PHYSICAL REVIEW A

ATOMIC, MOLECULAR, AND OPTICAL PHYSICS

THIRD SERIES, VOLUME 43, NUMBER 1 1 JANUARY 1991

Quasinormal mode expansion and late-time behavior of leaking systems

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(Received 30 April 1990)

The wave function of a particle escaping from a barrier, in general, would have its quasinormal modes, which decay like $\exp(-\gamma t/2)$, dominated by anomalous $t^{-\alpha}$ terms at asymptotically late times. One implication of the existence of these anomalous terms is that the wave function cannot be expressed as a sum of quasinormal modes. We show that these anomalous terms are related to classical motion linking the initial point $(x',t=0)$ to the final point (x,t). We then show that when the potential is unbounded from below, such terms do not appear, and more importantly, for such potential, the time development of a wave function initially concentrated in the finite region can be expressed in terms of a summation over quasinormal modes.

I. INTRODUCTION

In this paper we discuss the leakage of the wave function of a particle out of a potential barrier $V(x)$, with emphasis on the difference between potentials that approach a constant, say zero, as $x \to \infty$ [Fig. 1(a)], and potentials which are unbounded below [Fig. 1(b)], the latter including potentials for which the escape is classically allowed, e.g., $V(x)=-x^2/2$. For simplicity we consider only potentials with $V(x)=+\infty$ for $x<0$, i.e., the particle is confined to a half line and escapes only to the right, so that x represents a positive "radial" coordinate.

Imagine first "switching off" the leakage by making the barrier infinitely high, then the potential well, say of characteristic spatial dimension a, has a sequence of bound states with energy $E_n^{(0)}$ and wave functions $\phi_n^{(0)}(x)$, which are concentrated inside the well, i.e., $\phi_n^{(0)}(x)\sim 0$ for $x \gg a$. Now if the barrier height is finite, some of these states become quasinormal modes, leaking out at some rate γ_n, i.e., the energy becomes $E_n - i\gamma_n/2$, where $E_n \sim E_n^{(0)}$. The corresponding wave functions $\phi_n(x)$, instead of vanishing at infinity, obey the outgoing wave boundary condition, i.e., (outgoing wave amplitude)/(incoming wave amplitude)$=\infty$, which defines the complex energies as poles of the S matrix.

The quasinormal modes $\phi_n(x)$ are close to the bound states $\phi_n^{(0)}x$ inside the well ($x \lesssim a$), and the latter is a complete basis for wave functions that tend to zero sufficiently rapidly at infinity, so it is natural to examine the following propositions.

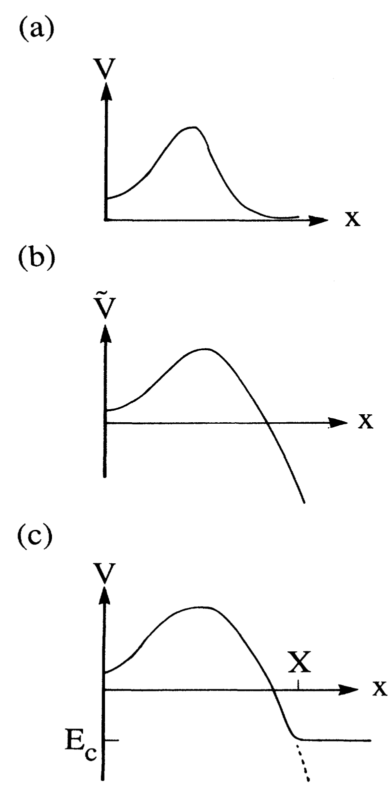

FIG. 1. (a) A typical bounded potential $V(x)$. (b) A potential unbounded below $\tilde{V}(x)$. (c) The potential $\tilde{V}(x)$ truncated to $V(x)$.

(A) Are the quasinormal modes (together with the bound states if any) complete for wave functions concentrated inside the well? More explicitly, we ask the question: For an initial wave function $\Psi(x,t=0)$ that is concentrated inside the well $[\Psi(x>a)=0]$, under what conditions would the time development of the wave function $\Psi(x,t)$ for finite x be given by

$$\Psi(x,t)=\sum_n a_n e^{-iE_n t} e^{-\gamma_n t/2}\phi_n(x)+\sum_i c_i\phi_i(x)e^{-iE_i t} \quad (1)$$

with the constants a_n being essentially the projection of $\Psi(x,t=0)$ on the quasinormal modes ϕ_n.[1] (In the presence of multiple poles,[2] in addition to the exponential time dependence, these terms may contain factor of powers of t.) The second sum in (1) represents the bound-state contribution. For simplicity of discussion, we henceforth do not consider the case where the potential inside the well is a global minimum, so that there is no bound-state contribution. The generalization to include bound states is trivial.

Before we study this question, we look in particular at a special case of (A).

(B) Under what conditions would the wave function $\Psi(x,t)$ for finite x as $t\to\infty$ be dominated by the lowest quasinormal mode with the smallest γ_n (in the absence of a bound state)? This question is interesting not only in its own right, but also guides us to the answer of (A). It is clear that the conditions in (B) are necessary conditions for (A), but not vice versa.

It has been shown,[3-5] at least for potentials vanishing at infinity, that for late times $\Psi(x,t)$ is dominated by anomalous terms going as $t^{-\alpha}$. The answer to (B) is negative and hence the expression (1) cannot be valid in general. Specific examples have been evaluated[3-5] and a similar phenomenon occurs in scattering.[6] These anomalous terms do not become important until very large times, and are in practice unobservable in experimental situations such as α decay. In Refs. 4–7, it has been pointed out that these terms are related to the long-wavelength components of the initial wave packet. We shall first study the physical origin of such terms, and show that they are absent for potentials that are unbounded below.

The renewed interest in this issue of t^{α} terms derives in part from quantum cosmology, where Ψ is the wave function of the universe in minisuperspace, x being the scale factor of the universe and t the conformal time, related to the proper time τ by $dt=d\tau/x(\tau)$. A model potential might be $V(x)=(x^2-\lambda x^4)/2$, where $\lambda>0$ is proportional to an effective cosmological constant generated by an inflaton field. The universe "escapes" from the quantum domain inside the well ($x\lesssim a\equiv 1/\sqrt{\lambda}$). To the extent that path integrals are equivalent to ordinary quantum mechanics, the problem is exactly analogous to that of a particle escaping from a barrier.[8,9] In quantum cosmology the proper time separation between three-geometries is unmeasurable in principle and has to be summed over in a path-integral sense, and the resulting wave function is dominated by paths with large t or τ.[9] If there is no $t^{-\alpha}$ terms, the wave function would be dominated by the slowest leaking quasinormal mode and hence would be completely determined by the dynamics of the system and independent of the initial data, a situation which is intuitively appealing. The issue therefore deserves a careful reexamination.

Apart from the motivation by quantum cosmology, potentials that are unbounded below arise naturally in another context. Field theory is often studied by perturbation in some coupling constant λ. The validity of the series in λ (or of any resummed series) relies crucially on analyticity in the λ plane. The case of time-independent perturbation for zero-dimensional field theory; i.e., the generalized anharmonic oscillator

$$H=-\frac{1}{2}\frac{d^2}{dx^2}+\tfrac{1}{2}(\omega^2x^2+\lambda x^{2N}) \quad (2)$$

has been studied in depth.[10] The behavior of high-order terms in the perturbation expansion is controlled by the tip of the cut for $\lambda=-|\lambda|$. A negative λ corresponds to an unbounded potential of the type of Fig. 1(b).

A potential that is unbounded from below [such as $\tilde{V}$ in Fig. 1(b)] can be regarded as the limit of a truncated potential [such as V in Fig. 1(c)], with $V(x)=E_c$ for $x>x_c$, where x_c is much larger than all relevant x values considered in the problem, and $|E_c|$ much larger than all energy scales in the problem. Then it is physically obvious that the truncation does not affect the physics. However, it is important to note that in this point of view, any $t\to\infty$ limit is to be taken *after* $x_c\to\infty$, $V_c\to\infty$.

The issue of completeness is important from another consideration. Imagine that a leaking potential $V(x)$ is slightly perturbed to $V(x)+\epsilon W(x)$, $|\epsilon|\ll 1$; assuming that the unperturbed problem is solvable, naturally one hopes to express the complex eigenvalues $E_n-i\gamma_n/2$ as a perturbative series in ϵ. The perturbation is nontrivial, since the quasinormal modes, with the outgoing wave condition at infinity, define a non-Hermitian system (in the sense that the total probability in any finite region of space is not conserved), to which the usual formalism of perturbation does not apply. The problem can be solved to *first* order in ϵ, both in quantum mechanics[11] and for the electromagnetic analog,[12] but in order to evaluate higher-order corrections one would need to expand the wave function in some complete set, preferably the discrete set of quasinormal-modes. The electromagnetic analog is of very general significance since any optical cavity permits the escapes of particles (i.e., photons) due to output coupling, so that there is always leakage of the wave function and the allowed frequencies are always complex.

The rest of this paper is organized as follows. Section II reviews the origin of $t^{-\alpha}$ terms for potentials that approach a constant at infinity, showing that these terms are related to the existence of a threshold. Section III then shows that for potentials unbounded below, there are not $t^{-\alpha}$ terms in the long-time behavior, essentially because there is no threshold. We prove that for wave functions concentrated in the finite region $[\Psi(x>x_0)=0$

for an $x_0 < \infty$], the quasinormal modes indeed form a complete set when the potential is unbounded from below, under the assumption that the scattering matrix S has no singularity other than poles. Section IV presents an example of unbounded potentials to illustrate the general properties. Section V is a brief conclusion.

II. BOUNDED POTENTIALS

We first study the case of a bounded potential in order to elucidate the origin of the anomalous $t^{-\alpha}$ terms,[3–7,13] concentrating on the $t \to \infty$ limit [proposition (B)] in this section. The existence of such terms in the $t \to \infty$ limit implies that the expansion (1) is impossible, while power-law behavior at finite t does not have the same implication. We assume $V(x) \to 0$ as $x \to \infty$; in fact, for simplicity we discuss only the case of $V(x)=0$ for $x > a$. The quantization volume is $(0,L)$, with $L \to \infty$ at the end being understood. Exact normalized eigenfunctions with *real* energy $E=q^2/2$ can be written as $(2L)^{-1/2}\phi(q,x)$, with

$$\phi(q,x)=S(q)\phi^{+}(q,x)-\phi^{-}(q,x) , \tag{3a}$$

where $\phi^{\pm}(q,x)$ are eigenfunctions with the same eigenvalue E, and

$$\phi^{\pm}(q,x)=e^{\pm iqx}, \quad x \geq a . \tag{3b}$$

In (3a), $S(q)=\exp[2i\delta(q)]$ and $\delta(q)$ is the phase shift. We have chosen $\hbar$ and the particle mass m to be unity. The initial wave function $\Psi_{\text{in}}(x)=\Psi(x,t=0)$ may be expanded as

$$\Psi_{\text{in}}(x)=\int_0^{\infty} dq \frac{1}{2\pi} c(q)\phi(q,x) , \tag{4}$$

where

$$c(q)=\int_0^a dx\, \Psi_{\text{in}}(x)\phi^*(q,x) , \tag{5}$$

and we assume that the particle is initially confined, i.e., $\Psi_{\text{in}}(x)=0$ for $x > a$. The subsequent evolution is given exactly by

$$\Psi(x,t)=\int_0^{\infty} dq \frac{1}{2\pi} c(q)\phi(q,x)e^{-iq^2t/2} \tag{6}$$

or, putting (5) into (6)

$$\Psi(x,t)=\int_0^a dx' G(x,x';t)\Psi_{\text{in}}(x') , \tag{7}$$

where the Green function is

$$G(x,x';t)=\int_0^{\infty} dq \frac{1}{2\pi}(e^{-2i\delta(q)}e^{-i\theta(q,x')}-e^{i\theta(q,x')}) \times (e^{2i\delta(q)}e^{i\theta(q,x)}-e^{-i\theta(q,x)}) \times e^{-iq^2t/2} , \tag{8}$$

and we have defined $\phi^{\pm}(q,x)=\exp[\pm i\theta(q,x)]$, so that $\theta(q,x)=qx$ for $x \geq a$.

A. Classical paths

At large t, $G(x,x';t)$ would be dominated by stationary phase contributions to (8), (see also Ref. 7) which occur at values of q satisfying

$$2\frac{d\delta(q)}{dq}+\frac{d}{dq}\theta(q,x)+\frac{d}{dq}\theta(q,x')-qt=0 , \tag{9a}$$

or

$$\frac{d}{dq}\theta(q,x)-\frac{d}{dq}\theta(q,x')-qt=0 . \tag{9b}$$

The other two possibilities are obtained by changing t to $-t$ in (9).

For the moment exclude the case $q=0$, which will be discussed separately below. Since $\Psi(x>a,0)=0$, we let $x' < a < x$, i.e., we consider only the propagation from inside the potential to the outside. (The case $x < a$ involves no complication and can be treated similarly.) We seek solutions to (9) for $t \to \infty$, and since $d\delta/dq$, $d\theta/dq$ are bounded for finite q, it remains to examine $q \to \infty$. In that limit, the WKB approximation is valid, so that

$$\theta(q,x')=qa-\int_{x'}^{a}\sqrt{2[E-V(y)]}dy , \tag{10}$$

$$\frac{d}{dq}\theta(q,x')=a-q\int_{x'}^{a}\frac{1}{\sqrt{2[E-V(y)]}}dy \equiv a-q\tau(q;x' \to a) , \tag{11}$$

where $\tau(q;x' \to a)$ is obviously the classical time for motion from x' to a at energy $q^2/2$. Recalling that $d\theta(q,x)/dq=x$ for $x>a$ and defining $\tau_D=(2/q)d\delta(q)/dq$, we see that (9a) can be written as

$$t=\frac{2a}{q}+\tau_D+\frac{x-a}{q}-\tau(q,x' \to a) . \tag{12a}$$

The interpretation of this formula is as follows. Consider a motion starting from a, moving to the left and returning to a. In the absence of a potential, the time taken is $2a/q$ and the potential introduces a delay τ_D. Now for motion from x' to x, we need to add the time from a to x, namely, $\tau(q,a \to x)=(x-a)/q$ and also subtract the time from x' to a, namely, $\tau(q,x' \to a)$. Thus the right-hand side of (12a) is the classical time for the motion shown in Fig. 2(a), in which the particle goes inwards to the potential before moving to large x. Likewise (9b) can be written as

$$t=\tau(q,x' \to a)+\frac{x-a}{q} , \tag{12b}$$

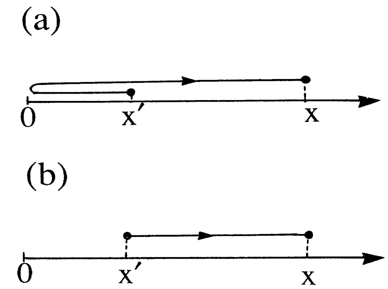

FIG. 2. Classical paths lining $(x',t'=0)$ and (x,t), corresponding to Eqs. (9a) and (9b).

which is the classical time for the motion shown in Fig. 2(b), which goes directly from x' to x, and for x' outside of the potential, would be independent of the potential. Thus the stationary phase contribution (if any) is related to classical paths linking $(x',t=0)$ to (x,t).

It is easy to see that these classical path contributions do give $t^{-\alpha}$ terms. For $t\to\infty$, the stationary phase appears at $q=q_x\equiv x/t$ for both (12a) and (12b). (Remember $q\neq 0$ and both a and x' are finite, hence τ is finite). Integration of (6) about $q=q_x$ produces terms of the form $t^{-1/2}\exp[ix^2/(2t)]$. However, these classical paths linking $(x',t=0)$ to (x,t) as $t\to\infty$ are possible only if $x\to\infty$ (for $q\neq 0$), again for both (12a) and (12b). Hence the $t\to\infty$ limit of $\Psi(x,t)$ with finite x [proposition (B)] contains no stationary phase contributions for $q\neq 0$.

It remains to examine the contribution at $q=0$. In this case, the WKB expressions (10) and (11) are not valid. We return to (8). The integral in (8) is carried along the positive q axis; as usual, we rotate the contour so that it passes along the line of steepest descent through the stationary point, i.e., along the line of $q=e^{-i\pi/4}p$, p real and positive, as illustrated in Fig. 3. The rotation of contour picks up the poles in the shaded region. Examining (8), we see that poles can only arise from $S=e^{2i\delta}$, and occur say at $E_n-i\gamma_n/2$ in the E plane. The residues are associated with exponential time dependence and produce no anomalous terms [they are, in fact, the quasinormal mode terms in (1)]; it remains to examine the $q=0$ threshold contribution along the rotated contour.

B. Threshold contribution

The threshold contributions depend on the $q\to 0$ behavior of $c(q)$ and $S(q)$. Setting

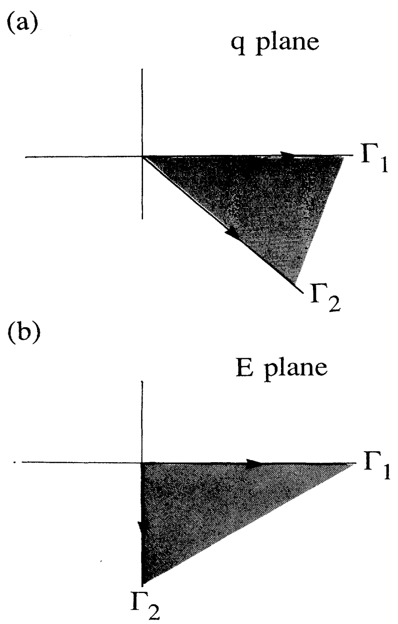

FIG. 3. The original path Γ_1 and the new contour Γ_2 in (a) the q plane, (b) the E plane.

$$c(q)\sim c_0 q^{\sigma},\quad S(q)-1\sim S_0 q^{\sigma'}$$

then the threshold contribution to (6), i.e., the integral along the rotated contour, can be written as the sum of two terms

$$\Psi_1(x,t)=\int_{\Gamma_2} dq\frac{1}{2\pi}c(q)(e^{iqx}-e^{-iqx})e^{-iq^2t/2}\sim A_1 x t^{-\mu} \tag{13a}$$

where

$$A_1=e^{-i\pi\sigma/4}2^{\sigma/2}\Gamma(\mu)(c_0/\pi) \tag{13b}$$

and $\mu=1+\sigma/2$;

$$\Psi_2(x,t)=\int_{\Gamma_2} dq\frac{1}{2\pi}c(q)[S(q)-1]e^{iqx}e^{-iq^2t/2}\sim A_2 t^{-\mu'} \tag{14a}$$

where

$$A_2=e^{-i\pi\mu'/2}2^{\mu}\Gamma(\mu')(c_0S_0/\pi) \tag{14b}$$

and $\mu'=(1+\sigma+\sigma')/2$.

Appendix A shows that the generic case is $\sigma=\sigma'=1$, so that both Ψ_1 and Ψ_2 vary as $t^{-3/2}$ as $t\to\infty$. Scattering can be treated in exactly the same manner, except that the integral (7) should be taken in the region $(R-b,R+b)$, where the incident wave packet occupies an interval of length $2b$ at a large distance R from the potential.[14]

III. UNBOUNDED POTENTIAL

For a potential unbounded from below, such as that in Fig. 1(b), there is simply no threshold, so from the discussion in the last section, we see that there will be no $t^{-\alpha}$ terms in the $t\to\infty$ limit [proposition (B)]. This suggests that the expression (1) might be valid [proposition (A)] for potential unbounded from below. In this section we prove that this is the case.

For potential unbounded from below, the wave number

$$k(x)\equiv\sqrt{2[E-V(x)]} \tag{15}$$

does not approach a constant at large x, and is not a convenient parameter to label the incoming and outgoing modes. We parametrize them in terms of their energy eigenvalue E, with $\phi^{\pm}(E,x)$ defined to be the solutions to the (time-independent) Schrödinger equation which satisfy the boundary conditions

$$\lim_{x\to\infty}\phi^{\pm}(E,x)\sqrt{k(x)}\exp\left[\mp i\int^{x}k(x)dx\right]=1\ . \tag{16}$$

For potential unbounded from below, the WKB approximation is valid for large x, with any fixed E. Hence Eq. (16) is the appropriate incoming and outgoing boundary conditions.

In energy representation, the Green function (8) becomes

$$G(x,x',t)=\int_{-\infty}^{\infty}dE\frac{1}{2\pi}[S(E)^{-1}\phi^{-}(E,x')-\phi^{+}(E,x')]\times[S(E)\phi^{+}(E,x)-\phi^{-}(E,x)]e^{-iEt}\ . \tag{17}$$

It is important to note that the range of integration in E starts from $-\infty$ (instead of from a finite constant, e.g., zero, as in the case of a bounded potential), since the potential $V(x)$ is unbounded from below. To evaluate (17) we shall let the lower limit be E_c and let $E_c \to -\infty$ at the end (the contour Γ_1 in Fig. 4).

The wave function is given by (7) with the Green function given by (17). To show that this wave function can be expanded as in (1), we deform the contour to $\Gamma_2+\Gamma_3$ in Fig. 4. The wave function is then given by the integrations along Γ_2, Γ_3, and the residues of the poles in the lower-half E plane. In the following, we evaluate these three contributions separately.

(i) On the contour Γ_3, E is given by $E_c+i\rho$, where $\rho\in(E_c,0)$ is a real number. The contribution of the integral along Γ_3 to $\Psi(x,t)$ in (7) is

$$\Psi_{\Gamma_3}=\lim_{E_c\to-\infty} i\int_{E_c}^{0}d\rho\frac{1}{2\pi}\phi(E_c+i\rho,x)\times\int_0^a dx'\Psi_{\rm in}(x')\phi^*(E_c+i\rho,x')\times e^{(-iE_c+\rho)t} . \tag{18}$$

In the limit $E_c\to-\infty$, $\phi^*(E_c+i\rho,x)$ in the dx' integration from 0 to a (where the potential is finite) is given by the WKB expression. Since ϕ^* is zero at $x'=0$ and is normalized by (16), it must be exponentially small (in $|E_c|$) in the interval $x'\in(0,a)$. Hence the dx' integration gives zero in the limit $E_c\to-\infty$. The physical reason for this is obvious: the initial wave function $\Psi_{\rm in}$ concentrated in the finite region has no overlap with the arbitrarily negative energy scattering states.

(ii) We next evaluate the integration along Γ_2. We first note that $S(E)$ remains finite on Γ_2 as $E_c\to-\infty$ [point (1) in Appendix B]. In (17), both $\phi^{\pm}(E,x)$ diverge at most like $|E|^{-1/4}\exp(b|E|^{1/2})$ as $|E|\to\infty$, with some constant b. Therefore the Green function (17) is dominated by the $\exp(-iEt)$ factor which goes to zero as $|E|\to\infty$ in the lower-half E plane.

(iii) Therefore the wave function (7) with the Green function (17) is determined solely by the singular points of the Green function in the lower-half E plane. It remains to show that they are the singular points of $S(E)$. We first express the Green function in terms of

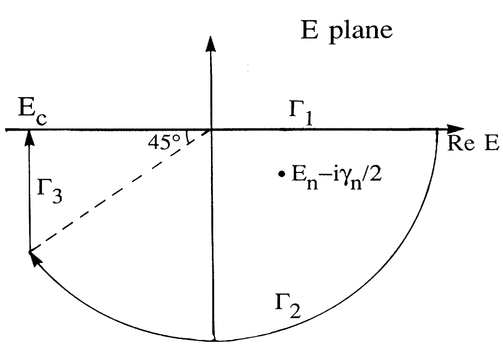

FIG. 4. Integration contour for potentials which are unbounded below.

$$\tilde{\phi}(E,x)\equiv f(E)\phi^+(E,x)-g(E)\phi^-(E,x) , \tag{19a}$$

where

$$f(E)\equiv\phi^-(E,0),\quad g(E)=\phi^+(E,0) \tag{19b}$$

are the Jost functions[15] in E representation. Since $\phi^{\pm}(E,x)$ satisfy the same equation with a real potential, and with the boundary conditions (16), we have $\phi^+(E,x)=[\phi^-(E^*,x)]^*$, which leads to

$$f(E)=g^*(E^*) . \tag{20}$$

From the ratio of the outgoing and incoming waves in (19), we we see that $S(E)$ is given by

$$S(E)=\frac{f(E)}{g(E)}=\frac{f(E)}{f^*(E^*)} . \tag{21}$$

$\tilde{\phi}(E,x)$ is a solution to the Schrödinger equation with the properties [point (2) in Appendix B]

$$\tilde{\phi}(E,0)=0 , \tag{22a}$$

$$\left.\frac{d\tilde{\phi}}{dx}\right|_{x=0}=2i . \tag{22b}$$

Therefore $\tilde{\phi}(E,x)$ satisfies the (time-independent) Schrödinger equation with boundary conditions independent of E, implying[16] that $\tilde{\phi}(E,x)$ is an entire function of E. In terms of $\tilde{\phi}(E,x)$ the Green function (17) is

$$G(x,x',t)=-\lim_{E_c\to-\infty}\int_{E_c}^{\infty}dE\frac{1}{2\pi}\frac{S(E)}{[f(E)]^2}\tilde{\phi}(E,x)\tilde{\phi}(E,x') . \tag{23}$$

It is easy to see that all the poles of $S(E)$ lie in the lower-half E plane [point (3) in Appendix B], therefore we have (i) by (21), $f(E)$ cannot have any zero in the lower-half plane, and the only singularities of the integrand in (23) are those of $S(E)$, and also (ii) all the poles of $S(E)$ are picked up by the distributions of contour $\Gamma_1\to\Gamma_2+\Gamma_3$.

Therefore we see that for a potential unbounded from below, a wave function initially concentrated in the finite region can be expanded as in (1), where each term represents the contribution of a pole of the scattering matrix $S(E)$, i.e., a quasinormal mode contribution. Notice that we have assumed that $S(E)$ has only poles. From the point of view that the unbounded potential is regarded as the limit of a truncated potential with $V(x>x_c)$ going to a constant [Fig. 1(c) at large x_c and $|V(E>x_c)|$ much larger than any energy scale in the problem, that $S(E)$ has only poles is guaranteed by the analytic property of $S(E)$, see, e.g. Newton.[15] We note that a truncation of the potential at large x cannot be used in the usual scattering problems with asymptotically flat potential, since (i) the region of interest in these scattering problems is exactly at large x, and (ii) the change involves an energy scale under consideration, namely, $E\approx0$. Such are not the case with the potential unbounded from below. We further note that even if $S(E)$ has singularities other than poles in the lower-half E plane, the expansion (1) still has the same form with the exponential time dependence, although they may not correspond to the usual quasinormal modes. If $S(E)$ has cut in the lower-half plane, part of the sum in (1) becomes an integral.

It must be emphasized that the derivation above is valid for *any* $t>0$. Taking $t\to 0+$ in (1) then shows that the quasinormal modes form a complete set for wave functions concentrated in the finite region.

IV. EXAMPLES AND DISCUSSIONS

In this section we use the example of the generalized anharmonic oscillator described by Eq. (2) to illustrate the expansion in quasinormal modes, some of its implications, and the late-time behavior of a leaky system. For $\lambda<0$ and $N>1$, Eq. (2) describes a leaky system; any initial wave function initially concentrated in the finite region will eventually escape to infinity. We shall look at various values of ω^2 and λ.

We first note that for $\lambda>0$, the potential $V(x)\to +\infty$ as $|x|\to\infty$, so there is a complete set of bound states $\phi_n(\lambda,x)$ with real energies $E_n(\lambda)$, satisfying $\phi_n(\lambda,x)\to 0$ as $|x|\to\infty$. Then[17]

$$\sum_n \phi_n(\lambda,x)\phi_n(\lambda,x')=\delta(x-x'),\quad \lambda>0\ , \tag{24}$$

this equation being understood in a weak sense over the space of functions satisfying the appropriate boundary condition. More generally

$$\sum_n \phi_n(\lambda,x)\phi_n(\lambda,x')e^{-iE_n(\lambda)t}=G(\lambda;x,x';t)\ , \tag{25}$$

where we have explicitly indicated the dependence of the Green function G on λ.

Now for $\lambda<0$, the potential becomes unbounded below; then by the derivation in Sec. III, we find G to be given by a sum of quasinormal-mode contributions. This implies that (25) holds for negative λ as well, provided each quantity in (25) [e.g., $E_n(\lambda)$] is defined by analytic continuation from $\lambda>0$. In particular, each quasinormal-mode function is the analytic continuation of the corresponding bound-state function $\phi_n(\lambda,x)$, $\lambda>0$.

That the quasinormal-mode functions are the analytic continuation of the bound-state functions for the generalized harmonic potential has been shown by Bender.[10] For a general potential containing a parameter λ, whether the continuation is possible or not depends on the singularities on the λ plane. It is interesting to ask under what conditions would such a continuation be possible, especially in view of our present result that the expansion (25) of the Green function (again in a weak sense) in terms of the quasinormal modes is valid for potentials unbounded from below.

To study the late-time behavior of the wave function, we look at a particular case of (2). We let $\lambda=0$ and $\omega^2=-\Omega^2<0$. This is an inverted harmonic oscillator, and the Green function is known analytically.

We start with the "usual" harmonic oscillator with $\omega^2>0$, whose Green function G' is

$$G'=\left[\frac{\omega}{2\pi i\,\sin\omega t}\right]^{1/2}\times\exp\left[\frac{i\omega}{2\,\sin\omega t}[\cos(\omega t)(x^2+x'^2)-2xx']\right]. \tag{26a}$$

The Green function G for the inverted oscillator is obtained by simply continuing to $\omega=-i\Omega$:

$$G=\left[\frac{\Omega}{2\pi i\,\sinh(\Omega t)}\right]^{1/2}\times\exp\left[\frac{i\Omega}{2\,\sinh(\Omega t)}[\cosh(\Omega t)(x^2+x'^2)-2xx']\right]. \tag{26b}$$

Despite the appearance of the square root in (26a), there is in fact no branch cut and analytic continuation is straightforward. Continuation to $\omega=+i\Omega$ gives the same result. The large t limit of (26b) is

$$G\sim\left[\frac{\Omega}{i\pi}\right]^{1/2}e^{-\Omega t/2}e^{i\Omega(x^2+x'^2)/2} \tag{27}$$

showing explicitly that there are no $t^{-\alpha}$ terms. Moreover, the resulting asymptotic $\Psi(x,t)$ has the form given by the leading term in the expansion (1), with

$$a_0=\left[\frac{\Omega}{i\pi}\right]^{1/2}\int dx'e^{i\Omega x'^2}\Psi_{\rm in}(x')\ , \tag{28}$$

$$\phi_0(x)=e^{i\Omega x^2/2}\ , \tag{29}$$

$$E_0=0,\quad \gamma_0=\Omega\ . \tag{30}$$

To show that ϕ_0 is a quasinormal mode with energy $E_0-i\gamma_0/2=-i\Omega/2$, one can verify directly that $H\phi_0=(-i\Omega/2)\phi_0$, which is just the analytic continuation $\omega\to -i\Omega$ of the Schrödinger equation for the ground state of the "usual" harmonic oscillator.

Two further properties of (28)–(30) are worthy of note. Since there is no barrier in this example, the particle does not "bounce" back and forth inside a well; the absence of periodic motion is the reason for $E_0=0$. Secondly, the projection (28) involves the integrand $\phi_0\Psi_{\rm in}$ and *not* $\phi_0^*\Psi_{\rm in}$, which is a general feature of such non-Hermitian systems.[11,12]

Incidentally, the inverted harmonic oscillator can also be regarded as a half-line problem if we start with an odd $\Psi_{\rm in}(x')$, thus ensuring that $\Psi(x=0,t)=0$ for all t. In this case a_0 in (28) is zero and the lowest quasinormal mode is the analytic continuation of the first excited state.

We should add that the Green function G' of the "usual" harmonic oscillator has no limit as $t\to\infty$ [see (24a)], because the particle oscillates indefinitely inside the well. The existence of the $t\to\infty$ limit is unique to leaking systems, and the corresponding path integral is well defined without the need for *ad hoc* prescriptions.[9]

V. CONCLUSION

We have shown that for potentials that are unbounded from below, the time development of a wave function initially concentrated in the finite x region can be expanded in terms of quasinormal modes [Eq. (1)]. We have proved the result for the case in which the particle is restricted to a half line ($x>0$), so that x represents a positive "radial" coordinate. The result can be extended to the full line, provided the potential is also bounded from *above;* or if the potential is unbounded from above, then $V(x)$ must tend to infinity rapidly enough. A detailed discussion of these generalizations will be given elsewhere.

We have also analyzed the origin of $t^{-\alpha}$ terms in the asymptotic wave function describing a particle escaping from a potential barrier. Mathematically, these terms arise from singularities on the real energy axis. Physically they are due to either an energy threshold or a classical path linking the initial point x' to the observation point x and taking time $t\to\infty$. The analysis of the classical motion shows that for potentials unbounded from below, anomalous terms do not exist. In such cases, the long-time behavior is dominated by the quasi-ground-state and one is in the fortunate position of knowing the eventual wave function even when the initial wave function is not specified in detail.

An analogy with classical mechanics may be suggestive. For a classical Hamiltonian system, if the initial state is specified only within a certain phase-space volume V, then by Liouville's theorem the final state is equally uncertain, and lies anywhere within a phase volume V' equal in magnitude to V: one cannot predict the future without knowing the past. However, for classical dissipative systems, the phase volume contracts, $V'\ll V$, and the final state can often be predicted even when the initial state is unknown. Quantum mechanics is of course a Hamiltonian system, but whenever the system is leaky, i.e., the wave function can escape to infinity, the finite parts of coordinate space behave essentially like a dissipative system, in that probability and energy are continually lost to infinity. The behavior may then resemble that of a classical dissipative system: we can predict the future without knowing the past. This point is of particular interest in quantum cosmology, since the initial condition of the universe is unknown.

ACKNOWLEDGMENTS

We thank Carl. M. Bender and Clifford M. Will for discussions, P. T. Leung for discussions and a reading of the manuscript. This work is supported in part by National Science Foundation Grant No. PHY 89-06286.

APPENDIX A

Here we show that generically $\sigma=\sigma'=1$. Let $\Phi(q,x)$ be the solution to the time-independent Schrödinger equation with energy $q^2/2$ and boundary condition $\Phi(q,0)=0$, $\Phi'(q,0)=1$. (This would be the appropriate condition if one imagines integrating the differential equation numerically from $x=0$.) Thus $\phi(q,x)$ in (3) differs only by a normalization factor: $\phi(q,x)=\alpha(q)\Phi(q,x)$. Now matching this definition to (3) at $x=a$ gives for $q\to 0$

$$\alpha(q)\Phi(q,s)=S(q)e^{iqa}-e^{iqa}\sim[S(0)-1]+O(q)\ , \qquad \text{(A1)}$$

$$\alpha(q)\Phi'(q,a)=iq[S(q)e^{iqa}+e^{-iqa}]\sim iq[S(0)+1]+O(q^2)\ . \qquad \text{(A2)}$$

Generically $\Phi'(q,a)\neq 0$ as $q\to 0$, thus we see from (A2) that $\alpha(q)=O(q)$, and hence from (A1) that $S(q)-1=O(q)$, or $\sigma'=1$.

Next from (5) we have

$$c(q)=\alpha(q)\int_0^a dx\ \Psi_{\rm in}(x)\Phi(q,x) \qquad \text{(A3)}$$

showing that as $q\to 0, c(q)\propto\alpha(q)=O(q)$, i.e., $\sigma=1$.

APPENDIX B

In this appendix we demonstrate the following properties of $\phi^{\pm}(E,x)$ and $S(E)$ used in Sec. III.

(1) $S(E)$ is finite in the $|E|\to\infty$ limit.

To see this, we first note that the WKB approximation for $\phi(E,x)$ is valid in the $|E|\to\infty$ limit. For small x, ϕ is a combination of exponentially increasing (in x) and decreasing components with nearly the same amplitudes [since $\phi(x=0)=0$]. For large $|E_c|$, the classical turning point (for $E_c<0$) is at large x, and the exponentially decreasing (in x) component is strongly suppressed there. From the WKB connection formula, we see that in the classically allowed region [where $E_c>V(x)$ with large but finite $|E_c|$], $\phi(E,x)$ cannot be purely outgoing wave. This implies that $S(E)$ cannot be infinite. Physically this means that there is not quasinormal mode with energy E_c much lower than the bottom of the potential well in the finite x region.

(2) The $\tilde{\phi}(E,x)$ defined by Eqs. (19) satisfied Eqs. (22).

The property (22a) follows directly from the definitions of $f(E)$ and $g(E)$ in (19b). Property (22b) can be obtained by considering the Wronskian of $\phi^{\pm}$,

$$\phi^{-}(E,x)\frac{d}{dx}\phi^{+}(E,x)-\phi^{+}(E,x)\frac{d}{dx}\phi^{-}(E,x)=2i\ .$$

Since $\phi^{\pm}(E,x)$ satisfies the same linear second-order equation, the Wronskian is x independent and we have evaluated it at $x\to\infty$ using Eq. (16). Likewise we have, using (19),

$$\phi^{-}(E,x)\frac{d}{dx}\tilde{\phi}(E,x)-\tilde{\phi}(E,x)\frac{d}{dx}\phi^{-}(E,x)=2if(E)\ .$$

Evaluating this equation at $x=0$ and using (22a), we get (22b).

(3) The poles of $S(E)$ are located in the lower-half E plane.

To see this we apply the continuity equation to $\tilde{\phi}(E,x)$ defined by Eqs. (19). Let

$$\tilde{\Psi}_E(x,t)=\tilde{\phi}(E,x)e^{-iEt}\ .$$

Then, by using the time-dependent Schrödinger equation,

$$i\frac{\partial}{\partial t}\int_0^x |\tilde{\Psi}_E|^2 dx = -\tfrac{1}{2}(\tilde{\Psi}_E^* \tilde{\Psi}_E' - \tilde{\Psi}_E^{*\prime} \tilde{\Psi}_E) \ . \qquad \text{(B1)}$$

Near a pole $E = E_1 + iE_2$ of $S(E)$ in the complex E plane,

$$\tilde{\Psi}_E \propto S(E)\phi^+(E,x)e^{-iEt} \approx S(E) \times \exp\left[i\int^x \sqrt{2(E-V)}dx \right] e^{-iEt} \ ,$$

where we have used the WKB approximation for ϕ^+ which is valid at large x. Putting this into the continuity equation (B1), we find

$$2E_2 \int_0^x |\phi^+|^2 dx = -\tfrac{1}{2}\{\sqrt{2(E-V)} + [\sqrt{2(E-V)}]^*\} \times \exp\left[-2\,\mathrm{Im}\int^x \sqrt{2(E-V)}dx \right] \ . \qquad \text{(B2)}$$

If we denote

$$\sqrt{2(E-V)} = |2(E_1 + iE_2 - V)|^{1/2} e^{i\theta_{E-V}}$$

then

$$\tan\theta_{E-V} = \frac{E_2}{E_1 - V}$$

goes to zero as $V(x) \to -\infty$. The term in parentheses in Eq. (B1) above is positive. Therefore from Eq. (B2), we see that E_2 must be negative and the pole is in the lower-half E plane.

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[14]The above discussion assumed that $\Psi_{\rm in}(x')$ has compact support, so that $c(q)$ as defined in (5) is analytic except at singularities of $S(q)=e^{2i\delta(q)}$. In contrast, one might imagine that $c(q)$ has compact support, say on (q_1,q_2); then $\Psi_{\rm in}(x')$ extends over all space. In that case, the behavior in t of $\Psi(x,t)$ is in general different from that at *fixed* x' just derived, because the $t\to\infty$ limit of (7) would not be uniform in x'. It would then be necessary to go back to (6) and evaluate the integral directly without contour rotation [since $c(q)$ is obviously nonanalytic in this case]. If $c(q)\sim(q-q_1)^{\sigma''}$ as $q\to q_1$, then there will be $t^{-\sigma}$ contributions to $\Psi(x,t)$ as $t\to\infty$, and similarly for the singularity at $q\to q_2$. However, this case is not physically relevant, since in all cases of interest, the particle is initially confined, i.e., $\Psi_{\rm in}(x')$ has compact support.

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