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Korteweg–de Vries soliton excitation in Bénard-Marangoni convection

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For a two-dimensional shallow liquid layer open to air and subjected to a thermal gradient (Bénard problem), the threshold condition is given for the onset of a Korteweg–de Vries soliton as the result of an instability triggered by the Marangoni effect, i.e., by the variation of the interfacial tension with temperature or concentration of a surfactant. In the absence of buoyancy, in contrast with Bénard convection, the heating for standard liquids must be done from the air side. However, for liquids exhibiting a minimum in surface tension with temperature the heating can also be done from the liquid side when operating past the minimum where the surface-tension coefficient increases with temperature.

Recently, we[1-3] have discussed the role of the Marangoni effect[4,5] i.e., the variation of the surface tension with temperature or surfactant concentration in triggering overstability and eventually sustaining transverse capillary-gravity waves or longitudinal oscillations at the open surface of a liquid or at the interface separating two liquids. Our approach relied on the construction of a *dissipative harmonic oscillator* that at vanishing damping provides the threshold conditions for overstability in the liquid layer.

In the present paper, we follow a similar analysis using the *inviscid nonlinear and dispersive* Korteweg–de Vries (KdV) *soliton equation*[6,7] as the "ideal" system. Thus an extension of the KdV analysis is provided in order to account for viscosity and gradients in surface tension. In our derivation we draw heavily on the derivation of the KdV equation provided in Sec. 6.1 of Ref. 7. Then we show that the Marangoni effect is able to trigger the soliton excitation. Here, however, we do not discuss the stability and eventual sustainment or destruction of this excitation.

We consider a shallow horizontal liquid layer of thickness h initially at rest and subjected to a transverse thermal gradient. Disturbances upon the quiescent state obey the continuity and Navier-Stokes equations to which we add a first approximation, either Fourier's heat equation or Fick's mass diffusion equation. These equations are supplemented with the corresponding boundary conditions (BC) at the bottom and at the open surface. For simplicity we consider only a two-dimensional geometry with disturbance velocity components (u,w) along horizontal and vertical coordinates (x,z), respectively. θ denotes either temperature or surfactant concentration disturbance so that κ is either heat or mass diffusivity and β is the thermal or mass concentration gradient, positive when imposed from the liquid side. η and ν are the dynamic and kinematic viscosities, respectively, related by the equation $\eta=\rho\nu$, with ρ the liquid density. ζ denotes the interfacial deformation of amplitude a and horizontal extent l. ∇^2 represents the Laplacian that with subscript Σ restricts its action to the open surface. The shallow layer approximation demands that $h \ll 1$ and we can safely disregard buoyancy effects in the Navier-Stokes equations. Then we have the following boundary-value problem

$$\nabla\cdot\mathbf{v}=0\ , \tag{1}$$

$$\frac{\partial}{\partial t}\mathbf{v}+\mathbf{v}\cdot\nabla\mathbf{v}=-\frac{1}{\rho}\nabla p+\nu\nabla^2\mathbf{v}\ , \tag{2}$$

$$\frac{\partial}{\partial t}\theta=\beta w+\kappa\nabla^2\theta\ , \tag{3}$$

with BC: at the rigid "insulating" bottom $z=0$,

$$w=u=0\ , \tag{4}$$

$$\frac{\partial\theta}{\partial z}=0\ ; \tag{5}$$

and at the open deformable surface $z=h+\zeta$,

$$\frac{\partial}{\partial t}\zeta=w-u\frac{\partial}{\partial x}\zeta\ , \tag{6}$$

$$p=\rho g\zeta+2\eta\frac{\partial}{\partial z}w-\sigma_0\nabla^2_\Sigma\zeta\ , \tag{7}$$

$$\left[\frac{\partial}{\partial T}\sigma\right]\nabla_\Sigma(\theta-\beta\zeta)-\eta\left[\frac{\partial}{\partial z}u+\frac{\partial}{\partial x}w\right]=0\ , \tag{8}$$

$$\frac{\partial\theta}{\partial z}=0\ . \tag{9}$$

These BC correspond to some of the experiments with liquids on glass plates conducted many years ago by Linde and collaborators.[8]

The velocity v can be decomposed into its potential (φ) and rotational (ψ) parts

$$\mathbf{v}=\nabla\varphi+\nabla\times\boldsymbol{\psi}\ . \tag{10}$$

This allows one to introduce viscosity in an iterative way.

For the two-dimensional problem

$$\boldsymbol{\psi}=\psi \mathbf{j} , \tag{11}$$

where $\mathbf{j}$ is the unit vector perpendicular to the (x,z) plane. Thus

$$u=\frac{\partial\varphi}{\partial x}-\frac{\partial\psi}{\partial z} , \tag{12}$$

$$w=\frac{\partial\varphi}{\partial z}+\frac{\partial\psi}{\partial x} . \tag{13}$$

Noting that $\nabla^2\varphi=0$ and using $\varphi=\sum_{j=0}^{\infty} z^j\varphi_j$ we set

$$\varphi=\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j)!}z^{2j}\varphi_0^{(2j)} , \tag{14}$$

where the superscript $(2j)$ accounts for the $2j$th derivative with respect to x. On the other hand,

$$\psi=\sum_{i=0}^{\infty} z^i\psi_i . \tag{15}$$

The simplest choices to fit the BC are

$$\psi_0=0 , \quad \psi_1=\varphi_{0,x} ,$$

and $\psi_2=\psi_3=\cdots=0$ or

$$\psi=z\varphi_{0,x} . \tag{16}$$

The subscripts x, z, and t denote derivatives with respect to those variables. Thus

$$u=-\frac{z^2}{2}\varphi_0^{(3)}+\frac{z^4}{24}\varphi_0^{(5)}-\cdots , \tag{17}$$

$$w=\frac{z^3}{6}\varphi_0^{(4)}-\frac{z^5}{120}\varphi_0^{(6)}+\cdots . \tag{18}$$

Introducing the coordinate expansion

$$\theta=\theta_0+z\theta_1+z^2\theta_2+\cdots$$

into the heat equation and its BC, one finds that even powers disappear and furthermore $\theta_1=\theta_3=0$. Thus

$$\theta=z^5\theta_5=z^7\theta_7+\cdots ,$$

and the solution satisfying the BC is given by

$$\theta=z^5(\tfrac{7}{5}h^2-z^2)\theta_7+\cdots . \tag{19}$$

Differentiating (3) with respect to z and using the continuity equation gives $\kappa\theta_{zzz}=\beta u_x$, at $z=h$, so that finally (19) becomes

$$\theta=z^5(\tfrac{7}{5}h^2-z^2)\frac{\beta}{252h^2\kappa}\varphi_0^{(4)}+\cdots . \tag{20}$$

The nonlinear kinematic BC (6) then yields

$$\zeta_t=\tfrac{1}{6}h^3\varphi_0^{(4)}+\tfrac{1}{2}h^2\zeta\varphi_0^{(4)}-\tfrac{1}{120}h^5\varphi_0^{(6)}+\tfrac{1}{2}h^2\varphi_0^{(3)}\zeta_x . \tag{21}$$

Using (17) and (18) together with (8) in the horizontal component of (2) taken at the open surface, and denoting $f=\varphi_0^{(3)}$, gives

$$f_t-\frac{2g}{h^2}\zeta_x+2\frac{\sigma}{\rho h^2}\zeta_{xxx}+\frac{2}{h}\zeta f_t-\tfrac{1}{12}h^2f_{txx}-\tfrac{1}{6}h^2ff_x =2\nu(f_{xx}+f/h^2) , \tag{22}$$

and

$$\zeta_t=\tfrac{1}{6}h^3f_x+\tfrac{1}{2}h^2\zeta f_x-\tfrac{1}{120}h^5f_{xxx}+\tfrac{1}{2}h^2f\zeta_x . \tag{23}$$

Using now the BC (8), Eq. (22) becomes

$$f_t-\frac{2g}{h^2}\zeta_x+2\frac{\sigma}{\rho h^2}\zeta_{xxx}+\frac{2}{h}\zeta f_t-\tfrac{1}{12}h^2f_{txx}-\tfrac{1}{6}h^2ff_x =\frac{2\nu}{h^2}\left[\tfrac{4}{3}h^2f_{xx}-\frac{\partial\sigma}{\partial T}\frac{1}{\rho\nu h}(\theta_x-\beta\zeta_x)\right] . \tag{24}$$

We now nondimensionalize all variables as follows:

$$\theta=\beta h\Theta, \quad \zeta=a\xi, \quad t=\tau\sqrt{6}/C_0, \quad x=ly,$$

$$f=-\sqrt{6}\frac{aC_0}{h^3}V ,$$

with $C_0^2=2gh$, $\epsilon=a/h$, and $\delta=h/l$. We always consider the "shallow water" limit $\epsilon\ll 1$ and $\delta^2\ll 1$. Introducing the Bond, $\mathcal{B}=\rho gh^2/\sigma_0$, Marangoni, $\mathcal{M}=-(d\sigma/dT)\beta h^2/\eta\kappa$, and Reynolds numbers, $\mathcal{R}=C_0l/\nu$, we obtain

$$\xi_\tau+V_y+3\epsilon\xi V_y+3\epsilon V\xi_y-\tfrac{1}{20}\delta^2V_{yyy}=0 \tag{25}$$

and

$$V_\tau+\xi_y+2\epsilon\xi V_\tau+\epsilon VV_y-\frac{\delta^2}{12}V_{\tau yy}-\frac{\delta^2}{\mathcal{B}}\xi_{yyy} =\frac{\sqrt{6}}{315\mathcal{R}}(840+\mathcal{M})V_{yy} . \tag{26}$$

At zeroth order we have

$$V_{(0)}=\xi, \quad V_\tau+V_y=0 ,$$

so that the general solution can be of the form

$$V=V_{(0)}+\epsilon V_{(1)}+\delta^2V_{(2)}+\gamma V_{(3)} \tag{27}$$

with

$$\gamma=\frac{\sqrt{6}}{315\mathcal{R}}(840+\mathcal{M}) ,$$

which is considered to be small either because $\mathcal{R}$ is large or $\mathcal{M}$ is near -840. The latter is the case of interest here as we just like to identify the threshold value for KdV solitary-wave excitaion due to the Marangoni effect.

Introducing (27) in (25) and (26) and subtracting the latter two equations yield

$$\xi_\tau+\xi_y+\tfrac{5}{2}\epsilon\xi\xi_y-\frac{30-\mathcal{B}}{60\mathcal{B}}\delta^2\xi_{yyy}=\frac{\gamma}{2}\xi_{yy} , \tag{28}$$

where ξ_y can be removed by a Galilean transformation. Thus we see that upon setting $\gamma=0$, Eq. (32) provides the Korteweg-de Vries equation.[6,7] $\gamma=0$ demands either $\nu=0$, i.e., an inviscid liquid, which is not of interest in the present analysis or $\mathcal{M}=-840$. The latter result indicates that for such a negative value of the Marangoni number the open surface of the liquid layer is excitable in the form of a KdV soliton. Whether or not the soliton is stable can be decided only by studying the role of the omitted nonlinear part in Eq. (3). However, we can safely say that, due to surface-tension gradients, $\mathcal{M}=-840$ defines the onset of a possible soliton excitation in a quiescent shallow liquid layer subjected to a transverse thermal gradient.

When $\mathcal{M}$ is positive, one has Bénard convection and we know[5,8] that at $\mathcal{M}=48$ there is the onset of steady Bénard convection with a polygonal (mostly hexagonal) planform. Thus an experiment with a liquid layer heated from below could provide a clear-cut test of our prediction. One only requires a liquid having a minimum in the surface tension versus temperature curve.[9–12] Before the minimum, we expect steady convection and past the minimum the present analysis suggests the possibility of a traveling solitary wave. Note that since the KdV equation corresponds to a *genuine nonlinear excitation*, the experimenter must abruptly excite the liquid with, say, quite a sudden jump in the temperature gradient or with strong adsorption or evaporation in the case of a volatile surfactant as done many years ago by Linde.[8] Unfortunately, there is not enough quantitative information in Linde's results to compare with our predictions. Yet a recent analysis of his old film and pictures qualitatively support our findings.[13] Another possibility is to mechanically excite the interface, then with heat or mass transfer the Marangoni effect is expected to help sustain this excitation.

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