Ultrarelativistic envelope solitons in a magnetized electron-positron plasma

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The nonlinear propagation of intense electromagnetic radiation in a pulsar magnetosphere is investigated. The radiation is considered to be a circularly polarized electromagnetic wave, in whose field electrons and positrons acquire ultrarelativistic velocities. The nonlinear frequency shift due to the wave-plasma interaction is found to cause the wave localization, and it produces a new kind of envelope soliton, which in coupling with the ambient magnetic field generates intense ambipolar field along the magnetic lines.

In this paper, we reconsider the nonlinear propagation of a field-aligned (parallel to the external magnetic field) ultrastrong electromagnetic wave in an electron-positron plasma. Our previous investigations are thus generalized to include the ultrarelativistic nonlinear effects on the wave propagation. The field amplitude is assumed to be so large that particles acquire ultrarelativistic velocities in this field. It is found that the frequency shift due to the wave-plasma interaction causes the wave to localize and a new kind of envelope soliton is produced. These localized radiations (solitons), in coupling with the ambient magnetic field, generate an intense ambipolar field along magnetic lines.

The basic equations describing the relativistic hydrodynamics in strong high-frequency fields are the relativistic two-fluid equations of motion, the continuity, the wave equation, and the Poisson equation:

$$\partial_t \mathbf{P}_j + \mathbf{v}_j \cdot \nabla \mathbf{P}_j = e_j \mathbf{E} + \frac{e_j}{C}(\mathbf{v}_j \times \mathbf{B}) - \frac{T_j}{n_j}\nabla n_j \ , \tag{1}$$

$$\partial_t n_j + \nabla \cdot n_j \mathbf{v}_j = 0 \ , \tag{2}$$

$$\left[\nabla^2 - \frac{1}{C^2}\partial_t^2\right]\mathbf{E} = \frac{4\pi}{C^2}\partial_t \mathbf{J} \ , \tag{3}$$

$$\nabla \cdot \mathbf{E} = 4\pi \sum_j e_j n_j, \quad \mathbf{J} = \sum_j e_j n_j \mathbf{v}_j, \quad \mathbf{P}_j = \frac{m_0 \mathbf{v}_j}{(1 - v_j^2/C^2)^{1/2}} \ , \tag{4}$$

where j refers to $p = e^+$ and $e = e^-$, $e_j = \pm e$ is the charge, $\mathbf{v}_j$ is the particle velocity, T_j is the temperature, and m_0 is the particle rest mass.

We consider the intense radiation as a circularly polarized electromagnetic wave which propagates along the ambient magnetic field $\mathbf{B}_0 = B_0 \hat{\mathbf{z}}$ and all the quantities do not depend on x and y but on z and time t,

$$\begin{aligned} E_x + iE_y &= \tilde{E} = E_0(z,t)e^{-i(\omega t - kz)} \ , \\ P_{jx} + iP_{jy} &= \tilde{P}_j = P_{0j}(z,t)e^{-i(\omega t - kz)} \ . \end{aligned} \tag{5}$$

Here, for simplicity the right-hand polarization of the wave is taken into account (the left-hand polarization may equally be treated) and E_0, P_{0j} are assumed to be slowly varying complex amplitudes. For such a case the transverse motion is easily integrable and one finds[1]

$$\tilde{P}_j = \frac{ie_j \tilde{E}}{\omega - \omega_{C_j}/\gamma_j}, \quad \omega_{C_j} = \frac{e_j B_0}{m_0 C} \equiv \frac{e_j}{|e|}\omega_C \ , \tag{6}$$

where

$$\begin{aligned} \gamma_j &= (1+v_j^2)^{1/2} \ , \\ v_j^2 &= \frac{|\tilde{P}_j|^2}{m_0^2 C^2} = \frac{\omega_E^2(1+v_j^2)}{[\omega(1+v_j^2)^{1/2} - \omega_{C_j}]^2} \ , \\ \omega_E^2 &= \frac{e^2|E|^2}{m_0^2 C^2} \ . \end{aligned} \tag{7}$$

The ponderomotive forces, exerted by the radiation pressure is given by

$$F^j_{\text{pond}} = \left\langle \frac{e_j}{C}(V_{jx}B_y - v_{jy}B_x) \right\rangle \ , \tag{8}$$

which, by using (6) and (7) attain the following:

$$F^e_{\text{pond}} = -\frac{e^2}{2m_0\omega[\omega(1+v_e^2)^{1/2} + \omega_C]}\partial_z |E|^2 \ , \tag{9}$$

$$F^p_{\text{pond}} = -\frac{e^2}{2m_0\omega[\omega(1+v_p^2)^{1/2} - \omega_C]}\partial_z |E|^2 \tag{10}$$

for electrons and positrons, respectively.

Thus the slow plasma motion along the ambient magnetic field is described by the equations

$$\partial_t P_{ez} = e\partial_z \phi - \frac{e^2}{2m_0\omega[\omega(1+v_e^2)^{1/2} + \omega_C]}\partial_z |E|^2 - \frac{T_e}{n_e}\partial_z n_e \ , \tag{11}$$

$$\partial_t P_{pz} = -e\partial_z \phi - \frac{e^2}{2m_0\omega[\omega(1+v_p^2)^{1/2} - \omega_C]}\partial_z |E|^2 - \frac{T_p}{n_p}\partial_z n_p \ , \tag{12}$$

$$\partial_t n_e + \partial_z n_e v_{ez} = 0 \ , \tag{13}$$

$$\partial_t n_p + \partial_z n_p v_{pz} = 0 \ , \tag{14}$$

$$\partial_z^2 \phi = 4\pi e (n_e - n_p) \ . \tag{15}$$

We consider the case of ultrarelativistic wave ($v_j^2 \gg 1$) propagating in the electron-positron plasma. For such a case, from Eq. (7), we see that

$$\gamma_e \simeq |v_e| = \frac{\omega_E - \omega_C}{\omega} \gg 1 \ , \tag{16}$$

$$\gamma_p \simeq |v_p| = \frac{\omega_E + \omega_C}{\omega} \gg 1 \ . \tag{17}$$

The above conditions imply that $\omega_E - \omega_C \gg \omega$ and the dispersion relation for the wave is

$$\frac{k^2 C^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega \omega_E} \ , \tag{18}$$

where $\omega_p^2 = 8\pi e^2 n_0 / m_0$ is the plasma frequency with n_0 being the background plasma density.

Let

$$|E| = a = a_0 + \delta a(z,t)$$

and

$$n_j = n_0 (1 + N_j), \quad N_j = \frac{\delta n_j}{n_0} \ ,$$

where a_0 is constant and $\delta a, \delta n_j$ are the small perturbations. Then for the case of ultrarelativistic wave, which satisfies the condition

$$\omega_E - \omega_C \gg \omega \gg \delta\omega_a \left[\delta\omega_a = \frac{e \delta a}{m_0 C} \right] \ ,$$

we have

$$\gamma_e \simeq \gamma_{0e} = \frac{\omega_{a_0} - \omega_C}{\omega}, \quad \gamma_p \simeq \gamma_{0p} = \frac{\omega_{a_0} + \omega_C}{\omega}, \quad \omega_{a_0} = \frac{e a_0}{m_0 C} \ . \tag{19}$$

The slow plasma response Eqs. (11)–(14) may be written as

$$\partial_t^2 N_e + \frac{1}{\gamma_{0e}} \partial_z^2 \left[\frac{e}{m_0} \phi - \frac{eC}{m_0 \omega} |E| - V_{te}^2 N_e \right] = 0 \ , \tag{20}$$

$$\partial_t^2 N_p + \frac{1}{\gamma_{0p}} \partial_z^2 \left[-\frac{e}{m_0} \phi - \frac{e_C}{m_0 \omega} |E| - V_{tp}^2 N_e \right] = 0 \ , \tag{21}$$

where

$$V_{te}^2 = \frac{T_e}{m_0}, \quad V_{tp}^2 = \frac{T_p}{m_0} \ .$$

Let us consider a linear space shift of the wave due to the interaction. In other words, we represent the complex field amplitude as

$$E_0 = |E| e^{-i\theta t + i\kappa z} \ , \tag{22}$$

where θ and κ are constants. Then it can be shown[5] that

$$|E| = f(\xi), \quad \xi = z - V_{0t}, \quad V_0 = \frac{kC^2}{\omega} \left[1 + \frac{\kappa}{k} \right] \ . \tag{23}$$

We assume that all the low-frequency perturbations (N_e, N_p, ϕ) are caused only by the ponderomotive force, which is the function of pump wave amplitude $|E| = f(\xi)$. Therefore, N_e, N_p, and ϕ are also some function $F(\xi)$. Thus the density perturbation of plasma species can easily be determined from Eqs. (20) and (21), which give

$$N_e = -\frac{1}{V_0^2 \gamma_{0e} - V_{te}^2} \left[\frac{e\phi}{m_0} - \frac{eC}{m_0 \omega} \delta a \right] \ , \tag{24}$$

$$N_p = -\frac{1}{V_0^2 \gamma_{0p} - V_{tp}^2} \left[-\frac{e\phi}{m_0} - \frac{eC}{m_0 \omega} \delta a \right] \ , \tag{25}$$

where we imply the boundary conditions

$$|E| \to a_0; \delta a, \Phi, N_e, N_p \to 0 \ \text{ as } |\xi| \to \infty \ . \tag{25a}$$

Thus using (24) and (25), from the wave equation (3) and the Poisson equation (15), we get the coupled system of equations

$$\beta d_\xi^2 \delta a = \Delta a_0 + (\Delta + \chi_{a_0}) \delta a + \chi_{B_0} \phi \ , \tag{26}$$

$$d_\xi^2 \phi = -\chi_{a_0} \phi - \frac{c^2}{\omega^2} \chi_{B_0} \delta a \ , \tag{27}$$

with

$$\beta = 1 - V_0^2 / C^2, \quad \Delta \equiv \frac{(\delta\omega)^2}{C^2} = \frac{2\omega\theta - \theta^2 + 2k\kappa^2 C^2 + x^2 C^2}{C^2} \ ,$$

$$\chi_{a_0} = \frac{\omega \omega_p^2}{V_0^2} \frac{2\omega_{a_0} - \omega (V_{te}^2 / v_0^2 + V_{tp}^2 / v_0^2)}{(\omega_{a_0} - \omega_c - \omega V_{tc}^2 / V_0^2)(\omega_{a_0} + \omega_C - \omega V_{tp}^2 / V_0^2)} \ ,$$

$$\chi_{B_0} = -\frac{\omega_p^2 \omega^2}{V_0^2 C} \times \frac{2\omega_C + \omega (V_{tc}^2 / V_0^2 - V_{tp}^2 / V_0^2)}{(\omega_{a_0} + \omega_C - \omega V_{tp}^2 / V_0^2)(\omega_{h_0} - \omega_C - \omega V_{te}^2 / V_0^2)} \ . \tag{27a}$$

The system of equations (26) and (27) describes the modulation of ultrarelativistic wave in magnetized electron-positron plasma.

To solve Eqs. (26) and (27), we introduce the dimensionless variables

$$\eta = \frac{\omega_p}{C} \xi, \quad \psi = \frac{\delta a}{a_0}, \quad \varphi = \frac{e\phi}{m_0 C^2}$$

and rewrite the system

$$d_\eta^2 \psi = \epsilon + \kappa_1 \psi + \lambda_1 \varphi \equiv G(\varphi, \psi) \ , \tag{28}$$

$$d_\eta^2 \varphi = \kappa_2 \varphi + \lambda_2 \psi \equiv F(\varphi, \psi) \ , \tag{29}$$

with

$$\epsilon=\frac{(\delta\omega)^2}{\omega_p^2\beta}, \quad \lambda_1=\frac{C^3}{\omega_p^2\omega_{a_0}\beta}\chi_{B_0},$$

$$\kappa_1=\epsilon+\frac{C^2}{\omega_p^2\beta}\chi_{a_0}, \quad \lambda_2=-\frac{C^3\omega_{a_0}}{\omega^2\omega_p^2}\chi_{B_0}, \tag{29a}$$

$$\kappa_2=-\frac{C^2}{\omega_p^2}\chi_{a_0}.$$

The above system has an "integral of motion"

$$\frac{\lambda_2}{2}(d_\eta\psi)^2+\frac{\lambda_1}{2}(d_\eta\varphi)^2 =\epsilon\lambda_2\psi+\frac{\kappa_1\lambda_2}{2}\psi^2+\frac{\kappa_2\lambda_1}{2}\varphi^2+\lambda_1\lambda_2\varphi\psi+E \equiv H(\varphi,\psi), \tag{30}$$

where the constant of integration $E=0$ can be obtained from the boundary conditions (25a) and we have introduced the functions $G(\varphi,\psi)$, $F(\varphi,\psi)$, and $H(\varphi,\psi)$ for further use.

Making use of the "energy integral" (30), it is possible to eliminate the independent variable η (see our earlier paper[5] and the references therein) between (28) and (29), yielding the following equation for φ in terms of ψ alone:

$$2H(\varphi,\psi)\frac{d^2\varphi}{d\psi^2}+\lambda_1G(\varphi,\psi)\left[\frac{d\varphi}{d\psi}\right]^3-\lambda_1F(\varphi,\psi)\left[\frac{d\varphi}{d\psi}\right]^2 +\lambda_2G(\varphi,\psi)\frac{d\varphi}{d\psi}-\lambda_2F(\varphi,\psi)=0. \tag{31}$$

As we assume that the density fluctuation and hence the ambipolar field generation are caused by the ponderomotive pressure of the high-frequency wave, we consider the electrostatic potential as a function of the driving field amplitude and introduce the expansion

$$\varphi=\sum_n C_n\psi^n, \quad n=1,2,3,\ldots, \tag{32}$$

provided the series converges rapidly (the convergency will be investigated later).

Making use of the expansion (32) and (31), a power series in ψ is obtained in terms of free parameters. This is done in the Appendix, where an argument on the convergence of the series is also given. It is found that $C_1=0$, $C_2\psi_{\max}^2\sim\epsilon$, $C_3\psi_{\max}^3\sim\epsilon^2$, and so on, where $\epsilon=(\delta\omega)^2/\omega_p^2\ll 1$ is a small parameter.

We then assume

$$\varphi(\xi)=C_2\psi^2, \tag{33}$$

and using (33) in (28), we get

$$(d_\eta\psi)^2=\alpha_1\psi+\alpha_2\psi^2+\alpha_3\psi^3, \tag{34}$$

where

$$\alpha_1=2\epsilon, \quad \alpha_2=\kappa_1, \alpha_3=\tfrac{2}{3}\lambda_1C_2. \tag{35}$$

For $\alpha_1>0$, Eq. (34) admits the localized solution[5]

$$\psi=\frac{\beta_1\beta_2\operatorname{sech}^2(\chi\eta)}{\beta_1-\beta_2\tanh^2(\chi\eta)}, \tag{36}$$

where

$$\chi=\tfrac{1}{2}\sqrt{\alpha_1}, \quad \beta_{1,2}=\frac{1}{2\alpha_3}[-\alpha_2\mp(\alpha_2^2-4\alpha_1\alpha_3)^{1/2}]. \tag{37}$$

The physical requirement of $\alpha_1>0$ for the localized solution (36) implies the condition $V_0^2/C^2<1$ (subluminous soliton). An analysis of the expression for V_0 [Eq. (23)] and the dispersion relation [Eq. (18)] shows that for the case of ultrarelativistic wave $V_0^2/C^2<1$ (always), since in this case $\omega\omega_{a_0}>\omega_p^2$. Therefore subluminous envelope solitons are the final state of the modulation of the ultrarelativistic wave.

The density perturbations are given by

$$\frac{\delta n_e}{n_0}=\frac{C^2}{V_0^2}\frac{\omega}{\omega_{a_0}-\omega_C-\omega V_{te}^2/V_0^2}\left[\frac{\omega_{a_0}}{\omega}-C_2\psi\right]\psi, \tag{38}$$

$$\frac{\delta n_p}{n_0}=\frac{C^2}{V_0^2}\frac{\omega}{(\omega_{a_0}+\omega_C-\omega V_{tp}^2/V_0^2)}\left[\frac{\omega_{a_0}}{\omega}+C_2\psi\right]\psi. \tag{39}$$

We calculate the charge density fluctuation along the magnetic lines. Assume that radiation pressure is much higher than the thermal pressure. Then, neglecting the thermal terms, we get

$$e\delta N\equiv e\left[\frac{\delta n_e}{n_0}-\frac{\delta n_p}{n_0}\right] =e\frac{2C^2}{V_0^2}\frac{\omega\omega_{a_0}}{(\omega_{a_0}^2-\omega_C^2)}\left[\frac{\omega_C}{\omega}-C_2\psi\right]\psi, \tag{40}$$

and the maximum density fluctuation is $(\delta N)_{\max}=\frac{2}{3}(\omega_c/\omega)\beta\epsilon$, which may generate an intense ambipolar field along the magnetic lines.

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APPENDIX

Here we give the first coefficients of the expansion (32) and discuss the convergence of the series.

Inserting the expansion (32) in (31), a power series in ψ is obtained and the C_n's can be determined.

$$C_1=0\ , \tag{A1}$$

$$C_2=\frac{1}{6}\frac{C^2}{V_0^2}(1-V_0^2/C^2)^{1/2}\frac{\omega_{a_0}\omega_p^2}{(\delta\omega)^2}\frac{2\omega_C+\omega(V_{te}^2/V_0^2-V_{tp}^2/V_0^2)}{(\omega_{a_c}+\omega_C-\omega V_{tp}^2/V_0^2)(\omega_{a_0}-\omega_C-\omega V_{te}^2/V_0^2)}\ , \tag{A2}$$

$$C_3=-\frac{1}{15}\left[4+\frac{C^2(1-V_0^2/C^2)^{1/2}}{(\delta\omega)^2}\left[1+\frac{4}{(1-V_0^2/C^2)^{1/2}}\right]\frac{\omega\omega_p^2}{V_0^2}\frac{2\omega_{a_0}-\omega(V_{te}^2/V_0^2+V_{tp}^2/V_0^2)}{(\omega_{a_0}-\omega_e-\omega V_{te}^2/V_0^2)(\omega_{a_0}+\omega_C-\omega V_{tp}^2/V_0^2)}\right]C_2\ . \tag{A3}$$

Since $V_0^2/C^2 \ll 1$ and $(\delta\omega)^2/\omega_p^2 \ll 1$, it is found that $C_2\psi_{\max}^2 \sim \epsilon$, $C_3\psi_{\max}^3 \sim \epsilon^2$, and so on, with $\epsilon=(\delta\omega)^2/\omega_p^2(1-V_0^2/C^2)^{1/2} \ll 1$. Hence the series (32) converges rapidly.

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