3.14)]. By solving Eqs. (3.11) or the corresponding mapped equations we obtain

$$
\begin{aligned}
&A_1=-\sqrt{w_3}/2\ ,\\
&A_0=-w_2/4\sqrt{w_3}\ ,
\end{aligned}
\tag{3.16}
$$

Notice that in the general formalism $a_0=a_2=0$ would be derived from Eqs. (3.11). The only constraint [mapped from Eqs. (3.12)] becomes

$$
\begin{aligned}
w_1&=2(2n+3)A_1+4A_0^2=w_2^2/4w_3-(2n+3)\sqrt{w_3}\\
&=w_1(w_2,w_3;n)
\end{aligned}
\tag{3.17}
$$

(in the general formalism $v_1=0$ is also a constraint au-

tomatically satisfied by definition). The quasiexactly solvable symmetric sextic potential takes the following compact form:

$$W(u;n)=w_3u(u+w_2/2w_3)^2-(2n+3)\sqrt{w_3}u\ . \quad (3.18)$$

The wave functions and corresponding eigenenergies are

$$\Psi_\mu(u;\nu)\propto C_\mu(u;\nu)u^{\delta/2}\exp\left[-\frac{w_2}{4\sqrt{w_3}}u-\frac{\sqrt{w_3}}{4}u^2\right], \quad (3.19)$$

$$E_m=\left[(n+1/2)\frac{w_2}{\sqrt{w_3}}+4\sqrt{w_3}\sum_{j=1}^{\nu}u_j\right]. \quad (3.20)$$

When looking for the zeros of $C_\mu(u;\nu)$ through the system of Eqs. (2.22) we find several solution sets. The μth set, say $\{u_j^{(\mu)}=(x_j)^2=(x_{n+1-j}^{(\mu)})^2;\ j=1,2,\ldots,\nu\}$, is characterized by having only a subset of $\mu\le\nu$ positive roots, i.e., $u_j^{(\mu)}>0$ for $j=1,2,\ldots,\mu$ while $\mu_j^{(\mu)}<0$ for $j=\mu+1,\ldots,\nu$. Associated with each solution set, a wave function with $m=2\mu+\delta$ nodes ($m\le n$) and $n=2\nu+\delta$ zeros is obtained. From symmetry considerations the nodes of a particular wave function $\psi_m(x;n)$, are located at $\pm[u_j^{(\mu)}]^{1/2}$, $1\le j\le\mu$ and when n is odd ($\delta=1$) at $x=0$ also. The remaining $(n-m)$ zeros of the wave function appear in complex-conjugate pairs on the imaginary axis at $\pm i|u_j^{(\mu)}|^{1/2}$, $\mu+1\le j\le\nu$.

For a fixed value of n, the constraint relation (3.17) holds, irrespective of the particular solution set considered; then we are left with a family of eigenfunctions and corresponding eigenenergies of the Hamiltonian whose potential is defined by Eq. (3.18). Let us write out these results for the first n values.

For $n=0$ ($\nu=0$) we have simply

$$\psi_0(x;n=0)\propto\exp\left[-\frac{w_2}{4\sqrt{w_3}}u-\frac{\sqrt{w_3}}{4}u^2\right]\equiv\tilde{\Psi}(u)\ , \quad (3.21)$$

$$E_0=w_2/2\sqrt{w_3}\ ,$$

i.e., for the potential $V(x,n=0)$, only the ground state has a closed expression.

For $n=1$ ($\nu=0$) there is only one node at the origin, $x_1=0$ and the wave function $\psi_1(x;n=1)\propto x\tilde{\Psi}(u)$, corresponds to a first excited state in the potential $V(x;n=1)$ with energy $E_1=3w_2/2\sqrt{w_3}$.

For $n=2,3$ ($\nu=1$) we explicitly find, after solving Eq. (2.22),

$$u_1=\begin{cases}\dfrac{w_2}{4w_3}\left[\pm\mathrm{sgn}w_2\left[1+\dfrac{8w_3\sqrt{w_3}}{w_2^2}\right]^{1/2}-1\right], & n=2 \quad (3.22)\\[2ex] \dfrac{w_2}{4w_3}\left[\pm\mathrm{sgn}w_2\left[1+\dfrac{24w_3\sqrt{w_3}}{w_2^2}\right]^{1/2}-1\right], & n=3\ . \quad (3.23)\end{cases}$$

In both cases there are two solution sets with only one member. Let us denote u_1^+ (u_1^-) the positive (negative) solution in Eqs. (3.22) and (3.23). The first set ($\mu=0$) provides two pure imaginary (conjugate pair) zeros at $x_{1,2}=\mp i(|u_1^-|)^{1/2}$, while the second ($\mu=1$) provides two real zeros at $x_{1,2}=\mp(u_1^+)^{1/2}$. For $n=2$ we obtain exact solutions for the ground state and the second excited state:

$$\psi_0(x;n=2)\propto(x^2+|u_1^-|)\tilde{\Psi}(x)\ ,$$

$$\psi_2(x;n=2)\propto(x^2-u_1^+)\tilde{\Psi}(x)\ ,$$

In a similar way, for $n=3$ we obtain exact solutions for the first and third excited states:

$$\psi_1(x;n=3)\propto x(x^2+|u_1^-|)\tilde{\Psi}(x)\ ,$$

$$\psi_3(x;n=3)\propto x(x^2-u_1^+)\tilde{\Psi}(x)\ .$$

For large values of n, it is no longer easy to solve Eqs. (2.22) for the zeros $\{u_j^{(\mu)}\}$ of the polynomial $C_\mu(u;\nu)$. Instead, one can obtain the coefficients of these polynomials by solving the system of Eqs. (B9) which has $\nu+1$ independent solution sets: $\{c_k^{(\mu)}\}$, $\mu=0,1,\ldots,\nu$. From the $\nu+1$ eigenfunctions can be constructed, with $m=n,n-2,\ldots,\delta$ nodes in the x real axis. The corresponding eigenenergies are readily obtained through Eq. (3.20) by noticing that $\sum_{j=1}^{\nu}u_j^{(\mu)}=-c_{\nu-1}^{(\mu)}$.

Another purely algebraic procedure has been recently developed in Ref. 11, that seems to be more adequate for computational purposes (see Appendix B).

IV. INVERSE POWER POTENTIALS ($L\neq0, M=0$)

Now we consider potentials of the form

$$V(x)=\sum_{j=-2L}^{0}v_jx^j\ , \quad (4.1)$$

with $v_{-1}\neq0$ (this excludes the possibility of analyzing symmetric potentials), $v_{-2}\ge-\frac{1}{4}$ and the remaining coupling constants $\{v_k\}$ ensuring the existence of bound states. This form is suitable to model atomic and molecular potentials.

From Eqs. (2.13) and (2.14) we have

$$\bar{g}(x)=\sum_{k=-L}^{0}a_kx^k\ , \quad (4.2)$$

$$\bar{\psi}(x)\propto x^{a_{-1}}\exp\left[\sum_{\substack{k=-L\\k\neq-1}}^{0}\frac{a_k}{k+1}x^{k+1}\right]. \quad (4.3)$$

From the requirement that $\bar{\psi}(x)$ be square integrable it follows that $a_{-L}<0$ when $L>1$ and $a_{-1}>-\frac{1}{2}$ when $L=1$. From Eqs. (2.10) and (4.2)

$$F_m(x;n)=\sum_{k=-L}^{-1}f_k^nx^k\ , \quad (4.4)$$

where the coefficients f_k^n are derived in Appendix A:

$$f_k^n=-\sum_{q=-L}^{k}a_qs_{q-k-1},\quad k=-L,-L+1,\ldots,-1\ . \quad (4.5)$$

Inserting Eqs. (4.4), (4.5), and (4.2) in the modified Ricatti equation (2.9) we obtain the following relations between the coefficients $\{a_k\}$ and $\{v_k\}$:

$$v_k=\begin{cases}\displaystyle\sum_{j=-L}^{k+L} a_j a_{k-j}, & -2L\le k\le -L-2 \quad (4.6a)\\ -La_{-L}+\displaystyle\sum_{j=-L}^{-1} a_j a_{-L-1-j}, & k=-L-1 \quad (4.6b)\\ -2\displaystyle\sum_{q=-L}^{k} a_q s_{q-k-1}+(k+1)a_{k+1}+\sum_{j=k}^{0} a_j a_{k-j}, & -L\le k\le -1\ . \quad (4.6c)\end{cases}$$

for $k=-1,-2,\ldots,-2L$.

In order to have a unified description we define $f_k^{n=0}\equiv 0$. The wave function and the energy are given by

$$\psi_m(x;n)\propto B_m(x;n)x^{a_{-1}}\exp\left[\sum_{\substack{k=-L\\ k\neq -1}} \frac{a_k}{k+1}x^{k+1}\right], \qquad E_m=-a_0^2 \quad (4.7)$$

so they are completely specified by the $L+1$ coefficients $\{a_k\}_{-L<k<0}$. The index m (the *radial* quantum number) denotes the number of zeros of the wave function in the $0<\mathrm{Re}(x)<\infty$.

The Eqs. (4.6a)–(4.6c) contain both the consistency relations and the definitions of the $\{a_k\}$ in terms of the $\{v_m\}$. The systematic procedure to obtain exact eigenfunctions when the potential is given by Eq. (4.1) can be implemented as follows. From Eqs. (4.6a) and (4.6b) (for $k=-L,\ -L+1,\ldots,-1$) it is possible to obtain the coefficients a_k as a function of $\{v_{-2L},v_{-2L+1},\ldots,v_{-L+k}\}$. The cases $L=1$ or $L>1$ require slightly different approaches:

For $L=1$ we must have[27] $v_{-2}=\lambda(\lambda+1)\ge -\frac{1}{4}$ and adopting $\lambda>-\frac{1}{2}$ we get

$$a_{-1}=\tfrac{1}{2}+\sqrt{(1/4)+v_{-2}}=\lambda+1 \quad (4.8)$$

and the other root, $a_{-1}=-\lambda$, leads to a wave function that does not satisfy the Schrödinger equation.[28]

On the other hand, when $L>1$ we have

$$a_{-L}=-\sqrt{v_{-2L}}\ ,$$
$$a_k=\frac{1}{2a_{-L}}\left[v_{k-L}-\sum_{j=-L+1}^{k-1} a_j a_{k-L-j}\right]+\frac{L}{2}\delta_{-1,k}\ ,$$
$$k=-L+1,-L+2,\ldots,-1\ . \quad (4.9)$$

The coefficient a_0 is defined through Eq. (4.6c) with $k=-1$:

$$a_0=\frac{1}{2a_{-1}}\left[v_{-1}-2f_{-1}^n\right]\ . \quad (4.10)$$

In order to write $\bar{g}(x)$ and the energy E_n in terms of the coupling constants $\{v_k\}$, it seems that we first need to obtain $f_{-1}^n(\{x_j\}_{j=1,2,\ldots,n})$. However, after Eq. (4.10), the set of Eqs. (2.11) that define the nodes can be rewritten as

$$\sum_{k=-L}^{-1} a_k x_j^k+\frac{v_{-1}-2f_{-1}^n}{2a_{-1}}=\gamma_{nj},\quad j=1,\ldots,n\ . \quad (4.11)$$

Thus, summing (4.11) over the index j and taking into account Eqs. (4.5) and $\sum_{j=1}^n \gamma_{n,j}=0$ we have

$$f_{-1}^n=\frac{nv_{-1}}{2(n+a_{-1})}\ , \quad (4.12)$$

which substituted in Eq. (4.10) gives

$$a_0=\frac{v_{-1}}{2(n+a_{-1})}\ . \quad (4.13)$$

In this way both $\bar{\psi}$ and the energy

$$E_n=-\left[\frac{v_{-1}}{2[n+a_{-1}(\{v_k\})]}\right]^2 \quad (4.14)$$

are completely specified by $L+1$ of the couplings v_k and and by n. For the remaining polynomial factor $B_m(x;n)$ the discussion of Sec. II applies.

For $L>1$ different solutions for $B_m(x;n)$ do not necessarily correspond to the same potential, as the remaining $L-1$ couplings are subject to the constraints given by Eq. (4.6c) that depend on each particular solution.

Through some simple examples we will now show how this approach works and briefly quote the results obtained for the cases $L=1$, 2, and 3.

a. Coulomb-Kratzer-type potential ($L=1$). The Coulomb-Kratzner potential is given by

$$V(x)=\frac{\lambda(\lambda+1)}{x^2}-\frac{2Z}{xa_B}\ , \quad (4.15)$$

where λ is not necessarily an integer and $a_B=2/e^2$ is the Bohr-radius-like parameter. In this case there are no consistency conditions, and we are confronted with a truly solvable potential, i.e., all the levels are expressible in closed form. From Eqs. (4.8) and (4.13)

$$a_{-1}=\lambda+1\ ,\qquad a_0=\frac{Z}{a_B(n+\lambda+1)}\ . \quad (4.16)$$

This leads to the well-known result

$$\psi_{n,\lambda}(x)=B_n(x)x^{\lambda+1}\exp\left[-\frac{Zx}{a_B(n+\lambda+1)}\right],\qquad E_{n,\lambda}=-\frac{Z^2}{a_B^2(n+\lambda+1)^2}\ , \quad (4.17)$$

where, for example, the nodes for $n=1,2$ are located at

$$x_1=(\lambda+1)(\lambda+2)a_B/Z, \quad n=1 \tag{4.18}$$

$$x_{1,2}=(2\lambda+3\pm\sqrt{2\lambda+3})(n+\lambda+1)a_B/(2Z), \quad n=2 \tag{4.19}$$

b. $L=2$. The potential

$$V(x)=\frac{\hbar^2}{2\mu}\left[\frac{v_{-4}}{x^4}+\frac{v_{-3}}{x^3}+\frac{v_{-2}}{x^2}+\frac{v_{-1}}{x}\right] \tag{4.20}$$

with $v_{-4}>0$ models a repulsive interaction at short distances. As before we assume that the remaining coupling constants $\{v_k\}$ are such that the potential admits bound states. For it we get

$$\bar{\psi}(x)=x^{(1+v_{-3}/2\sqrt{v_{-4}})}\times\exp\left[\frac{v_{-1}}{2(n+1+v_{-3}/2\sqrt{v_{-4}})}x-\frac{\sqrt{v_{-4}}}{x}\right], \tag{4.21}$$

$$E_n=-\left[\frac{v_{-1}}{2(n+1+v_{-3}/2\sqrt{v_{-4}})}\right]^2,$$

when the coupling are related by the constraint

$$v_{-2}=\frac{v_{-1}\sqrt{v_{-4}}}{(n+1+v_{-3}/2\sqrt{v_{-4}})}+\left[1+\frac{v_{-3}}{2\sqrt{v_{-4}}}\right]\frac{v_{-3}}{2\sqrt{v_{-4}}}-2\sqrt{v_{-4}}\,s_{-1}. \tag{4.22}$$

c. $L=3$. Finally for the potential

$$V(x)=\frac{\hbar^2}{2\mu}\left[\frac{v_{-6}}{x^6}+\frac{v_{-5}}{x^5}+\frac{v_{-4}}{x^4}+\frac{v_{-3}}{x^3}+\frac{v_{-2}}{x^2}+\frac{v_{-1}}{x}\right] \tag{4.23}$$

with $v_{-6}>0$ we get

$$\bar{\psi}(x)=x^{a_{-1}}\exp\left[\frac{v_{-1}}{2(n+a_{-1})}x-\frac{v_{-5}}{2\sqrt{v_{-6}}}\frac{1}{x}-\frac{\sqrt{v_{-6}}}{2}\frac{1}{x^2}\right],$$

$$E_n=-\left[\frac{v_{-1}}{2(n+a_{-1})}\right]^2, \tag{4.24}$$

with $a_1=\frac{3}{2}+(v_{-4}-v_{-5}^2/4v_{-6})/2\sqrt{v_{-6}}$, while the consistency relations for the potential to be solvable are given by

$$v_{-2}=a_{-1}(a_{-1}-1)+\frac{v_{-5}}{\sqrt{v_{-6}}}\left[\frac{v_{-1}}{2(n+a_{-1})}-s_{-1}\right]-2\sqrt{v_{-6}}\,s_{-2}, \tag{4.25}$$

$$v_{-3}=\frac{v_{-5}}{\sqrt{v_{-6}}}(a_{-1}-1)+\sqrt{v_{-6}}\left[\frac{v_{-1}}{n+a_{-1}}-2s_{-1}\right].$$

V. FINITE LAURENT-TYPE POTENTIALS ($L\neq0, M\neq0$)

For the general potential considered in Eq. (2.12) only some minor modifications need to be addressed. With respect to the range of the coupling constants of the potential the considerations made in the previous sections apply. The only significant difference is that the maximum power M in the potential is not restricted to be odd as the wave function (2.14) is normalizable in any case.

Proceeding as before the relations between the coefficients $\{a_k\}$ and $\{v_k\}$ now read

$$v_k=\begin{cases}\displaystyle\sum_{j=-L}^{k+L}a_ja_{k-j}, & -2L\le k\le -L-2 \quad (5.1a)\\ -La_{-L}+\displaystyle\sum_{j=-L}^{-1}a_ja_{-L-1-j}, & k=-L-1 \quad (5.1b)\\ 2f_k^n+(k+1)a_{k+1}+\displaystyle\sum_{j=-\min(L,M-k)}^{\min(L+k,M)}a_ja_{k-j}, & -L\le k\le M-1 \quad (5.1c)\\ \displaystyle\sum_{j=k-M}^{M}a_ja_{k-j}, & M\le k\le 2M \quad (5.1d)\end{cases}$$

for $k\neq0$. In these equations f_k^n is the coefficient of the finite Laurent expansion of $F_m(x;n)$ defined through Eqs. (2.10):

$$F_m(x;n)=\sum_{k=-L}^{M-1}f_k^n x^k, \tag{5.2}$$

with the coefficients f_k^n defined in Eq. (3.1) for $k\ge0$ and in Eq. (4.5) for $k<0$.

Finally the wave function and its energy are given by

$$\psi_m(x;n)=B_m(x;n)x^{a_{-1}}\exp\left[\sum_{\substack{k=-L\\k\neq-1}}^{M}\frac{a_k}{k+1}x^{k+1}\right],$$

$$E_m=-\left[2[n+\lambda+3/2]a_1+\sum_{\substack{j=-\min(L,M)\\j\neq1}}^{\min(L,M)}a_ja_{-j}+2\sum_{m=1}^{M-1}a_{m+1}s_m\right]. \tag{5.3}$$

As before, the polynomial factor $B_m(x;n)$ can be determined from the knowledge of the coefficients $\{a_k\}_{-L<k<M}$ and thus both the wave function and its energy are now completely specified by $L+M+1$ free parameters. The systematic procedure can also be adapted for this case as follows. From Eqs. (5.1a) and (5.1b) it is possible to obtain the coefficients a_k for $k=-L, -L+1,\ldots,-1$, in terms of $\{v_{-2L},v_{-2L+1},\ldots,v_{-L+k}\}$ and from Eq. (5.1d) the coefficients a_k for $k=0,1,\ldots,M$, in terms of $\{v_{2M},v_{2M-1},\ldots,v_{M+k}\}$. The relations are the same as those given in Eqs. (3.9) and

Eq. (4.8) for $L=1$ or Eq. (4.9) for $L>1$.

The discussion of Sec. II applies to the polynomial factor $B_m(x;n)$. As before different solutions for $B_m(x;n)$ do not necessarily correspond to the same potential, as the remaining $M+L-1$ couplings are subject to the constraints given by Eq. (5.1c) that depend on each particular solution

$$v_k = v_k(v_{-2L}, v_{-2L+1}, \ldots, v_{-1}, v_M, v_{M+1}, \ldots, v_{2M}; f_k^n\{x_j\}), \quad k=-L,\ldots,M-1 . \tag{5.4}$$

As an illustration of the method we explicitly give the nodeless function $\bar{\psi}(x)$, the energy E_n, and the consistency relations between the coupling constants for the Coulomb-like potential plus polynomial terms up to the sixth ($L=1$ and $M=1,2,3$), e.g,

$$V(x)=\left[\frac{\lambda(\lambda+1)}{x^2}+\frac{v_{-1}}{x}+\sum_{k=1}^{M} v_k x^k\right] . \tag{5.5}$$

After some elementary algebra we obtain, for $M=1$,

$$\begin{aligned}
\bar{\psi}(x) &= x^{\lambda+1}\exp\left[-\frac{v_1}{2\sqrt{v_2}}x-\frac{\sqrt{v_2}}{2}x^2\right] , \\
E_n &= \left[2(n+\lambda+3/2)\sqrt{v_2}-\frac{v_1^2}{4v_2}\right] , \\
v_{-1} &= -(\lambda+1)\left[\frac{v_1}{\sqrt{v_2}}+2s_{-1}\right] ,
\end{aligned} \tag{5.6}$$

for $M=2$

$$\begin{aligned}
\bar{\psi}(x) &= x^{\lambda+1}\exp\left[\frac{v_3^2/4v_4-v_2}{2\sqrt{v_4}}x-\frac{v_3}{4\sqrt{v_4}}x^2-\frac{\sqrt{v_4}}{3}x^3\right] , \\
E_m &= \left[(n+\lambda+3/2)\frac{v_3}{\sqrt{v_4}}-\frac{1}{4v_4}\left[\frac{v_3^2}{4v_4}-v_2\right]^2 -2\sqrt{v_4}\,s_1\right] , \\
v_{-1} &= 2(\lambda+1)\left[\frac{1}{2\sqrt{v_4}}\left[\frac{v_3^2}{4v_4}-v_2\right]-s_{-1}\right] , \\
v_1 &= \frac{v_3}{2v_4}\left[v_2-\frac{v_3^2}{4v_4}\right]-2(n+\lambda+2)\sqrt{v_4} ,
\end{aligned} \tag{5.7}$$

for $M=3$,

$$\begin{aligned}
\bar{\psi}(x) &= x^{\lambda+1}\exp\left\{-\frac{1}{2\sqrt{v_6}}\left[\frac{v_5}{2v_6}\left[\frac{v_5^2}{4v_6}-v_4\right]+v_3\right]x+\frac{1}{4\sqrt{v_6}}\left[\frac{v_5^2}{4v_6}-v_4\right]x^2-\frac{v_5}{6\sqrt{v_6}}x^3-\frac{\sqrt{v_6}}{4}x^4\right\} , \\
E_n &= \left\{(n+\lambda+3/2)\frac{v_4-v_5^2/4v_6}{\sqrt{v_6}}-\frac{1}{4v_6}\left[\frac{v_5}{2v_6}\left[\frac{v_5^2}{4v_6}-v_4\right]+v_3\right]^2+\frac{v_5}{\sqrt{v_6}}s_1+2\sqrt{v_6}\,s_2\right\} , \\
v_{-1} &= -2(\lambda+1)\left\{\frac{1}{2\sqrt{v_6}}\left[\frac{v_5}{2v_6}\left[\frac{v_5^2}{4v_6}-v_4\right]+v_3\right]+s_{-1}\right\} , \\
v_1 &= -\frac{1}{2v_6}\left[\frac{v_5^2}{4v_6}-v_4\right]\left[\left[\frac{v_5^2}{4v_6}-v_4\right]\frac{v_5}{2v_6}+v_3\right]-(n+\lambda+2)\frac{v_5}{\sqrt{v_6}}-2\sqrt{v_6}\,s_1 , \\
v_2 &= \frac{(v_5^2/4v_6-v_4)^2}{4v_6}+\frac{v_5}{2v_6}\left[\frac{v_5}{2v_6}\left[\frac{v_5^2}{4v_6}-v_4\right]+v_3\right]-(2n+2\lambda+5)\sqrt{v_6} .
\end{aligned} \tag{5.8}$$

From Eqs. (5.6)–(5.8) it is immediately possible to state the consistency relations, energy, and wave function for the ground states. From a formal point of view there is no problem to continue with the excited states, but in general they involve rather cumbersome algebraic (and sometimes numerical) manipulations in order to find explicitly the nodes by solving Eq. (2.11). For the sake of brevity we explicitly show only the solution for the first excited state in the case $M=L=1$, e.g.,

$$V(x)=\left[\frac{\lambda(\lambda+1)}{x^2}+\frac{v_{-1}}{x}+v_1 x+v_2 x^2\right] . \tag{5.9}$$

For $n=1$ Eq. (2.11) yields two solutions

$$x_1^{\pm}=\pm\left[\left[\frac{v_1}{4v_2}\right]^2+\frac{\lambda+1}{\sqrt{v_2}}\right]^{1/2}-\frac{v_1}{4v_2} , \tag{5.10}$$

being $x_1^+>0$ (inside the physical domain) and $x_1^-<0$ (outside the physical domain). Recalling the discussion in Sec. II we obtain two wave functions $\psi^+(x)\equiv\psi_{m=1}(x;n=1)$ and $\psi^-(x)\equiv\psi_{m=0}(x;n=1)$ with one and zero nodes, respectively, both with the same eigenvalue E

$$\psi^{\pm}(x) \propto x^{\lambda+1}(x-x_1^{\pm})\exp\left[-\frac{v_1}{2\sqrt{v_2}}x-\frac{\sqrt{v_2}}{2}x^2\right], \tag{5.11}$$

$$E=\left[2(\lambda+5/2)\sqrt{v_2}-\frac{v_1^2}{4v_2}\right].$$

They are eigenfunctions of the Hamiltonians $H^{\pm}$ with the potentials $V^{\pm}$ given by Eq. (5.9) having their "charge" parameter fixed to the value

$$v_{-1}^{\pm}=-(\lambda+1)\left[\frac{v_1}{\sqrt{v_2}}+\frac{2}{x_1^{\pm}}\right]. \tag{5.12}$$

For the symmetric Laurent-type potentials, the mapping considered in Sec. III B lets us recover the discussion above with minor modifications. As an illustration we quote some interesting results which can be derived for the 3D version of the anharmonic sextic polynomial oscillator[10]

$$W(u)=\frac{\lambda(\lambda+1)}{u}+w_1u+w_2u^2+w_3u^3. \tag{5.13}$$

The coefficients $A_0=-w_2/4\sqrt{w_3}$ and $A_1=-\sqrt{w_3}/2$ coincide with those derived in the 1D case, while $2A_{-1}=\delta=\lambda+1$ should be considered. Notice that μ is now the *radial* quantum number, equal to the number of nodes in the physical domain $\mathrm{Re}(x)>0$ which at most can be equal to ν. Then, the 3D results are easily obtained by the substitutions $n\to 2\nu+\lambda+1$ in the 1D expressions derived in Sec. III B (here m and n have no physical meaning). The only constraint for the potential to be quasiexactly solvable now reads

$$\begin{aligned} w_1&=2(4\nu+2\lambda+5)A_1+4A_0^2 \\ &=w_2^2/4w_3-(4\nu+2\lambda+5)\sqrt{w_3},\quad \nu=0,1,\ldots \end{aligned} \tag{5.14}$$

and the eigenfunctions and eigenvalues read

$$\psi_{\mu,\lambda}(u;\nu)\propto C_\mu(u;\nu)u^{(\lambda+1)/2}\times\exp\left[-\frac{w_2}{4\sqrt{w_3}}u-\frac{\sqrt{w_3}}{4}u^2\right], \tag{5.15}$$

$$E_{\mu,\lambda}=\left[(2\nu+\lambda+3/2)\frac{w_2}{\sqrt{w_3}}+4\sqrt{w_3}\sum_{j=1}^{\nu}u_j^{(\mu)}\right].$$

For the remaining polynomial factor $C_\mu(u;\nu)$, the discussion in Appendix B applies (see also Sec. III B for general considerations).

In conclusion, for the 3D symmetric sextic anharmonic polynomial oscillator given by Eq. (5.13) and subject to the constraint relation (5.14), the first $\nu+1$ eigenfunctions and the corresponding energies can be obtained by purely algebraic methods.

VI. GENERALIZATION OF THE MODIFIED RICATTI APPROACH

The modified Ricatti approach can be also suitable for looking at a more general class of potentials as was anticipated in Sec. II. There we considered the transformation $u=x^2$ when $W(u)$ is an even finite Laurent series in u. This particular case leads to a large family of partially solvable potentials having as a particular exactly solvable member the harmonic oscillator (1D and 3D). We also found the very interesting quasiexactly solvable potential: the 1D and 3D sextic symmetric anharmonic polynomial oscillator.

In this section we investigate the occurrence of partial and quasiexact solubility in families related to already known exactly solvable potentials. The latter are particular cases in the broader context of the partial solubility analysis.

Suppose that the potential is regular almost everywhere and can be written as some simple function of $u(x)$, say $V(x)=W(u)$. Proceeding as before it is natural to write the wave functions as

$$\psi_m(x;n)=\Psi_\mu(u;\nu) \tag{6.1}$$

The indexes μ and ν are related to the number of zeros of the wave functions (except those at the boundaries) in the x and u complex plane. For instance, if $u=x^2$, $\nu=[n/2]$ and $\mu=[m/2]$. According to (6.1), the logarithmic derivative of the wave function is

$$g_m(x;n)=\frac{du}{dx}\frac{d}{du}[\ln\Psi_\mu(u;\nu)]\equiv\frac{du}{dx}G_\mu(u;\nu). \tag{6.2}$$

The replacement of Eq. (6.2) in Eq. (2.4) modifies the Ricatti equation to

$$\alpha(u)[G_\mu^2(u;\nu)+\dot{G}_\mu(u;\nu)]+\beta(u)G_\mu(u;\nu)=W(u)-E_m, \tag{6.3}$$

where $\alpha(u)=[(u'(x)]^2$, $\beta(u)=u''(x)$, and the dot denotes the derivative with respect to u. Decomposing as before $G_\mu(u;\nu)$ in its regular almost everywhere and singular components

$$G_\mu(u;\nu)=\bar{G}(u)+\sum_{j=1}^{\nu}\frac{1}{u-u_j} \tag{6.4}$$

and replacing this expression in Eq. (6.3) the generalized modified Ricatti equation results:

$$\alpha(u)[\bar{G}^2(u)+\dot{\bar{G}}(u)]+\beta(u)\bar{G}(u)+2\Phi_\mu(u;\nu)=W(u)-E_m, \tag{6.5}$$

where it is easy to derive

$$\Phi_\mu(u;\nu)=\sum_{j=1}^{\nu}\frac{\alpha(u)[\bar{G}(u)-\Gamma_{\nu,j}]+\beta(u)/2}{u-u_j} \tag{6.6}$$

and $\Gamma_{\nu,j}\equiv\sum_{k=1,k\neq j}^{\nu}(u_k-u_j)^{-1}$ for $j=1,2,\ldots,n$, together with $\Gamma_{0,0}\equiv\Gamma_{1,0}\equiv 0$ and $\Phi_0(x,0)\equiv 0$.

Provided that $W(u)$, $\alpha(u)$, and $\beta(u)$ are regular almost everywhere, the left-hand side of Eq. (6.6) must also be regular in the same domain. Therefore we have

TABLE I. The mapping $u(x)$ for some exactly solvable potentials $V^s(x)=W^s(u)$ together with the corresponding regular function $\bar{G}^s(u)$ [see Eq. (6.4)].

Potential	$V^s(x)$	$u(x)$	$W^s(u)$	$\bar{G}^s(u)$
1D harmonic oscillator	$\frac{x^2}{4}$	x^2	$\frac{u}{4}$	$\frac{\delta}{2u}-\frac{1}{2}$
3D harmonic oscillator	$\frac{x^2}{4}+\frac{\lambda(\lambda+1)}{x^2}$	x^2	$\frac{u}{4}+\frac{\lambda(\lambda+1)}{u}$	$\frac{\lambda+1}{2u}-\frac{1}{2}$
Morse	$\sigma^2(e^{-2x}-2e^{-x})$	e^{-x}	$\sigma^2(u^2-2u)$	$\frac{\sigma-\nu-1/2}{u}-\sigma$
Pöschl-Teller	$\frac{\xi(\xi-1)}{\sin^2 x}+\frac{\eta(\eta-1)}{\cos^2 x}$	$\sin^2 x$	$\frac{\xi(\xi-1)}{u}+\frac{\eta(\eta-1)}{1-u}$	$\frac{\xi}{2u}-\frac{\eta}{2(1-u)}$
Modified Pöschl-Teller	$\frac{\xi(\xi-1)}{\sinh^2 x}-\frac{\eta(\eta+1)}{\cosh^2 x}$	$\sinh^2 x$	$\frac{\xi(\xi-1)}{u}-\frac{\eta(\eta+1)}{1+u}$	$\frac{\xi}{2u}-\frac{\eta}{2(1+u)}$

$$\left[\bar{G}(u)+\frac{\beta}{2\alpha}\right]_{u=u_j}=\Gamma_{\nu,j},\quad j=1,\ldots,\nu\ . \tag{6.7}$$

If $\alpha(u)$ is a polynomial in u with their zeros at $\{z_i\}$, i.e., $\alpha(u)\propto\prod_{i=1}^{q}(u-z_i)$, we can apply the considerations of Sec. II A. It is clear that the generalized Ricatti equation (6.5) is well suited provided that both the potential $W(u)$ and $\beta(u)/\alpha(u)=-(d/du)[\ln(dx/du)]$ can be written as finite Laurent-like-series potentials in the variables $\{u-z_i,\ i=1,\ldots,q\}$.[21]

Certainly, for some of the well-known[29] exactly solvable potentials $V^s(x)$, a transformation $u(x)$ with the desired properties is available. In Table I we explicitly show the proper choice $u(x)$, and the resulting $W^s(u)$ and $\bar{G}^s(u)$.

In order to construct a family of new partially solvable potentials we will consider the addition of regular almost everywhere terms to the solvable $\bar{G}^s(u)$, i.e.,

$$\bar{G}(u)=\bar{G}^s(u)+\sum_{\mathbf{j}} A_{\mathbf{j}}\prod_{i=1}^{q}(u-z_i)^{j_i}\equiv\bar{G}^s(u)+\delta G\ , \tag{6.8}$$

where $\mathbf{j}=(j_1,\ldots,j_q)$.

A detailed analysis of these families, which closely resembles the discussion of the previous sections, will be given elsewhere.[21] From Eq. (6.8) it is immediate to see that the candidates for partial solvability will be of the form

$$W(u)=W^s(u)+\alpha(u)\left[2G^s\delta G+(\delta G)^2+\frac{d}{du}(\delta G)\right]+\beta(u)\delta G\ . \tag{6.9}$$

With the particular choice $u=x$ and $u=x^2$, Eqs. (6.1)–(6.9) are consistent with the formalism developed in the previous sections.

Our restrictions to almost regular potentials considered here can be removed leading to minor modifications of the formalism. For example, by redefining the concept of *regularity almost everywhere* in order to include as exceptional points the singularities of the potential, we are able to generate partially solvable potentials associated with the Rosen-Morse and Manning-Eckart ones.[21]

VII. FINAL REMARKS

We have developed a systematic procedure to study the problem of partial solubility in quantum mechanics for a large class of potentials. The method is based on isolating the singularities of the logarithmic derivative of the wave function. We have shown that the regular component $\bar{g}(x)$ characterizes univocally the solvable subspace, fixes the position of the zeros of the wave function, and satisfies a modified Ricatti equation [see Eq. (2.9)] which, for a very wide class of potentials, is simpler to solve than the corresponding one for $g_m(x)$. The regularity (almost everywhere) of the function $\bar{g}(x)$ is the key to determine its functional form. This gives a great generality to the approach here presented, providing a unifying scope for the several partially solvable systems discussed in the literature. In that sense we have shown the effectiveness of the scheme in finding exact solutions for both general and symmetric finite Laurent-type potentials in the 1D and 3D cases. This includes the anharmonic polynomial oscillators, the Coulomb-like plus polynomial terms, and the Lennard-Jones-type potentials.

By means of an adequate transformation $x\rightarrow u$, we generalized the method providing a constructive scheme to generate new families of partially solvable potentials, associated with the well-known exactly solvable ones. A detailed analysis is planned to be developed elsewhere.[21]

For simplicity we have only outlined the skeleton of the method, and restricted ourselves to those $\bar{g}_m(x;n)$ with only simple poles outside the physical domain and potentials regular almost everywhere. The removal of these conditions enlarges the class of Hamiltonians on which partial solubility can be investigated. We plan to present the corresponding results elsewhere.[21]

Lastly, we want to remark that the problem of exact solubility arises as a very particular case of partial solubility and thus we have given an alternative perspective

to the Infeld and Hull[2] approach.

The possibilities of the method are far from being exhausted and we mention now two of them. The first involves the construction of the isospectral supersymmetric partner of the already found partially solvable potentials, through the factorization approach.[2,5] The second one consists in using the results so far obtained to obtain better approximations to unsolvable problems through perturbation[30] theory as well as by semiclassical methods.[31]

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APPENDIX A

In this appendix we will derive explicitly the expression for $F_m(x;n)$ as defined through Eqs. (2.10), for the functional form of $\bar{g}(x)$ discussed in Sec. IV (the case of Sec. III is implicitly considered by making $L=0$). Let us write

$$\bar{g}(x)=\sum_{m=-L}^{M} a_m x^m = x^{-L}\sum_{m=0}^{M+L} a_{m-L}\, x^m \tag{A1}$$

and introduce the auxiliary polynomials

$$P_{n,j}(x)=x^L[\bar{g}(x)-\bar{g}(x_j)]=\sum_{m=0}^{M+L} p_m^{n,j} x^m \tag{A2}$$

such that

$$F_n(x;n)=x^{-L}\sum_{j=1}^{n}\frac{P_{n,j}(x)}{x-x_j}\ . \tag{A3}$$

From Eqs. (2.10) and (2.11) it follows that

$$p_m^{n,j}=a_{m-L}-\bar{g}(x_j)\delta_{m,L},\quad 0\le m\le M+L\ . \tag{A4}$$

In order to evaluate the quotient in Eq. (A3) we expand $P_{n,j}(x)$ around $x=x_j$, yielding

$$P_{n,j}(x)=\sum_{l=0}^{M+L-1}\left[\sum_{m=l+1}^{M+L}\binom{m}{l+1}p_m^{n,j}x_j^{n-l-1}\right]\times(x-x_j)^{l+1}\ , \tag{A5}$$

where we have used $P_{n,j}(x_j)=0$. If we now expand $(x-x_j)^l$ and make use of the identity

$$\sum_{l=k}^{m}(-1)^{l-k}\begin{bmatrix}1\\k\end{bmatrix}\begin{bmatrix}m+1\\l+1\end{bmatrix}=1 \tag{A6}$$

we obtain

$$P_{n,j}(x)=(x-x_j)^{L+M-1}\sum_{k=0}\left[\sum_{m=k+1}^{L+M}p_m^{n,j}x_j^{m-k-1}\right]x^k\ , \tag{A7}$$

and therefore from Eq. (A3)

$$F_m(x;n)=\sum_{k=-L}^{M-1}\left[\sum_{m=k+1}^{M}\sum_{j=1}^{n}p_{m+L}^{n,j}(x_j)^{m-k-1}\right]x^k\ . \tag{A8}$$

According to this, the coefficients f_k^n can be written

$$f_k^n=\sum_{m=k+1}^{M}\sum_{j=1}^{n}p_{m+L}^{n,j}x_j^{m-k-1},\quad -L\le k\le M-1 \tag{A9}$$

and using Eq. (A4) we finally get

$$f_k^n=\begin{cases}\displaystyle\sum_{m=k+1}^{M}a_m\sum_{j=1}^{n}(x_j)^{m-k-1}, & 0\le k<M-1\\ \displaystyle-\sum_{m=-L}^{k}a_m\sum_{j=1}^{n}(x_j)^{m-k-1}, & -L\le k\le -1\end{cases} \tag{A10}$$

as assessed in Eqs. (3.2) and (4.5).

Similarly, for the symmetric situation, $V(x)=W(u=x^2)$, and from Eqs. (2.21) and (2.23) it is possible to obtain

$$\Phi_\mu(u;\nu)=\sum_{k=-(L+1)/2}^{(M-1)/2}\varphi_k^\nu u^k\ , \tag{A11}$$

where the coefficients φ_k^ν are given by

$$\varphi_k^\nu=\begin{cases}\displaystyle 4\sum_{m=k}^{(M+1)/2}A_m\sum_{j=1}^{\nu}(u_j)^{m-k}, & 0\le k\le(M-1)/2\\ \displaystyle -4\sum_{m=-(L+1)/2}^{k-1}A_m\sum_{j=1}^{\nu}(u_j)^{m-k} & \\ & -(L+1)/2\le k\le -1\ .\end{cases} \tag{A12}$$

APPENDIX B

The set of coupled Eq. (2.11) can be difficult to solve for $n\ge 2$. In an application of the method it could be useful to have an alternative route to solve the posed problem: Instead of looking for the nodes $\{x_j\}$ we can try to solve for the coefficients $\{b_k\}$ defined by

$$B_m(x;n)=\prod_{j=1}^{n}(x-x_j)=\sum_{k=0}^{n}b_k x^k\ . \tag{B1}$$

These coefficients are related to appropriate sums over all the possible products of $n-k$ distinct nodes x_j, i.e.,

$$b_k=(-1)^{n-k}\sum_{i\ne j\ne\cdots\ne s}x_i x_j\cdots x_s=b_k(\{x_j\})\ ,\quad 0\le k\le n-1 \tag{B2}$$

$$b_n=1\ .$$

In our approach to solubility, the constraints and the energy both require the knowledge of the f_k^n, which according to Eq. (A10) depend on the sums

$$s_m\equiv\sum_{j=1}^{n}(x_j)^m,\quad m=-L,..,M-1\ . \tag{B3}$$

The relations between s_m and $\{b_j\}$ are explicitly given in Ref. 32 so it is always possible to write $f_k^n = f_k^n(\{b_j\})$. Thus the $\{b_j\}$ together with the function $\bar{g}(x)$ specify completely the wave function, its energy and consistency relations to attain solubility.

By replacing both Eq. (2.7) and Eq. (2.9) in the Schrödinger Eq. (2.3) the following differential equation is obtained for the polynomial $B_m(x;n)$ (we omit the indices m and n):

$$B''(x)+2B'(x)\bar{g}(x)-2F_n(x)B(x)=0 . \tag{B4}$$

For the particular potentials considered in Secs. II–IV we assume for $\bar{g}(x)$ a finite Laurent series [see Eq. (4.2)]; thus by equating to zero the coefficients of the different powers in x in Eq. (B4) we obtain

$$\sum_{j=\max\{-L,k-n\}}^{\min\{M-1,k\}} f_j^n b_{k-j} - \sum_{j=\max\{-L,k+1-n\}}^{\min\{M,k\}} (k+1-l)a_j b_{k-j} = \begin{cases} 0, & n-1\le k\le M+n-1 \\ \dfrac{(k+1)(k+2}{2} b_{k+2}, & 0\le k\le n-2 \end{cases} \tag{B5}$$

i.e., a set of nonlinear equations for the coefficients $\{b_j\}$ equivalent to the system of Eqs. (2.11) for the $\{x_j\}$

For the symmetric situation considered in Sec. II B, i.e., with the mapping $u=x^2$, and $B_m(x,n)=C_\mu(u,\nu)x^\delta$, we obtain, as a generalization of Eq. (B4), the following differential equation for the polynomial $C_\mu(u;\nu)=\sum_{k=0}^{\nu} c_k^{(\mu)} u^k$ (omitting indices μ,ν):

$$4u\ddot{C}+[8u\bar{G}(u)+2]\dot{C}-\Phi_\mu(u;\nu)C=0 . \tag{B6}$$

The replacement of Eqs. (2.23) and (A11) into (B6) would lead to a set of equations similar to (B5), i.e., a three-term recursion relation for the c_k's whose coefficients depend on $\{\varphi_k^\nu\}$ and $\{A_m\}$. This system would be in general *nonlinear* as the coefficients φ_k^ν [see Eq. (A11)] also depend on the unknowns $\{c_k\}$, i.e., $\varphi_k^\nu=\varphi_k^\nu(\{\sigma_m\})$ where $\sigma_m\equiv\sum_{j=1}^n (u_j)^m$ and through them[30] $\varphi_k^\nu=\varphi_k^\nu(\{c_k\})$. Thus, if $C(u)$ is a polynomial solution of Eq. (B6), $KC(u)$ with K an arbitrary constant is not.

Here we will quote the results only for the interesting situations

For the harmonic oscillator in its 1D (3D) version, i.e., $L=0$ (1), $M=1$ and $W=\omega u/4+\delta(\delta-1)/u$, we obtain $\bar{G}(u)=\delta/2u+A_0$ with $A_0=-\omega/4$ and $\delta=n-2\nu$ if $L=0$ or $\delta=\lambda+1$ if $L=1$. From Eq. (A11) results $\Phi_\mu(u;\nu)=4A_0\nu$, independent of the position of the nodes, so we are confronted with an exactly solvable potential. In effect, after changing variables to $y=-2A_0u=\omega u/2$, Eq. (B6) reduces to the *linear* Kummer's differential equation:

$$y\frac{d^2C}{dy^2}+(\delta+1/2-y)\frac{dC}{dy}+\nu C=0 . \tag{B7}$$

As $C(u)$ is a polynomial with $c_\nu=1$, only the regular solution should be considered resulting in $C_\nu(u;\nu)=(-1)^\nu\nu!L_\nu^{\delta-1/2}(\omega u/2)$ where $L_n^\alpha(y)$ denotes the generalized Laguerre polynomial.

The symmetric sextic polynomial oscillator in its 1D (3D) version corresponds to $L=0$ (1) and $M=3$. When the couplings are subject to the constraint (3.17), we obtain $\bar{G}(u)=\delta/2u+A_0+A_1u$, with A_0 and A_1 given in (3.16) and $\delta=n-2\nu$ if $L=0$ or $\delta=\lambda+1$ if $L=1$. Now $\Phi_\mu(u;\nu)=(4A_0\nu-4A_1c_{\nu-1})+(4A_1\nu)u$, where we have used $\sum_{j=1}^n u_j=-c_{\nu-1}$. Now Eq. (B6) reduces to

$$\frac{u}{2}\ddot{C}+[u(A_0+A_1u)+\delta/2+1/4]\dot{C} -(A_0\nu-A_1c_{\nu-1}+A_1\nu u)C=0 , \tag{B8}$$

clearly nonlinear in the coefficients c_k. Grouping equal powers in u the following three-term recursion relation is obtained:

$$\tfrac{1}{2}(\delta+k+1/2)(k+1)c_{k+1}+[A_1c_{\nu-1}-A_0(\nu-k)]c_k +(k-\nu-1)c_{k-1}=0, \quad k=0,\ldots,\nu+1 . \tag{B9}$$

With $c_{\nu+2}=c_{\nu+1}=0$ as the initial condition, Eqs. (B9) can be backwards recurred. For $k=\nu$, (B9) reads $A_1c_{\nu-1}(c_\nu-1)=0$, which for a nontrivial solution requires $c_\nu=1$. For $k=\nu-1$ we can write $c_{\nu-2}$ as a quadratic function of $c_{\nu-1}$. In general, from the kth relation it is possible to obtain $c_{\nu-k-1}$ as a polynomial of degree $k+1$ in $c_{\nu-1}$. Thus the $k=0$ relation reads

$$\tfrac{1}{2}(\delta+1/2)c_1(c_{\nu-1})+(A_1c_{\nu-1}-A_0\nu)c_0(c_{\nu-1}) =P(c_{\nu-1})=0 ,$$

where $P(c_{\nu-1})$ is a polynomial of degree $\nu+1$. The $\nu+1$ roots of P will give place to $\nu+1$ eigenfunctions and their corresponding eigenenergies [see Eq. (3.20)].

An alternative algorithm (more adequate from the computational point of view) to solve Eq. (B6) has been recently developed[11] for the 1D case ($L=0$) with arbitrary M. However, the usefulness of such an approach is apparent for $M=3$. The scheme can be easily extended to cope with the 3D case ($L=1$) with minor modifications.[21] Closely following Ref. 11, the unknown polynomial $C(u)$ can be expanded in a basis of generalized polynomials, i.e., $C(u)=\sum_k q_k Q_k(u)$, where $Q_k(u)$ are defined through a Rodrigues-like formula. When the constraint relation (3.17) holds, the first $\nu+1$ coefficients of such an expansion satisfy a system of linear equations and the determinant of the associated matrix coefficients vanishes. Consequently the system has $\nu+1$ distinct solutions.

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