Time evolution of plasmas using the exact solution of the Boltzmann equation in the presence of an intense laser field

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In a recent paper, using the Volkov solution of the minimally coupled Dirac equation, we developed the concept of the quasifree states of the electron from a kinematical point of view [Rashid, Phys. Rev. A **38**, 2525 (1988)]. Using these concepts, we present a coupling (decoupling) scheme that separates the kinematical factors from the dynamical ones. When the electrons in a gas are assumed to be in quantum states having quasi-four-momentum, then the Boltzmann equation reduces to a form that lends itself easily to the methods of separation of the variables. The solution is obtained as the product of a time-dependent part and a time-independent part. The latter is assumed to be just the initial distribution function, in which case the time-dependent part is found to be linked to the scattering cross section of the electrons. We assume a high-Z plasma where electron-ion collisions dominate over electron-electron collisions. The complete solution is finally obtained by normalizing the product solution. The first case we consider is a cold plasma, so that the energy of the photon is much greater than the mean kinetic energy of the electron and the electrons are assumed to form a Maxwellian gas. The cross section calculated under the Born approximation from our previous paper is used and the laser intensity is assumed to be such that the Kibble parameter is much smaller than unity. The time-dependent solution is surprisingly found to be Maxwellian still but with the temperature increasing with time. The second case concerns a hot plasma with a mean electron energy much greater than the photon energy but still less than its rest mass energy. The final normalization gives us a non-Maxwellian distribution but is reduced to an analytically closed form so that it is easily tractable. The temperature is still found to be increasing, but now we get a difference in the time dependence between the amplitude part and the exponential part.

I. INTRODUCTION

In the field of laser fusion the optimization of the heating rate of the electrons due to multiphoton absorption is an important area of research.[1] Prior to Langdon's analysis[2] authors have calculated the heating rate W of the electrons in the plasma[3-8] basically using the Maxwellian distribution for both the initial and the final states of the electron without taking into account the nonlinear modifications introduced by the high-intensity laser field. In a recent paper, we[1] discovered that, due to the effect of the spin of the electron, the heating rate increases substantially with the temperature of the plasma when $k_B T > 100$ eV. This discovery is of importance in the field of laser fusion and also in the area of laser diagnostics of high-temperature plasma. The computation for the isotropic part of the electron distribution is assumed to be Maxwellian, which has been shown by Jones and Lee[9] to be a self-similar solution if the anisotropic part of the electron distribution is ignored.

The fact remains that as of yet we do not know of any solution to the Boltzmann equation when the laser field is intense enough so that multiphoton absorption sets in. It is characterized by the parameter $\chi = E_q / E_{\text{ph}}$, where E_q is the average electron energy in the radiation field and $E_{\text{ph}} = \hbar\omega$ is the energy of one photon. The latest efforts are still based on the expansion of the distribution function in an infinite series[10] or using the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy.[11] In either case, the end result is a string of expressions that obscures rather than clarifies the physics of the problem. Further restrictions and approximations do reduce the final answer to a decent form but then the result is valid only for a narrow range of values of the parameters.

The present analysis is an attempt to break away from this line of thought and to approach the problem of solving the Boltzman equation for a plasma in an intense laser field from another angle that is based on our previous presentation.[1] In Sec. II, instead of considering the free electron and the laser field we look at the dressed electron, which has the quasimomentum $\mathbf{p}^*$ and the effective mass m^*. The quantities marked with an asterisk are actually a time average of the equivalent quantum-mechanical observables so that now the electrons can be considered to be in quasifree states. We then introduce the modified Boltzmann equation based on our picture of quasifree electrons interacting with the Coulomb field of the ions. Assuming the ions to be relatively stationary, the modified Boltzmann equation then reduces to a form that can easily be solved using the separation of variables technique. The solution is obtained as the product of a time-independent part and a time-dependent part that we also assume to be dependent on the initial quasimomentum p^*.

In Sec. III, we apply the solution to the case of a nonrelativistic electron gas having a modified Maxwellian distribution,[1] the modification being due to the presence of the intense laser field and is taken in a time-averaged manner. Using the fact that the plasma is cold and that the laser field is intense, so that the absorbed photon energy is much greater than the average electron energy, we can compute the time-dependent part of the Boltzmann equation solution and finally get the time-evolving distribution.

In Sec. IV we present the complete result and the time development of the modified Maxwellian distribution for different laser-field intensities. We compare our result with that obtained by Jones and Lee.[9] In conclusion we discuss the implication of our result and present some possible problems to be tackled and solved in the future.

II. MODIFIED BOLTZMANN EQUATION AND ITS EXACT SOLUTION

We assume that the plasma is cold and quasineutral so that the electrons have random free motion between collisions with the ions within the plasma but the ions are relatively stationary. Hindsight helps us in assuming a shielded Coulomb field for the ions so that our final answer is finite. We are justified in doing this since the Debye shielding effect should be taken into account. We further assume that, since the laser field is designed to cover the whole of the plasma interaction region, the electrons are inside the laser field before and after the collision with the ion and hence we consider the electrons to be in Volkov states.[1] We treat the electron relativistically in order to take into account the relativistic effect of the laser field. We use the Lorentz-Heaviside units[12] with $\hbar=c=1$ and the metric $g^{\mu\nu}=(1,-1,-1,-1)$. The laser beam can be represented by a classical monochromatic field the amplitude of which is

$$A_\mu=(0,\mathbf{A})=A_0\cos(k\cdot x)\ , \qquad (2.1)$$

where

$$k_\mu=(k_0,\mathbf{k})=|\mathbf{k}|(n_0,\mathbf{n}) \qquad (2.2)$$

and the gauge is $\epsilon\cdot k=0$. The exact solution for the electron in the intense laser field A_μ is given by the Volkov state[13]

$$\Psi_i=(m/E_i)^{1/2}\left[1+\frac{e}{2n\cdot p_i}\hat{n}\hat{A}\right]u_i e^{-i(p_i\cdot x-S_i)} \qquad (2.3)$$

where u_i is a spinor satisfying the normalization condition

$$u_i^\dagger\cdot u_i=|E_i|/m \qquad (2.4)$$

and

$$S_i=(2n\cdot p_i)^{-1}\int_{-\infty}^{+n\cdot x}[2ep_i\cdot A-(eA)^2]dy\ . \qquad (2.5)$$

The subscript i indicates the incident electron and the caret over the four-vector implies a dot product between the vector and the γ matrices.[12] The expression (2.3) is an exact solution to the minimally coupled Dirac equation.

We now compute the observables for the electron inside the laser field by introducing the time-averaging method.[1] For the velocity of the electron we use the α-matrix operators in the following way,[12]

$$\langle\alpha\rangle_t=\langle\Psi_p^\dagger\alpha\Psi_p\rangle_t\ , \qquad (2.6)$$

where the subscript t denotes taking the average over the time period and ψ_p is the Volkov solution (2.3). One can easily show that

$$\langle\boldsymbol{\alpha}\rangle_t=(1/E)\{\mathbf{p}+[e^2A_0^2/(4k\cdot p)]\mathbf{k}\}\ . \qquad (2.7)$$

If we equate this to $\mathbf{p}^*/E$ we would define the quasi-three-momentum $\mathbf{p}^*$. Similarly, one can prove that

$$\langle\mathbb{1}\rangle_t=\langle\Psi_p^\dagger\mathbb{1}\Psi_p\rangle=1+[e^2A_0^2/(4k\cdot p)]k^0p^0=p^{*0}/p^0\ , \qquad (2.8)$$

where a quasienergy p^{*0} has been defined. Combining (2.7) and (2.8) we write the relativistic quasi-four-momentum p^* as

$$p^{*\mu}=p^\mu+[e^2A_0^2/(4k\cdot p)]k^\mu \qquad (2.9)$$

so that we can define the effective mass m^* as[1]

$$m^*=m[1+\tfrac{1}{2}(eA_0/m)^2]^{1/2}\ . \qquad (2.10)$$

Since the Kibble parameter

$$\varepsilon_K=[E_q/(mc^2)]^{1/2}=eA_0/(2m) \qquad (2.11)$$

is relevant whenever the quiver energy of the electron is comparable to or larger than its rest mass energy we see from (2.9) and (2.10) that the radiation field could give rise to relativistic effects to a nonrelativistic electron. Hence, once this relativistic contribution has been taken into account in a time-averaged manner we can always go back to the nonrelativistic picture if the bare electron's energy is nonrelativistic.

If the electrons in the plasma are in states specified by the quasi-four-momentum $p^{*\mu}$ as defined in (2.9) and form a gas, then the modified Boltzmann equation can be written as

$$\frac{\partial f^*}{\partial t}=n\int dv_1^*\int d\Omega\frac{d\sigma^*}{d\Omega}(f^{*\prime}f_1^{*\prime}-f^*f_1^*)|\mathbf{v}^*-\mathbf{v}_1^*|\ , \qquad (2.12)$$

where the quantities with subscript 1 indicates the ion's parameters and those without any subscript as those of the electron's parameters. The asterisk indicates that these depend on quasimomentum $\mathbf{p}^*$ and $\mathbf{v}^*$ are the quasivelocities. The primes denote that final velocities $v^{*\prime}$ are to be used. $d\sigma^*/d\Omega$ is the scattering cross section for the dressed electron by the ion. We assume a high-Z plasma when electron-ion collisions dominate over electron-electron collisions.[2,9]

Since the ions are assumed to be stationary we can do the $\mathbf{v}_1$ integral so that we get

$$\frac{\partial f^*}{\partial t}=n\int d\Omega(f^{*\prime}-f^*)v^*\frac{d\sigma}{d\Omega}\ . \qquad (2.13)$$

We solve this by assuming that

$$f^* = f(v^*) f(v^*,t) , \tag{2.14}$$

$$f^{*'} = f(v^{*'}) f(v^*,t) . \tag{2.15}$$

Hence, the time-dependent part of the evolving distribution function depends only on the initial velocity v^*. Using the method of the separation of variables and the restriction that

$$f(v^*,0)=1 , \tag{2.16}$$

we get the solution of the time-dependent part as

$$f(v^*,t)=e^{\Lambda t} \tag{2.17}$$

where

$$\Lambda(v^*)=\sum_l \left[n\int d\Omega [f(v^{*'})-f(v^*)] v^* \frac{d\sigma^l}{d\Omega} \right] \Big/ f(v^*) . \tag{2.18}$$

Substituting (2.17) and (2.18) back into (2.14) gives us the general solution of the Boltzmann equation where $f(v^*)$ is any initial distribution function for a quasi-free-electron gas. Since Λ is not a function of the angles of $\mathbf{v}^*$ we are justified in pulling out the time-dependent part $f(v^*,t)$ out of the integral in (2.13).

III. EVOLUTION OF THE MODIFIED MAXWELLIAN DISTRIBUTION IN THE PRESENCE OF INTENSE LASER FIELD

We now use the result of the preceding section to evaluate the evolution of a Maxwellian gas when the laser radiation is very intense. In the presence of intense laser field a Maxwellian electron gas is changed so that it follows the modified Maxwellian distribution given by[1]

$$f(v^*)=[m^*/(2\pi k_B T)]^{3/2} e^{-m^*(v^*)^2/(2k_B T)} \tag{3.1}$$

The scattering cross section of the dressed electron for l-photon absorption has been derived in our previous paper[1] and is given by

$$\frac{d\sigma^l}{d\Omega}=\int (E_i^*/p_i^*) dR^{*l} , \tag{3.2}$$

where the transition rate dR^{*l} from a quasifree state to a quasifree state has been derived in Ref. 1 [Eq. (2.31)].

Substituting (3.1) and (3.2) into (2.18), taking the nonrelativistic approximation, and doing the E_f integration (see Ref. 1 for details), we get (Fig. 1)

$$\Lambda(v^*)=-4n[e^2 Z/(4\pi)]^2 \sum_{l=1}^{+\infty} \int d\Omega_i \int d\Omega_f p_f^* E_f^* \frac{e^{-l\hbar\omega/k_B T}-1}{|\mathbf{p}_f^*-\mathbf{p}_i^*-l\hbar\mathbf{k}|^4} J_l^2(e\mu_1) , \tag{3.3}$$

where J_l is the Bessel function of order l and its argument $e\mu_1$ is defined as

$$e\mu_1=[eA_0/(\omega mc)][\epsilon\cdot(\mathbf{p}_i^*-\mathbf{p}_f^*)] ; \tag{3.4}$$

ϵ is the polarization vector of the laser field. We have assumed that the laser intensity is not high enough to make the Kibble parameter ε_K greater than unity. In (3.3), Ω_i and Ω_f are the solid angles in the initial and the final momenta directions.

Because of the nonrelativistic approximation, (3.3) becomes

$$\Lambda(v^*)=-4n[e^2 Z/(4\pi)]^2 m^* c^2 \sum_{l=1}^{+\infty} p_f^*(e^{-l\hbar\omega/k_B T}-1)\cdot I_l , \tag{3.5}$$

where the integral

$$I_l=\int d\Omega_i \int d\Omega_f J_l^2(e\mu_1)/|\mathbf{p}f^*-\mathbf{p}i^*-l\hbar\mathbf{k}|^2 . \tag{3.6}$$

Assuming that the Kibble parameter ε_K is negligible compared to unity, the argument of the Bessel function will also be small, and hence we can use the following approximation,

$$J_l(x)\approx x/(2l!) . \tag{3.7}$$

The integral of (3.6) then becomes

$$I_l=[eA_0/(2l!mk)]^2 \int_{-1}^{+1} dx_i \int_{-1}^{+1} dx_f \int_0^{2\pi} d\phi_i \int_0^{2\pi} d\phi_f \frac{(p_f x_f - p_i x_i)^2}{(p_i^2+p_f^2-2p_i p_f \cos\gamma)^2} , \tag{3.8}$$

where γ is the angle between $\mathbf{p}_i$ and $\mathbf{p}_f$. The latters have coordinates (p_i,θ_i,ϕ_i) and (p_f,θ_f,ϕ_f), respectively, with laser polarization ϵ along the z-direction. The x's are defined by

$$x_i=\cos\theta_i, \quad x_f=\cos\theta_f . \tag{3.9}$$

The angle γ is given in terms of x_i, x_f, ϕ_f as

$$\cos\gamma=x_i x_f+[(1-x_i^2)(1-x_f^2)]^{1/2}\cos(\phi_i-\phi_f) . \tag{3.10}$$

If the plasma is cold such that

$$2m^*/\hbar\omega \gg p^{*2} , \tag{3.11}$$

then the integrations could easily be done. (See Appendix A.) The final answer is then

$$\Lambda(v^*)=\alpha p^{*2} \tag{3.12}$$

where

$$\alpha=-[25n/(3\hbar k)][e^4 Z A_0/(2m^*\hbar k)]^2 \times \sum_{l=1}^{+\infty}(e^{-l\hbar\omega/k_B T}-1)/[(l)(l!)^2] . \tag{3.13}$$

Combining with (2.14) and (2.17) and introducing a normalization constant $C(t)$, we get the evolving quasi-Maxwellian distribution as

$$f^*=C(t)[m^*/(2\pi k_B T)]^{3/2} \times e^{-p_i^{*2}/(2m^* k_B T)[1-\alpha 2m^* k_B T t]} . \tag{3.14}$$

Defining a time-dependent temperature $T^*(t)$ given by

$$T^*(t)=T[1-\alpha(2m^* k_B T)t]^{-1} , \tag{3.15}$$

we find the evolving distribution as

$$f^*(v^*,t)=\{m^*/[2\pi k_B T^*(t)]\}^{3/2} e^{-p_i^{*2}/[2m^* k_B T^*(t)]} . \tag{3.16}$$

The α of (3.13) has two parts of opposite signs and one part is due to the contribution from the absorption of photons and the other part is due to the stimulated emission process of the electron in the laser field. Since $l\hbar\omega \gg k_B T$ we see from (3.13) that α is positive and hence from (3.15) our treatment is valid for time

$$t<1/(\alpha_R) \tag{3.17}$$

where now the new coefficient α_R is given by

$$\alpha_R \approx 28.125 n(2m^* k_B T)/(3\hbar k)[e^3 Z A_0/(2m^*\hbar k)]^2 . \tag{3.18}$$

The expression (3.16) tells us that if we start with a modified Maxwellian distribution when the laser field is turned on then the distribution remains Maxwellian, with only the temperature increasing with respect to time, according to the expression (3.15). This is a surprising result since we have not considered any relaxation process in our treatment. If we expand (3.16) in a Taylor series, with respect to the parameter $\alpha_R t$, then we get back the Maxwellian distribution as the first term that corroborates what Jones and Lee[9] have obtained earlier.

The time-dependent heating rate W for the complete Maxwellian distribution (3.16) is exactly that of our earlier treatment,[1] except that the temperature T there is replaced by $T^*(t)$ as given in (3.15). For the nonrelativistic case that we have treated, the spin contribution would be negligible.

IV. EVOLUTION OF A HOT PLASMA IN INTENSE LASER RADIATION

Here, we again use the expression (3.3) but with the following approximations,

$$mc^2 \gg p_{f,i}^2/(2m) \gg l\hbar\omega . \tag{4.1}$$

When the intensity is such that the Kibble parameter ε_K is much smaller than unity, then the argument $e\mu_1$, given by (3.4), of the Bessel function in (3.3) is also small compared to unity. Hence, we still can use (3.5) together with the integral I_l given by (3.8). Doing the ϕ_i and the ϕ_f integrations and assuming that $p_f \approx p_i = p$, the integral (3.8) reduces to (see Appendix A)

$$I_l=A_l\int_{-1}^{+1}dx_i\int_{-1}^{+1}dx_f\frac{(x_f-x_i)^2(1-x_i x_f)}{[a_l^2+(x_i-x_f)^2]^{3/2}} , \tag{4.2}$$

where

$$a_l^2=(m^* l\hbar\omega/p^2)^2 \tag{4.3}$$

and

$$A_l=[\pi e A_0/(2l!mkp)]^2 . \tag{4.4}$$

The factor a_l in the denominator of the argument of the integrals in (4.2) saves it from becoming infinite and is crucial to our analysis. Performing the rest of the integrations in (4.2) (see Appendix B) we get

$$I_l=A_l\tfrac{4}{3}[\ln(a_l)-\tfrac{19}{3}] . \tag{4.5}$$

As $\ln(a_l)$ is small on the average, hence, I_l is negative in value. Substituting back into (3.5), we get

$$\Lambda=-n_0[e^2 Z/(2\pi)]^2\hbar\omega/(k_B T)(m/p)[\pi e A_0/(2mk)]^2 S , \tag{4.6}$$

where

$$S=\sum_{l=1}^{\infty}[\tfrac{19}{3}-\ln(a_l)]/[l!(l-1)!] . \tag{4.7}$$

Substituting (4.6) back into (2.17) and assuming the initial distribution for the electron gas to be Maxwellian given by (3.1), we get [after combining with (2.14)],

$$f(t)=C(t)[A/(2\pi)]^{3/2}e^{-Av^2-B(t)/v} , \tag{4.8}$$

where

$$A=m/(2k_B T) ,$$
$$B(t)=n_0[e^2 Z/(2\pi)]^2[\hbar\omega/(k_B T)][\pi e A_0/(2mk)]^2 S t . \tag{4.9}$$

The quantity $C(t)$ is a time-dependent normalization factor. We first consider the case when the time t is such that $B(t)$ is much smaller than the average value of the velocity v so that the usual normalization of (4.8) can be written as

$$C(t)4\pi(A/\pi)^{3/2}\int_0^{\infty}dv\, e^{-Av^2}v(v-B)=1 . \tag{4.10}$$

Therefore the distribution function then becomes

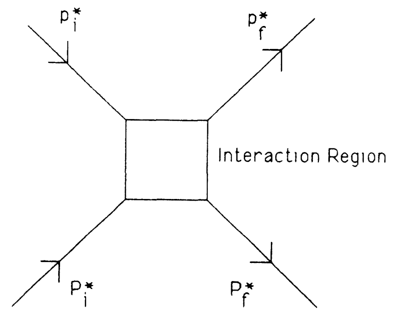

FIG. 1. Quasi-free-electron scattering from a quasi-free-ion via Coulomb interaction.

$$f(t)=[1-2AB(t)/\pi^{1/2}]^{-1}[A/(2\pi)]^{3/2}e^{-Av^2-B(t)/V} . \tag{4.11}$$

A plot of this evolving distribution function can easily be done by a small computer.

V. CONCLUSION

The usual method employed to solve the problem of the time evolution of a plasma in the presence of a weak radiation field has been to use the Boltzmann equation and expand the distribution function in an infinite series. When the laser field is intense, this scheme does not work mainly because the series cannot be truncated fast enough. Moreover, our analysis clearly show that the four momentum of the electron gets modified by the laser field. Taking this modification into consideration by introducing quasi-four-momentum we have shown that, in fact, the Boltzmann equation becomes simpler insofar as to lend itself to the technique of the separation of variables. This leads to a solution that is the product of the initial distribution function and a time evolution part which depends on the differential cross section of the scattering process involved in the heating of the plasma. Also, this solution is true for the relativistic case (except for the case where the ions are assumed to be in motion in which case a factor of $\frac{1}{2}$ has to be introduced).

We applied the result for the nonrelativistic case of a Maxwellian electron gas interacting with stationary ions and calculated the time evolving part of the distribution function. Finally, we showed that this amounts to the fact that the Maxwellian distribution remains Maxwellian but the temperature increases as given in Eq. (3.9). A plausible explanation of this could be that we are dealing with an infinite plasma and hence instead of disjoint phase-space heating as with a finite plasma[2] we get a uniform increase in temperature due to the availability of large sample space.

In the future, we would like to investigate the time development of other distribution functions and explore the relativistic case too.

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APPENDIX A

The integral in (3.8) is of the type

$$I=\int_{-1}^{+1}dx_i\int_{-1}^{+1}dx_f\int_0^{2\pi}d\phi_i\int_0^{2\pi}d\phi_f\frac{(p_fx_f-p_ix_i)^2}{(p_i^2+p_f^2-2p_ip_f\cos\gamma)^2} , \tag{A1}$$

where γ is defined by (3.10) and (3.9). The first integration we try in (A1) is over ϕ_i and is given by the type

$$\begin{aligned}I_1&=\int_0^{2\pi}d\phi_i[A+B\cos(\phi_i-\phi_f)]^{-2} ,\\&=2\int_0^{2\pi}d\phi_i[A+B\cos\phi_i]^{-2} ,\end{aligned} \tag{A2}$$

where

$$\begin{aligned}A&=p_i^2+p_f^2-2p_ip_fx_ix_f ,\\B&=-2p_ip_f[(1-x_i^2)(1-x_f^2)]^{1/2} .\end{aligned} \tag{A3}$$

In (A2), we have used Relation (3.032.2) of Ref. 14. Using Eq. (2.554.3) of Ref. 14, we get

$$I_1=2\pi A[A^2-B^2]^{-3/2} . \tag{A4}$$

As A and B are independent of ϕ_f, the integration over the latter is just 2π. Hence, the integration of I_1 over the variable x_f becomes

$$I_2=\int_{-1}^{+1}dx_f(A'x_f^3+B'x_f^2+C'x_f+D')R^{-3/2} , \tag{A5}$$

where

$$\begin{aligned}&A'=-2p_ip_f^3x_i,\quad B'=p_f^2(p_f^2+p_i^2)+4p_f^2p_i^2x_i^2 ,\\&C'=-2p_i^3p_fx_i^3-2p_fp_i(p_f^2+p_i^2)x_i ,\\&D'=p_i^2(p_f^2+p_i^2)x_i^2,\quad R=a+bx_f+cx_f^2 ,\\&a=(p_f^2-p_i^2)^2-(2ml\hbar\omega)^2,\quad b=-4p_ip_f(p_i^2+p_f^2)x_i ,\\&c=4p_i^2p_f^2 .\end{aligned} \tag{A6}$$

Under the approximation (3.11), the factor as defined in (A6) will dominate in the denominator of the integrand in (A5). Hence the overall integral

$$\begin{aligned}I=[(2\pi)^2/(2ml\hbar\omega)^3]\int_{-1}^{+1}dx_i\int_{-1}^{+1}dx_f(p_ix_i-p_fx_f)^2&\\\times(p_i^2+p_f^2&\\-2p_ip_fx_ix_f) .&\end{aligned} \tag{A7}$$

This integral is straight forward, and retaining only factors relevant for normalization we get (3.12) of the text.

APPENDIX B

To perform the integration in (4.2), we first use the transformation

$$x_i = x_f + x \ , \tag{B1}$$

where x is the new variable. The integral in (4.2) then becomes

$$I = \int_{-1}^{+1} dx_f \int_{-(1+x_f)}^{(1-x_f)} dx \frac{x^2[1-(x+x_f)x_f]}{[a_l^2+x^2]^{3/2}} \ . \tag{B2}$$

Using Eqs. (2.261) and (2.264) from Ref. 14 and canceling terms that are antisymmetric, we get

$$I = \int_{-1}^{+1} dx_f \{ -2 + (1-x_f^2)[\ln(1+x_f)+\ln(1-x_f) - \ln(a_l^2/4)] \} \ . \tag{B3}$$

The last integration in (B3) can easily be done using Eq. (4.293) of Ref. 14 to get (4.5) of the text.

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[1] S. Rashid, Phys. Rev. A **38**, 2525 (1988).
[2] A. B. Langdon, Phys. Rev. Lett. **44**, 575 (1980).
[3] H. Brehme, Phys. Rev. C **3**, 837 (1971).
[4] Ehlotsky, Opt. Commun. **13**, 1 (1975).
[5] I. Goldman, Phys. Lett. **2**, 103 (1964).
[6] G. J. Pert, J. Phys. A **5**, 506 (1972).
[7] N. M. Kroll and K. M. Watson, Phys. Rev. A **8**, 804 (1972).
[8] B. K. Osborn, Phys. Rev. A **5**, 1660 (1972).
[9] R. D. Jones and K. Lee, Phys. Fluids **25**, 2307 (1982).
[10] I. P. Shkarofsky, T. W. Johnston, and M. P. Bachynski, *The Kinetics of Plasmas* (Addison-Wesley, Reading, MA, 1966).
[11] D. C. Montgomery and D. A. Tidman, *Plasma Kinetic Theory* (McGraw-Hill, New York, 1964).
[12] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).
[13] V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, *Relativistic Quantum Theory* (Part I) (Pergamon, New York, 1971).
[14] I. S. Gradsteyn and I. M. Ryzik, *Table of Integrals, Series, and Products* (Academic, New York, 1980).